2.4. Multi-Dimensional Diffusion Problems

Reference: Tannehill et al 4.3.9-4.2.11.
C.F. Fletcher 8.1, 8.2, 8.5.

For multi-dimensional (MD) problems, one can use

1) Direct extension of 1-D operators. It's the most straightforward method but may not have the best stability property. There can be problems with neglecting cross directive terms in the Taylor expansion.

2) Direction splitting method – we build up a MD problem by successive 1-D passes through the grid in alternating coordinate directions. Each time solving a 1-D problem.

3) Full MD methods designed specifically for MD problems.

We will only discuss the first two methods.

Consider 2-D diffusion equation on a regular x-y domain:

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (K > 0 \text{ and constant}). \qquad (1)$$

2.4.1. Direct extension of FTCS method

$$\delta_{+t}u_{ij}^n = K[\delta_{xx}u_{ij}^n + \delta_{yy}u_{ij}^n] \qquad (2)$$

It is consistent and $\tau = O(\Delta t, \Delta x^2, \Delta y^2)$.

We can find out (show it for yourself), using Neumann stability analysis (assuming $u_{ij}^n = U_k \lambda^n e^{i(kx+ly)}$) that the stability condition is

$$\Delta t \le \frac{1}{2K}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)^{-1}. \qquad (3)$$

For Δx = Δy, $\Delta t \le \frac{\Delta^2}{4K}$, which is <u>twice as restrictive</u> as that for 1-D problem!

2.4.2. Direct extension of general method

$$\frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t} = \alpha K[\delta_{xx}u_{ij}^{n+1} + \delta_{yy}u_{ij}^{n+1}] + (1-\alpha)K[\delta_{xx}u_{ij}^n + \delta_{yy}u_{ij}^n] \qquad (4)$$

Reorganizing →

$$[1 - \alpha \Delta t K(\delta_{xx} + \delta_{yy})]u_{ij}^{n+1} = [1 - \alpha \Delta t K(\delta_{xx} + \delta_{yy})]\, u_{ij}^n \qquad (5)$$

One can find out that the stability condition is

$$\Delta t \le \frac{1}{(2-4\alpha)K}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)^{-1}$$ for $0 \le \alpha < 1/2$ (again more restrictive than the

corresponding 1-D case), and unconditionally stable for $1 \ge \alpha \ge 1/2$.

When $\alpha \ne 0$, the above scheme is implicit, as in the 1-D case.

This system of equations is more difficult to solve, however, due to the involvement of unknowns at more than 3 grid points – in fact, five grid points (five unknown) are involved for this 2-D problem. This can be more clearly seen if (5) is rewritten into the following form:

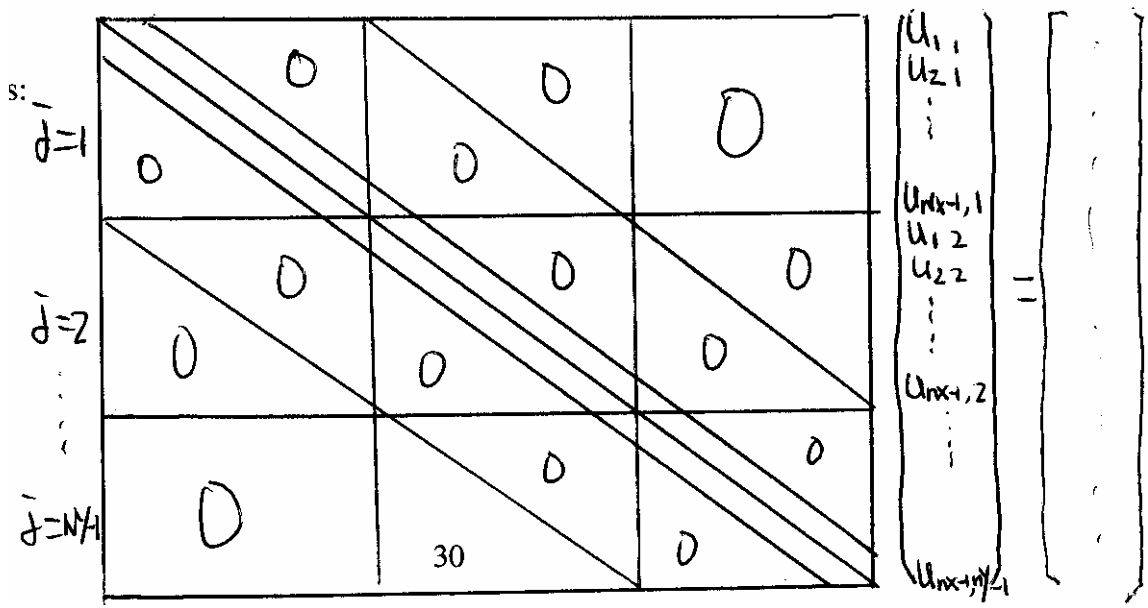
$$a u_{i,j-1}^{n+1} + b u_{i,j+1}^{n+1} + c u_{i,j}^{n+1} + a u_{i+1,j}^{n+1} + b u_{i-1,j}^{n+1} = d_{ij}^{n} \qquad (6)$$

assuming K is constant and Δx=Δy in (5).

In matrix form, we can write (5) as

$$\{[I]-\alpha[A]\}\vec{U}^{n+1} = \{[I]-(1-\alpha)[A]\}\vec{U}^{n} \qquad (7)$$

where [I] is an identity matrix and [A] is block tridiagonal. $\vec{U}$ is a vector consisting of u at all grid points.



This system cannot be solved as efficiently as the tridiagonal system from 1-D problem. Similar system arises from the discretization of elliptic equation $\nabla^2 u = F$. We will discuss methods for solving it at a later time.

2.4.3. Directional Splitting Method

Goal: We look for ways to avoid having to solve the block tridiagonal matrix – we want to get back to a single tridiagonal matrix.

a) Alternating Direction Implicit (ADI) method

One of the best known way of doing this is the alternating direction implicit (ADI) scheme due to Peaceman and Rachford.

The basic idea is to write the single full time step as a sum of two half steps, each representing a single coordinate direction:

$$\frac{u_{ij}^{n+1/2} - u_{ij}^{n}}{\Delta t/2} = K[\delta_{xx} u_{ij}^{n+1/2} + \delta_{yy} u_{ij}^{n}]$$

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+1/2}}{\Delta t/2} = K[\delta_{xx} u_{ij}^{n+1/2} + \delta_{yy} u_{ij}^{n+1}] \qquad (8)$$

→ $$[1 - s\delta_{xx}]u_{ij}^{n+1/2} = [1 + s\delta_{yy})]\, u_{ij}^{n} \qquad (9a)$$

$$[1 - s\delta_{yy}]u_{ij}^{n+1} = [1 + s\delta_{xx})]\, u_{ij}^{n+1/2} \qquad (9b)$$

where s = KΔt/2.

The left hand side of the equations can be written in the form of

$$A_i u_{i-1}^{n+1/2} + B_i u_i^{n+1/2} + C_i u_{i+1}^{n+1/2} \quad \text{and} \; A_j u_{j-1}^{n+1} + B_j u_j^{n+1} + C_j u_{j+1}^{n+1}$$

therefore they form two systems of tridiagonal equations. (9a) is first solved for all *j* indices then (9b) is solved for all *i* indices.

Stability: The amplification factor of each full step is simply the product of the amplification factor of the two individual steps.

We can see

$$u^{n+1/2} = \lambda_a u^n$$
$$u^{n+1} = \lambda_b u^{n+1/2}$$

therefore $u^{n+1} = \lambda_a \lambda_b u^n = \lambda u^n$.

Stable if $|\lambda| = |\lambda_a \lambda_b| \le 1$.

We can easily show that

$$\lambda_a = \frac{1 - 2\mu \sin^2(l\Delta y/2)}{1 + 2\mu \sin^2(k\Delta x/2)}$$
$$\lambda_b = \frac{1 - 2\mu \sin^2(k\Delta x/2)}{1 + 2\mu \sin^2(l\Delta y/2)}$$

therefore $|\lambda| \le 1$ for all μ!   The scheme is absolutely stable.

Comment: ADI is unconditionally stable for the 2D diffusion equation, but conditionally stable in 3D. The condition is $K_x \Delta t/\Delta x^2 \le 1, K_y \Delta t/\Delta y^2 \le 1$ and $K_z \Delta t/\Delta z^2 \le 1$.

To overcome the conditionally stability problem with the above 3D version of ADI, Douglas and Gunn (1964) developed a general method for deriving ADI that are unconditionally stable for all dimensions. The method is called approximate factorization.
This is discussed in section 4.2.10 of Tennehill or section 8.2.2 of Fletcher.

b) Local 1-D or Fractional Step Method

There are many ways to split MD problems into a series of 1D problems. So far, we have been splitting the FDE. One can also split the PDE, into a pair of equations for 2D case, which each of them being a <u>local</u> 1D equation. This method was developed by Soviet mathematicians in the early seventies (see Yanenko 1971).

In a sense, the method splits equation

$$\frac{\partial u}{\partial t} = K_x \frac{\partial^2 u}{\partial x^2} + K_y \frac{\partial^2 u}{\partial y^2} \qquad (K_x, K_y > 0 \text{ and constant})$$

into two equations:

$$\frac{1}{2}\frac{\partial u}{\partial t} = K_x \frac{\partial^2 u}{\partial x^2}$$
$$\frac{1}{2}\frac{\partial u}{\partial t} = K_y \frac{\partial^2 u}{\partial y^2}.$$

They can be solved using an <u>explicit</u> scheme:

$$u_{ij}^{n+1/2} = (1 + K_x \Delta t \delta_{xx}) u_{ij}^n$$
$$u_{ij}^{n+1} = (1 + K_y \Delta t \delta_{yy}) u_{ij}^{n+1/2}.$$

When Δx = Δy this scheme is stable for $\mu \le 1/2$, which is only half as restrictive as our full one-step 2-D explicit scheme (FTCS which has $\mu \le 1/4$).

When the implicit Crank-Nicolson scheme is used to solve those two equations, i.e.,

$$(1-0.5K_x\Delta t\delta_{xx})u_{ij}^{n+1/2} = (1+0.5K_x\Delta t\delta_{xx})u_{ij}^{n}$$
$$(1-0.5K_y\Delta t\delta_{yy})u_{ij}^{n+1} = (1+0.5K_y\Delta t\delta_{yy})u_{ij}^{n+1/2}.$$

This scheme is stable for all μ, and we solve two tridiagonal systems per time step.

In practice, we often use the xyyx ordering to avoid directional bias.

The direction splitting method can also be applied to hyperbolic equations, which is the subject of our next chapter.