Chapter 4. Nonlinear Hyperbolic Problems

4.4 System of Hyperbolic Equations – Shallow Water Equation model

Ref: Chapter IV of Mesinger and Arakawa (1976) GARP Report

4.4.1. Introduction

It is assumed that you are familiar with the shallow water equations and associated theories. If not, consult Holton or Haltiner and Williams book.

The following is a set of linear 1D shallow water equations:

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + \frac{\partial \phi'}{\partial x} = 0 \qquad (21a)$$

$$\frac{\partial \phi'}{\partial t} + \bar{u}\frac{\partial \phi'}{\partial x} + \Phi\frac{\partial u'}{\partial x} = 0 \qquad (21b)$$

$\bar{u}$ = constant base state flow
$\Phi = gH$ = $g \times$ mean depth of the water = constant
$u \rightarrow u'$ = perturbation velocity
$\phi = gh'$ = perturbation geopotential height

<u>Issues to consider with respect to numerical solution</u>

1) More than 1 variable
2) Equations coupled
3) Can support multiple physical modes
4) There are more possibilities of grid layout (see figure below)

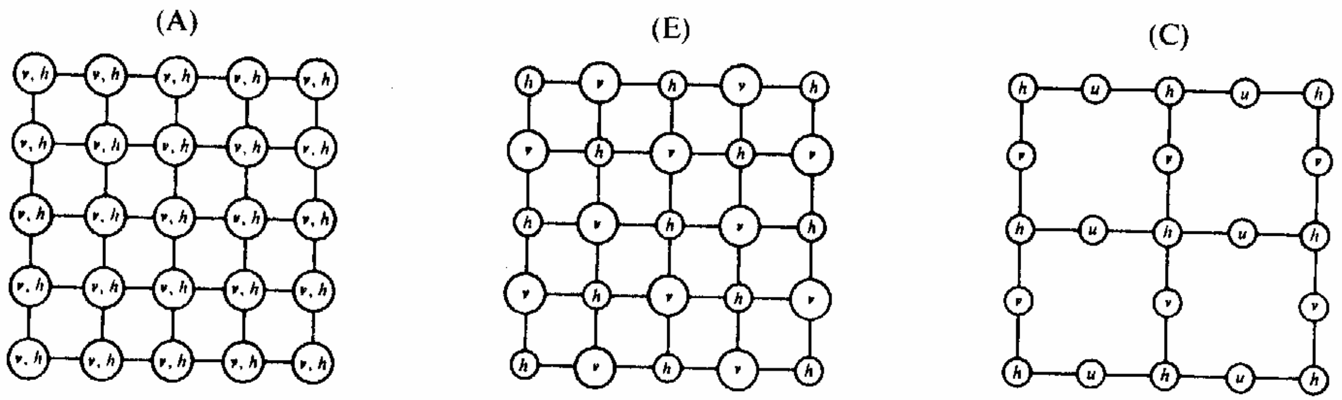

Figure 2.1 Three types of lattice considered for the finite difference solution of 2.1.

4.4.2. The differential solution

Performing standard analysis by assuming

$$\psi = \Psi \exp[i(kx - \omega t)] \qquad (22)$$

gives $\omega = k(\bar{u} \pm \sqrt{\Phi})$ (23)

which is called the dispersion relation.

From (23) →

$$c = \frac{\omega}{k} = \bar{u} \pm \sqrt{\Phi}\ .$$

In the phase speed, there are slow mode represented by $\bar{u}$ (advection) and fast mode given by $\sqrt{\Phi}$ (surface gravity waves). Since c is constant, the waves are non-dispersive.

Group velocity

$$c_g = \frac{\partial \omega}{\partial k} = \bar{u} \pm \sqrt{\Phi}$$

represents the speed of wave energy propagation.

What about the characteristics (we have seen this before – see example problem given at the end of Chapter 1). Make use of the auxiliary equations, we have the following equations in matrix form:

$$\begin{bmatrix} 1 & \bar{u} & 0 & 1 \\ 0 & \Phi & 1 & \bar{u} \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{bmatrix} \begin{pmatrix} u_t \\ u_x \\ \phi_t \\ \phi_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ du \\ d\phi \end{pmatrix} \qquad (24)$$

Setting the determinant of the coefficient matrix to zero gives

$$\left(\frac{dx}{dt}\right)^2 - 2\bar{u}\frac{dx}{dt} + (\bar{u}^2 - \Phi) = 0 \rightarrow$$

$$\frac{dx}{dt} = \bar{u} \pm \sqrt{\Phi}$$

which is the characteristics equations.

The compatibility equations can be found to be

$$u \pm \frac{\phi}{\sqrt{\Phi}} = \text{constant along } \frac{dx}{dt} = \bar{u} \pm \sqrt{\Phi} \,. \tag{25}$$

(25) can be rewritten as

$$\left[\frac{\partial}{\partial t} + \left(\bar{u} + \sqrt{\Phi}\right)\frac{\partial}{\partial x}\right]\left(\bar{u} + \frac{\phi}{\sqrt{\Phi}}\right) = 0 \tag{26a}$$

$$\left[\frac{\partial}{\partial t} + \left(\bar{u} - \sqrt{\Phi}\right)\frac{\partial}{\partial x}\right]\left(\bar{u} - \frac{\phi}{\sqrt{\Phi}}\right) = 0 \tag{26b}$$

which are two decoupled equations describing wave disturbances 'advected' by the respective propagation speeds. $\bar{u} \pm \phi / \sqrt{\Phi}$ are known as the Riemann invariants, as said before.

Equations (26) can also be obtained using matrix method (see Chapter 1).

4.4.3. Discretization for the Shallow Water Equations

4.4.3.1. Forward-backward scheme

We know that FTCS is unstable for pure advection equations, and this is also true to the shallow water equations.

But, we can obtain a stable scheme if we use <u>backward</u> scheme for the second equation. Let's look at the simper case of $\bar{u}$ =0, i.e., there is not mean flow:

$$\begin{aligned} &\delta_{+t} u + \delta_{2x} \phi^n = 0 \\ &\delta_{+t} \phi + \Phi \delta_{2x} u^{n+1} = 0 \end{aligned} \tag{27}$$

Since forward scheme is used for the first eq. and backward scheme used for the second, the overall scheme is called <u>forward-backward</u> scheme. We can show that it is conditionally stable.

<u>Stability Analysis</u>

Assume that

$$\begin{aligned} u_j^n &= A\lambda^n \exp(ikx_j) \\ \phi_j^n &= B\lambda^n \exp(ikx_j) \end{aligned} \tag{28}$$

Note here A and B could be complex so as to account for possible phase difference between u and ϕ.

Plug (28) into (27) →

$$(\lambda^{n+1}-\lambda^{n})A+B\lambda^{n}\frac{\Delta t}{2\Delta x}(e^{ik\Delta x}-e^{-ik\Delta x})=0$$
$$(\lambda^{n+1}-\lambda^{n})B+\Phi A\lambda^{n+1}\frac{\Delta t}{2\Delta x}(e^{ik\Delta x}-e^{-ik\Delta x})=0 \qquad (29)$$

or

$$(\lambda-1)A+iB\frac{\Delta t}{\Delta x}\sin(k\Delta x)=0$$
$$(\lambda-1)B+i\Phi A\frac{\Delta t}{\Delta x}\lambda\sin(k\Delta x)=0 \qquad (30)$$

or

$$\begin{pmatrix} \lambda-1 & i\frac{\Delta t}{\Delta x}\sin(k\Delta x) \\ i\Phi\frac{\Delta t}{\Delta x}\lambda\sin(k\Delta x) & \lambda-1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (30')$$

(30') is a simultaneous linear system of equations for A and B. It has non-trivial solutions if and only if the determinant of the coefficient matrix equals to zero. →

$$\lambda^2-\lambda[2-\Phi a^2]+1=0 \quad [\text{where } a=\Delta t/\Delta x\sin(k\Delta x)]$$

$$\lambda_{\pm}=\frac{\Phi a^2-2\pm\sqrt{(2-\Phi a^2)^2-4}}{2} \qquad (31)$$

If the radical is negative, then $|\lambda_\pm| \equiv 1$. I.e., if

$$(2-\Phi a^2)^2 \le 4$$
$$|2-\Phi a^2| \le 2$$
$$-2 \le 2-\Phi a^2 \le 2$$

$2-\Phi a^2 \le 2$ is always satisfied, in addition,

$\Phi a^2 \le 4$ →

$\Delta t \le \dfrac{2\Delta x}{\sqrt{\Phi}\sin(k\Delta x)}$, for it to be valid for all *k*, we require

$$\Delta t \le \frac{2\Delta x}{\sqrt{\Phi}} \quad \text{or} \quad \mu = \frac{\Delta t\sqrt{\Phi}}{\Delta x} \le 2 \qquad (32)$$

which is the stability condition! Here $\sqrt{\Phi}$ is the disturbance propagation speed in the absence of base-state advective flow.

When the mean flow is non-zero, the condition is

$$\Delta t \le \frac{2\Delta x}{|\overline{u}|+\sqrt{\Phi}}.$$

Note the factor of 2 in the condition – the use of forward-backward scheme actually allows a Courant number of 2 to be used! This is due to the fact the backward scheme is actually a kind of 'implicit' scheme.

4.4.3.2. Centered-in-time (leapfrog) Center-in-Space (CTCS) scheme

$$\delta_{2t}u + \bar{u}\delta_{2x}u + \delta_{2x}\phi = 0$$
$$\delta_{2t}\phi + \bar{u}\delta_{2x}\phi + \Phi\delta_{2x}u = 0$$

(here we assume a non-staggered grid)

Similar stability analysis leads to:

$$\Delta t \le \frac{\Delta x}{|\bar{u}| + \sqrt{\Phi}} \qquad (33)$$

which is twice as restrictive as that for forward-backward scheme. Also it contains a computational mode.

Grid Splitting

The above CTCS scheme used non-staggered grid.

When using non-staggered grid for the above equations, we can run into the grid-splitting problem. We discussed this issue in the past.

One way of avoiding grid splitting is to use staggered grid – in which different variables are located at different points of a grid mesh.

Let's stagger u and ϕ (h in the figure) in the following way:

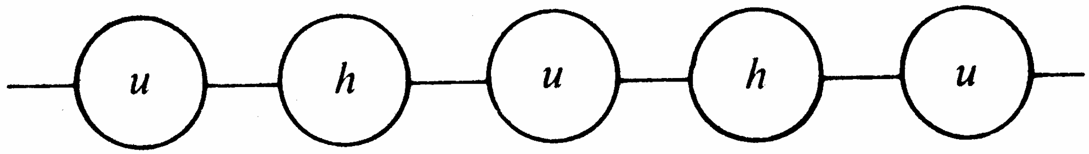

Figure 1.2 A grid with two dependent variables that are carried at alternate grid points.

Our FD equation using CTCS scheme is then

$$\begin{aligned} &\delta_{2t}u + \bar{u}\delta_{2x}u + \delta_x\phi = 0 && \text{at u point} \\ &\delta_{2t}\phi + \bar{u}\delta_{2x}\phi + \Phi\delta_x u = 0 && \text{at } \phi \text{ point} \end{aligned} \qquad (34)$$

Note the key difference in the third term of each equation from the previous non-staggered CTCS scheme. Also the equations are solved at different grid point.

Stability analysis will show the stability condition is

$$\Delta t \le \frac{\Delta x}{|\bar{u}| + 2\sqrt{\Phi}}$$

which, for zero mean flow case, is twice as restrictive as the non-staggered version (because of the factor of 2 in front of phase speed $\sqrt{\Phi}$).

However, since the pressure gradient force and velocity divergence terms are differenced over one Δx interval, and these are terms responsible for the gravity wave propagation, the solution should be more accurate, since the effective grid spacing is half as large.

4.4.3.3. Treatment of insignificant fast modes

(Reading: Durran Chapter 7 – Physically insignificant fast waves)

We obtained earlier the phase speed of shallow water waves:

$$c = \bar{u} \pm \sqrt{gH}$$

it contains two modes. The slower advective mode and the faster gravity wave (GW) mode:

$$\bar{u} \sim 10 m/s$$

$$\sqrt{gH} \sim \sqrt{10 \times 10000} \sim 200 m/s \text{ for external gravity waves}$$

$$|\bar{u}| \ll \sqrt{gH} \text{ for many problems.}$$

Gravity waves are not important in global coarse-resolution (effective grid spacing > 100 km) models in which the resolutions are usually too coarse to resolve them adequately anyway.

GWs are often important for mesoscale flows. For mesoscale models, often, compressible equations are used which support fast sound waves – so sound wave play a similar role as the gravity waves in large scale model in limiting the time step size (when using explicit schemes).

When the fast mode is not important, we don't want it to be the one that limits the time step size.

There are in general two ways to deal with this problem – one is to treat the terms responsible for the fast modes implicitly, and the other uses different time step sizes for fast and slow modes and the method is called mode splitting method. ARPS uses the latter to deal with fast sound waves (hence the large and small time steps, dtbig and dtsml you find in arps.input).

4.4.4.4. Semi-implicit method

Since the PGF term in u equation and the velocity divergence term in ϕ equation are responsible for gravity waves, we can treat them implicitly, so that hopefully the gravity wave mode no longer limit the time step size.

Again we look at the non-staggered case:

$$\begin{aligned} &\delta_{2t}u + \bar{u}\delta_{2x}u + \overline{\delta_{2x}\phi}^{2t} = 0 \\ &\delta_{2t}\phi + \bar{u}\delta_{2x}\phi + \Phi\overline{\delta_{2x}u}^{2t} = 0 \end{aligned} \qquad (35)$$

The time averages makes the scheme implicit. Since only some of the terms are treated implicitly, the scheme is called semi-implicit.

Stability of the system – only the advective velocity $\bar{u}$ appears in the stability condition therefore much larger time step can be used (see Durran 7.2.3; Mesinger and Arakawa Chapter 4 section 6).

Analysis shows that the fast mode in the numerical solution is actually slowed down – i.e., there is a lagging phase error with this mode – it is okay if this mode is consider unimportant, like the sound waves in the atmosphere or the gravity waves in large-scale models.

Solution procedure for (35)

1) Computer ϕ^{n+1} for all j by eliminating u^{n+1} from the 2nd equation using the first:

$$\phi_j^{n+1} - \frac{\Phi \Delta t^2}{4\Delta x^2}\left[\phi_{j-2}^{n+1} - 2\phi_j^{n+1} + \phi_{j+2}^{n+1}\right] = f \ ,$$

the right hand side is known.

2) Two effectively decoupled tridiagonal system of equations have to be solved, one for even *j* and one for odd *j* (can lead to grid splitting).

3). Once ϕ^{n+1} is known, we can plug it into u equation to obtain u^{n+1}.

4) If a staggered grid is used, then only one tridiagonal system of equations has to be solved. The total amount of calculation is about the same as the non-staggered case since because twice any many grid points are now involved.

5) For 2D or 3D problems, the semi-implicit scheme results in a Helmholtz equation that can't be as easily solved as the 1D tridiagonal equation.

Tapp and White is one of the first to use semi-implicit method in a compressible mesoscale model of the UK Met Office (Tapp and White 1976 QJRMS).

4.4.4.5. Mode-splitting Method

For info on mode-splitting method for compressible model, see Klemp and Wilhemson (1978) and Durran Section 7.3.2.

Skamarock and Klemp (1982) discuss that stability issues associated with the mode-splitting methods as applied to compressible system of equations.

References:

Klemp, J. B., and R. B. Wilhelmson, 1978: The simulation of three-dimensional convective storm dynamics. *J. Atmos. Sci.*, **35**, 1070-1096.

Skamarock, W. C., and J. B. Klemp, 1992: The stability of time-split numerical methods for the hydrostatic and nonhydrostatic elastic equations. *Mon. Wea. Rev.*, **120**, 2109-2127.).

4.4.4. The Arakawa Grids

(p.47 in Mesinger and Arakawa 1976)

Arakawa (Arakawa and Lamb 1977) introduced a variety of staggered grids when trying to find the most accurate method for handling geostrophic adjustment process, which we know relies on inertia gravity waves. Inertia gravity waves are dispersive, they disperse ageostrophic energy.

To describe inertia GW, we need to include rotational effect into the shallow water equations:

$$\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} - fv = 0$$
$$\frac{\partial v}{\partial t} + g\frac{\partial h}{\partial y} + fu = 0$$
$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

For 1-D version of this problem, i.e. for $\frac{\partial}{\partial y} = 0$ case, the dispersion equation for the exact solution is

$$\omega = (f^2 + k^2 gH)^{1/2}.$$

Arakawa defined 5 different grids, all of which has the same number of dependent variables per unit area – so that the computational time is about the same.

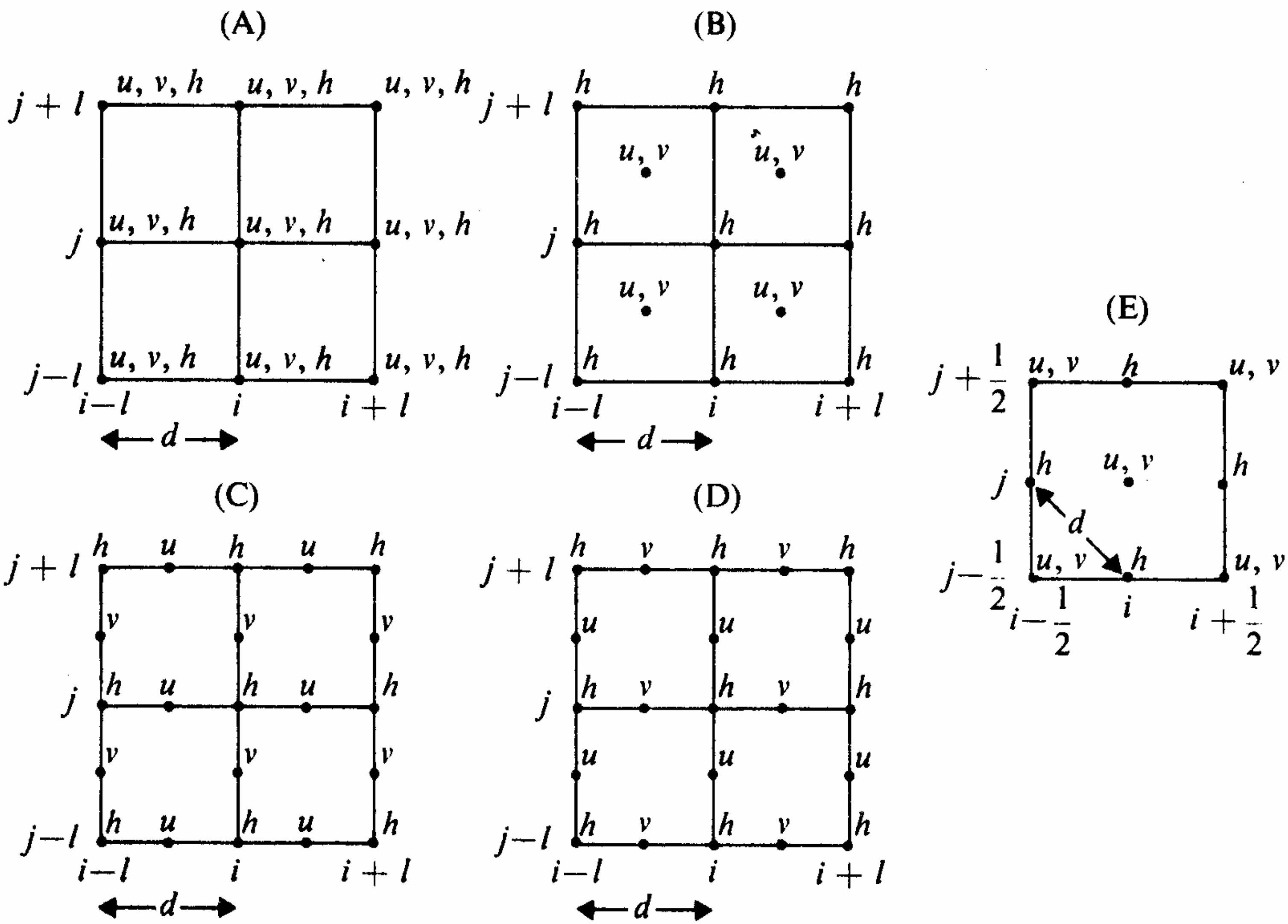

Figure 3.1 Five types of lattice considered for the finite difference solution of (3.1).

For each of the above grid, the finite difference equation can be written as

$$\frac{\partial u}{\partial t} = -g\,\overline{\delta_x h}^{x} + fv, \quad \frac{\partial v}{\partial t} = -g\,\overline{\delta_y h}^{y} - fu, \qquad (3.2)_A$$
$$\frac{\partial h}{\partial t} = -H\left(\overline{\delta_x u}^{x} + \overline{\delta_y v}^{y}\right),$$

$$\frac{\partial u}{\partial t} = -g\,\overline{\delta_x h}^{y} + fv, \quad \frac{\partial v}{\partial t} = -g\,\overline{\delta_y h}^{x} - fu,$$
$$\frac{\partial h}{\partial t} = -H\left(\overline{\delta_x u}^{y} + \overline{\delta_y v}^{x}\right), \qquad (3.2)_B$$

$$\frac{\partial u}{\partial t} = -g\,\delta_x h + f\bar{v}^{xy}, \quad \frac{\partial v}{\partial t} = -g\,\delta_y h - f\bar{u}^{xy},$$
$$\frac{\partial h}{\partial t} = -H\left(\delta_x u + \delta_y v\right), \qquad (3.2)_C$$

$$\frac{\partial u}{\partial t} = -g\,\overline{\delta_x h}^{xy} + f\bar{v}^{xy}, \quad \frac{\partial v}{\partial t} = -g\,\overline{\delta_y h}^{xy} - f\bar{u}^{xy}, \qquad (3.2)_D$$
$$\frac{\partial h}{\partial t} = -H\left(\overline{\delta_x u}^{xy} + \overline{\delta_y v}^{xy}\right),$$

$$\frac{\partial u}{\partial t} = -g\,\delta_x h + fv, \quad \frac{\partial v}{\partial t} = -g\,\delta_y h - fu,$$
$$\frac{\partial h}{\partial t} = -H\left(\delta_x u + \delta_y v\right). \qquad (3.2)_E$$

We want to find the numerical dispersion relations and compare them with the exact solution. For 1-D problem, the dispersion relations are (note ν is our ω, d = Δx, the time derivative terms are not differenced, i.e., remain in their continuous form):

$$\left(\frac{\nu}{f}\right)^2 = 1 + \left(\frac{\lambda}{d}\right)^2 \sin^2 kd, \qquad (3.6)_A$$

$$\left(\frac{\nu}{f}\right)^2 = 1 + 4\left(\frac{\lambda}{d}\right)^2 \sin^2 \frac{kd}{2}, \qquad (3.6)_B$$

$$\left(\frac{\nu}{f}\right)^2 = \cos^2 \frac{kd}{2} + 4\left(\frac{\lambda}{d}\right)^2 \sin^2 \frac{kd}{2}, \qquad (3.6)_C$$

$$\left(\frac{\nu}{f}\right)^2 = \cos^2 \frac{kd}{2} + \left(\frac{\lambda}{d}\right)^2 \sin^2 kd, \qquad (3.6)_D$$

$$\left(\frac{\nu}{f}\right)^2 = 1 + 2\left(\frac{\lambda}{d}\right)^2 \sin^2 \frac{kd}{\sqrt{2}}. \qquad (3.6)_E$$

They are plotted in the following figure:

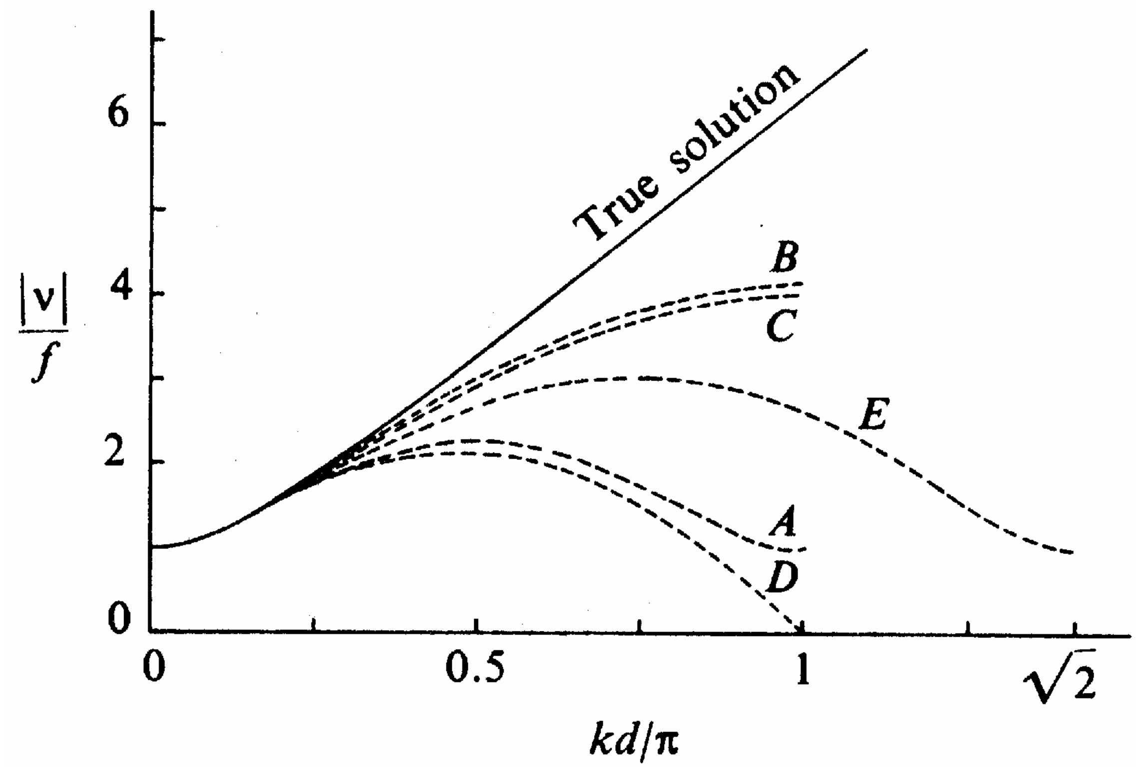

Figure 3.2 The functions $|\nu|/f$ given by (3.4) and (3.6), with $\lambda/d = 2$.

The phase speed and group velocities for each of these grids can be plotted together with the exact solution:

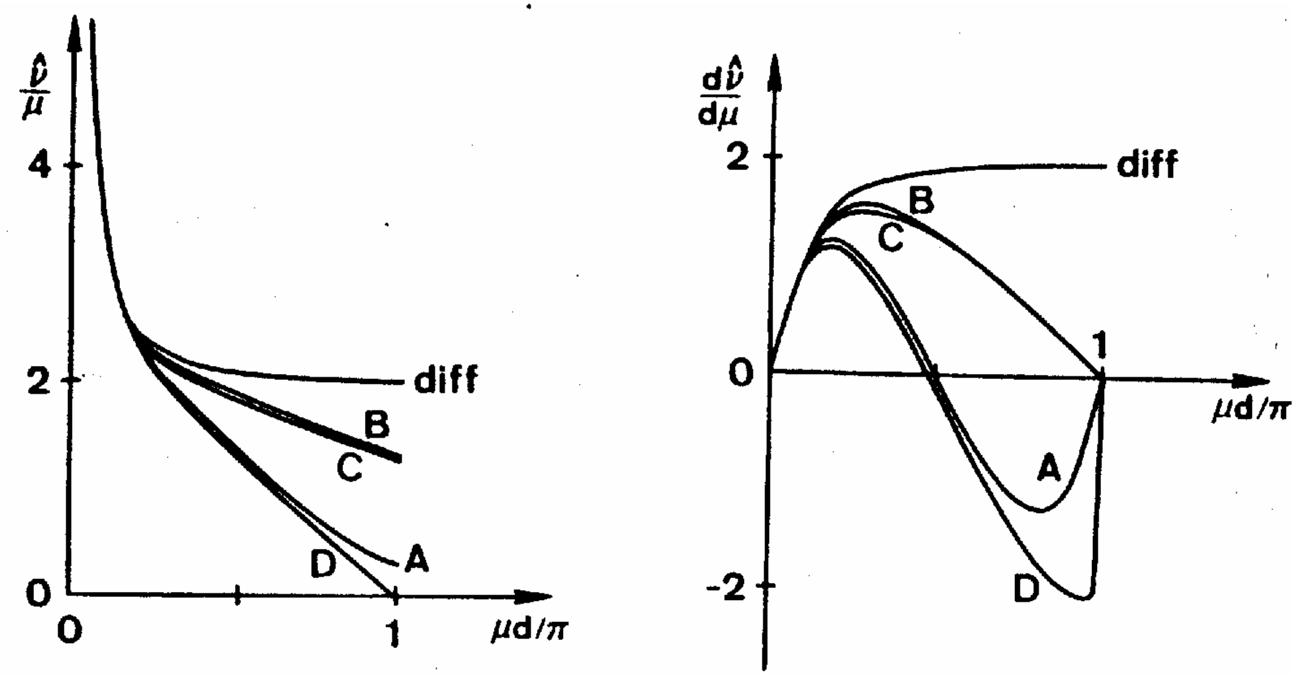

Figure 7.5 The phase velocity $c = \hat{\upsilon}/\mu$ and the group velocity $d\hat{\upsilon}/d\mu$ from Schoenstadt (1978) as functions of $\mu d/\pi$ for the four grids as indicated. The differential solution is also included. These results use the values: $gH = 10^4$ m^2 sec^{-2}, $f = 10^{-4}$ sec^{-1}, $d = 500$ km.

We can see that for the 1-D problem, B and C grid perform the best.

A and D are not good at all. Energy of waves shorter than 4Δx propagates in the wrong direction.

E is reasonable good.

For 2-D problem, the ω/f is plotted in the following:

Exact Solution **B grid** **C grid**

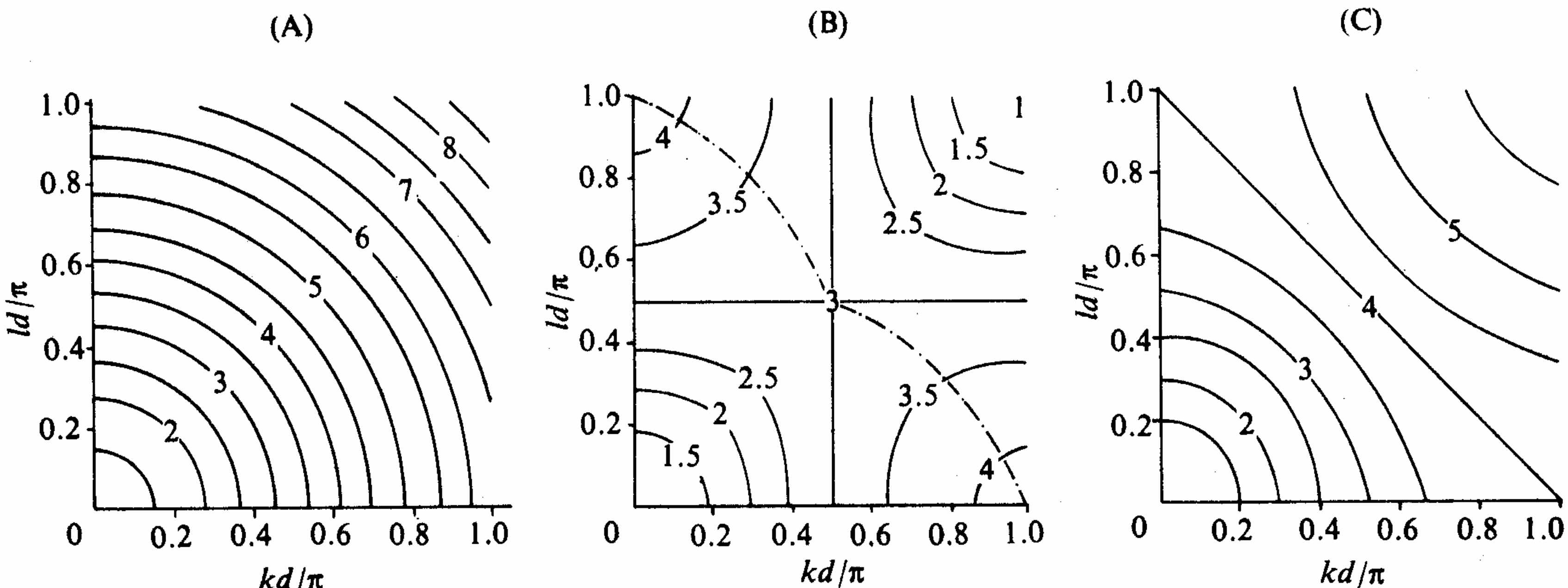

Figure 3.3 The functions $|\nu|/f$, for the true solution and for solutions of systems $(3.2)_B$ and $(3.2)_C$, with $\lambda/d = 2$.

We can see C grid is closest to the exact solution given in (A), and B grid is not as good in 2-D, especially along the diagonal direction in the plot.