Meteorology 5344, Fall 2003
Computational Fluid Dynamics

Computer Problem #5: Linear Convective Transport
Distributed: Monday, October 27, 2003
Due: Monday, November 10, 2003
Updated on Nov 4, 2003.

Consider the 1-D linear convection equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

where c is a positive and constant advection speed. This equation can be solved numerically using the two-step MacCormack method:

Predictor: $$(u_i^{n+1})^* = u_i^n - c\Delta t\frac{u_{i+1}^n - u_i^n}{\Delta x}$$

Corrector: $$u_i^{n+1} = \frac{1}{2}\left[u_i^n + (u_i^{n+1})^* - c\Delta t\frac{(u_i^{n+1})^* - (u_{i-1}^{n+1})^*}{\Delta x}\right]$$

a. Derive the modified equation for this two-step scheme and determine the anticipated error type (dispersive or dissipative).

$$u_t + cu_x = -c\frac{(\Delta x)^2}{6}(1-\mu^2)u_{xxx} - \frac{c(\Delta x)^3}{8}\mu(1-\mu^2)u_{xxxx+....}$$

When trying to eliminate $(\ )_t$, make sure you use of FDE not PDE.

Leading order truncation error is in the form of third (odd) order derivative – dispersive errors should dominate.

b. Use the von Neumann technique to assess the stability of this scheme, and plot the phase and amplitude errors as a function of $k\Delta x$ for several Courant numbers, including a few for which linear stability is violated.

c. Write a computer code for this scheme, using as initial conditions the following function:

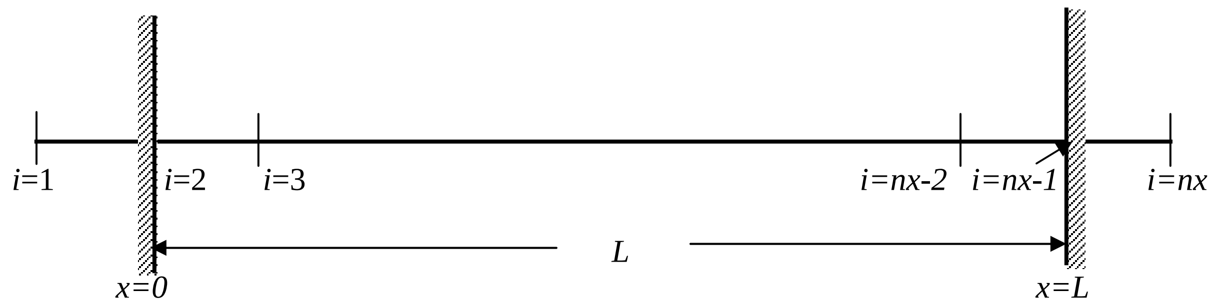
$$u(x,t=0) = 2 + u_0(x)\left[1 + 0.3\sin\left(\frac{2\pi x}{9\Delta x}\right)\right]\left[1 + 0.4\sin\left(\frac{2\pi x}{10\Delta x}\right)\right]$$

where

$$u_0(x) = \begin{cases} -1 & \text{if } 8 \le x \le 28 \\ 1 & \text{if } 28 < x \le 39 \\ 0 & \text{otherwise} \end{cases}$$

with $\Delta x = 1.0$ in a periodic domain of length 50. This somewhat unconventional initial condition provides a stringent test of advective schemes because it contains sharp gradients and other spatial irregularities.

The periodic boundary condition means that $u(x=0)=u(x=L)$, $u(x=-l)=u(x=(L-l))$ and $u(x=l)=u(x=(L+l))$, where L is the length of the physical domain.
To facilitate the implementation of periodic conditions at the lateral boundaries, we usually define an extra grid point outside each physical boundary, so for a physical domain of length L, we need $nx = L/\Delta x + 3$ number of grid points, as illustrated below:



In the discrete form, the boundary conditions are: $u(1) = u(nx\text{-}2)$ and $u(nx) = u(3)$. When these conditions are used, $u(2) = u(nx\text{-}1)$) should be automatically satisfied (note a typo with the previous version of this formula). For actual implementation, you integrate the finite difference equation forward in time for $i = 2$ to nx-1, and set boundary conditions at $i = 1$ and nx.

Note that this version of question is different from the previous version in that 53 points is used here instead of 50. If you had already done this problem with nx=50, it's equally good.

Please show the plots for the numerical solutions. Should point out that the numerical solutions are indeed dominated by phase errors – consistent with what the modified equation suggests. The solution with $\mu = 1$ is exact.

d. Run your code with Courant numbers of 0. 10, 0.25, 0.50, and 1.00. Use Takacs' method to compute, for solutions at 50, 100, 200, and 400 time steps, the amplitude and phase errors relative to the exact solution (which is simply the initial condition shifted to the right by the number of time steps multiplied by the Courant number), and compare them with the theoretical predictions made in part b. Is the predominant error type similar to that anticipated from the modified equation? Comment on your findings.

The estimation of errors using Tackacs' method should show that dispersion errors in the numerical solutions are indeed larger than the corresponding dissipation errors.

Think about the following questions and make sure you know the expected answers. Read Durran (1991 MWR). You do not need to hand in answers to these questions.

Consider once again, the 1-D advection equation in the previous problem:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

where c is a positive and constant advection speed.

Use a periodic domain containing 32 points with Δx 1/32 and an initial condition given by

$$u_0(x) = \begin{cases} \{64[(x-1/2)^2 - 1/64\}^2 & \text{if } 3/8 \le x \le 5/8 \\ 0 & \text{otherwise} \end{cases}$$

Also, let c = 0.25. With these conditions, the feature being advected completes one circuit through the domain in a time of 1.0.

a. Using the leapfrog time-differencing scheme and a second-order centered-in-space discretization for the advection term, run the solution to t = 3 using Courant numbers of 0.7, 0.5, and 0. 1. Compute the dispersion, dissipation, and diffusion errors using Tackacs' method.

b. Repeat part a, this time adding the Asselin-Robert time filter with a filter coefficient of 0. 1. Discuss the results.

c. Repeat part a, this time using a fourth-order centered-in-space difference for the advection term, given by

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = -c\left[\frac{4}{3}\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} - \frac{1}{3}\frac{u_{i+2}^n - u_{i-2}^n}{4\Delta x}\right]$$

Before running the program, perform a linear stability analysis and determine the phase and amplitude errors of this scheme relative to that in part a.

d. Repeat-part c, this time using the third-order Adams-Bashforth time differencing scheme. Do not perform a stability analysis. Discuss your results and comment on the possible advantages and disadvantages of the third-order Adams-Bashforth scheme.