2.2. Reynolds averaging and Reynolds averaged equations

We will study several aspects of the PBL. Before that, we need to develop a set of equations that are suitable for studying turbulent flows.

2.2.1. Reynolds Averaging

(Holton p.119).

A time series plot of the wind speed can look like the above. There are many high-frequency (fast) fluctuations.

Such fluctuations are due to small turbulent eddies, and they do not reliably represent the mean flow.

To obtain a wind speed measurement representative of the large-scale flow, we obtain take an average over a time period long enough to smooth over the fluctuations but still short enough for keep the trend. The trend can be early identified in the above.
Such averaging was first proposed by Reynolds, and is therefore named after him.

We use overbar to denote the mean value and prime to denote the perturbation.

For any quantity \( A \) (could be e.g., vertical velocity \( w \) and potential temperature \( \theta \)), we have

\[
A = \bar{A} + A'
\]

where

\[
\bar{A} = \frac{1}{T} \int_0^T A(x,t)dt
\]

for a continuous function \( A \), or

\[
\bar{A} = \frac{1}{N} \sum_{n=1}^N A_n(x)
\]

if we have discrete values of \( A \). Here \( N \) is the number of points in the time series in time the chosen time interval \( T \).

Note that \( \langle \cdot \rangle \) can be thought of as the integration operator, thus we can apply it as follows:

\[
\langle A \rangle = \langle \bar{A} + A' \rangle = \bar{A} + A' = \bar{A} + \bar{A}' = 2\bar{A}' + \bar{A}
\]

\[
\Rightarrow \quad \bar{A}' = 0 \quad \text{(mean of the fluctuation is zero).} \quad (2.3)
\]

Why \( \bar{A} = \bar{A} \)? It's just the average of the average, which is the average (for a given time averaging interval, the average is no longer a function of time therefore further averaging has no effect). The statement that \( \bar{A}' = 0 \) means that as much area lies above the \( \bar{A} \) line as blow it. Consider a since wave:
Other rules of averaging:

\[ \overline{(A + B)} = \overline{A} + \overline{B} \quad (2.4) \]

\[ \overline{cA} = c\overline{A} \quad (2.5) \]

where \( c \) is a constant

\[ \overline{A} = \overline{A} \quad (2.6) \]

as discussed before. In another word, the averaged value acts like a constant. Further averaging has no effect. For this reason, we have

\[ \overline{(AB)} = \overline{AB} \quad (2.7) \]

The above equations can be easily proven by using definition (1) (do it yourself!).

It can also be shown that

\[ \frac{d\overline{A}}{dt} = \frac{d\overline{A}}{dt}. \quad (2.8) \]

Now let's examine the mean of the product of two variables, \( A \) and \( B \), each of which can be split into the mean and perturbation parts, therefore:

\[ \overline{AB} = (A + A')(\overline{B} + B') \]
\[ = (\overline{AB} + A'\overline{B} + \overline{AB}' + A'B') \]
\[ = (\overline{AB}) + (A'\overline{B}) + (\overline{AB}') + (A'B') \quad (2.9) \]
\[ = \overline{AB} + 0 + 0 + A'B' \]
\[ = \overline{AB} + A'B' \]

In the above, \( (A'\overline{B}) = A'B = 0 \cdot \overline{B} = 0 \).
We call AB a nonlinear term since both A and B are time-dependent variables. A'B' is also a nonlinear term whose average is not necessarily zero!

\[ \overline{A'B'} \] is called the covariance of A and B (defined as \( \overline{A'B'} = \frac{1}{T} \int_{0}^{T} A'B' \, dt \)) for continuous A' and B' and \( \overline{A'B'} = \frac{1}{N} \sum_{n=1}^{N} A'_n B'_n \) for discrete values of A' and B'). This term is actually extremely important for boundary layer studies, as we will see later. For example, when A is w (vertical velocity) and B is \( \theta \) (potential temperature), the \( w'\theta' \) represent vertical turbulent flux of potential temperature (heat).

When A' = B', then the covariance becomes variances \( \overline{A'^2} \).

Terms like \( \overline{A'B'} \), \( \overline{A'^2} \), and \( \overline{B'^2} \), are called second-order moments because they are covariances or variances.

In the above, we have chosen the averaging to be performed in time. If a turbulence field is measured simultaneously by many sensors distributed in space, then a spatial mean should be used. For statistically stationary turbulence, the time mean is equal to the spatial mean.

### 2.2.2. Reynolds Averaged Equations

Now we have decided how to characterize the turbulent flow (in terms of mean and perturbations), we want to derive a set of equations that can describe the mean flow while still include the effect of the perturbations.

We do this by performing Reynolds averaging on the equations.

We start from the basic equations of motion (you learned them in Dynamics I)
\[
\frac{du}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv
\]
\[
\frac{dv}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu
\]
\[
\frac{dw}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g
\]  

(2.10)

In the above, we neglected the effect of molecular viscosity.

**Boussinesq approximation**

For boundary layer problems, the air density typically does not change more than 10% of the total, so it is possible to assume the density to be constant for in the equations, except in the terms where the density variational is critical, i.e., in the buoyancy term.

We make what is called the **Boussinesq approximation**.

With **Boussinesq approximation**, density is assumed to be constant ($\rho \approx \rho_0$) except when it contributes directly to the buoyancy (you know that buoyancy is directly related to density of matters, think of a wood block in water – the net buoyancy is upward). The horizontal equations become

\[
\frac{du}{dt} \approx -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + fv
\]
\[
\frac{dv}{dt} \approx -\frac{1}{\rho_0} \frac{\partial p}{\partial y} - fu
\]  

(2.11)

The vertical equation involves the buoyancy effect, if you simply set $\rho = \rho_0$, then the buoyancy effect due to density different is lost. We need to do something else. Let's decompose the total pressure $p$ into the base-state and perturbation parts, i.e.,

\[
p = p_0(z) + p'(x, y, z, t)
\]  

(2.12)

and we require the base state pressure satisfy the hydrostatic relation
\[ \frac{\partial p_0}{\partial z} = -\rho_0 g \] (2.13)

The right hand side (RHS) of the vertical momentum equation is

\[ -\frac{1}{\rho} \frac{\partial p}{\partial z} - g = -\frac{1}{\rho} \left[ \frac{\partial p}{\partial z} - \rho g \right] = -\frac{1}{\rho} \left[ \frac{\partial p_0}{\partial z} + \frac{\partial p'}{\partial z} + \rho_0 g + \rho' g \right] = -\frac{1}{\rho} \left[ \frac{\partial p'}{\partial z} + \rho' g \right] \]

\[ \Rightarrow \quad -\frac{1}{\rho} \frac{\partial p}{\partial z} - g = -\frac{1}{\rho} \frac{\partial p'}{\partial z} - \frac{\rho'}{\rho} g. \]

\[ \Rightarrow \quad \frac{dw}{dt} = -\frac{1}{\rho} \frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0} g \] (2.14).

It is now safe to approximate \( \rho \) in (2.14) with \( \rho_0 \) because the (first-order) effect of density perturbation on the buoyancy has been taken into account. The vertical equation with Boussinesq approximation is then

\[ \frac{dw}{dt} \approx -\frac{1}{\rho_0} \frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0} g \] (2.15).

(2.15) says that positive density perturbation (heavier air) creates negative buoyancy.

Since with Boussinesq approximation, we assume the density is constant, the mass conservation equation is also simplified, from

\[ \frac{1}{\rho} \frac{d\rho}{dt} = -\left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] \]

to

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \approx 0 \] (2.16)
i.e., the flow is non-divergent.

The thermodynamic energy equation is

\[ \frac{d\theta}{dt} = S_\theta \]  \hspace{1cm} (2.17)

where \( S_\theta \) represents heat source or sink and potential temperature \( \theta \) is defined as

\[ \theta = T \left( \frac{1000mb}{p} \right)^{R/C_p} \]  \hspace{1cm} (2.18).

It is easy to show (Holton p200) that

\[ \frac{\theta'}{\theta_0} \approx \frac{1}{\gamma} \frac{p'_0}{\rho_0} - \frac{\rho'}{\rho_0} \]  \hspace{1cm} (2.19)

where \( \gamma = C_p/C_v \) is the ratio of the specific heat of dry air at constant pressure and constant volume. For typical atmospheric flows, the pressure perturbation part in (2.19) is much smaller than the density part, therefore

\[ \frac{\theta'}{\theta_0} \approx -\frac{\rho'}{\rho_0} \]  \hspace{1cm} (2.20)

With (2.20), we can replace density perturbation (which is not directly measured) by the potential temperature perturbation \( \theta' \)

\[ \frac{dw}{dt} \approx -\frac{1}{\rho_0} \frac{\partial p'}{\partial z} + \frac{\theta'}{\theta_0} g \]  \hspace{1cm} (2.21)

(2.21) says that positive potential temperature perturbation (warmer air) creates positive buoyancy, consistent with our physical understanding.
Equations (2.11), (2.16), (2.17) and (2.21) form a set of equations that we will use to derive the Reynolds averaged equations for describing the mean flow.

**Reynolds averaged equations**

We divide all dependent variables in the above set of equations into the mean and perturbation parts. E.g., \( u = \bar{u} + u' \) and \( p = \bar{p} + p' \). The terms on the right hand side of the equations are all linear in terms of these dependent variables, the mean of these terms will be equal to these terms of the mean variables only, i.e., the perturbation terms will drop out after the mean is taken. For example,

\[
\frac{1}{\rho_0} \frac{\partial \bar{p}}{\partial x} + f\bar{v} = -\frac{1}{\rho_0} \frac{\partial (\bar{p} + p')}{\partial x} + f(\bar{v} + v')
\]

\[
= -\frac{1}{\rho_0} \frac{\partial \bar{p}}{\partial x} + f\bar{v} - \frac{1}{\rho_0} \frac{\partial \bar{p}'}{\partial x} + f\bar{v}' = -\frac{1}{\rho_0} \frac{\partial \bar{p}}{\partial x} + f\bar{v}
\]  \hspace{1cm} (2.22)

because the mean of mean is equal to the mean itself and the mean of perturbation is zero. Therefore, as long as a term is linear, the mean of the entire term is equal to the term written for the mean variables. This is not true for nonlinear terms, such as the advection terms, however.

Let's look at the left hand side of the \( u \) momentum equation, i.e., the \( \frac{du}{dt} \) term:

\[
\frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}
\]

\[
= \frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} + \frac{\partial uw}{\partial z} - u \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] \]  \hspace{1cm} (2.23)

\[
= \frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} + \frac{\partial uw}{\partial z}
\]

Chain rule of differentiation and zero divergence equations were used in the above.
Taking a Reynolds average of (2.23), we obtain

\[
\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} + \frac{\partial wu}{\partial z} = \frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v} \bar{u}}{\partial y} + \frac{\partial \bar{w} \bar{u}}{\partial z} + \frac{\partial u'u'}{\partial x} + \frac{\partial v'u'}{\partial y} + \frac{\partial w'u'}{\partial z} \quad (2.24)
\]

In the above, we again used the zero divergence condition for the mean velocity.

By defining \( \frac{\partial}{\partial t} \frac{\partial}{\partial x} + \frac{\bar{v}}{\partial y} + \frac{\bar{w}}{\partial z} \) which is the total time derivative for the mean flow (equal to the local time derivative plus advection by the mean flow), the equations of motion become

\[
\begin{align*}
\frac{\partial \bar{u}}{\partial t} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + f_v \left[ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right], \\
\frac{\partial \bar{v}}{\partial t} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial y} - f_v \left[ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right], \\
\frac{\partial \bar{w}}{\partial t} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + g \frac{\theta'}{\theta_0} \left[ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right].
\end{align*}
\] (2.25a, 2.25b, 2.25c)

The thermodynamic energy equation (2.17) becomes

\[
\frac{\partial \bar{\theta}}{\partial t} = -\left[ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right] + \bar{S}_\theta. \quad (2.25d)
\]
The mass continuity equation for the mean flow is

\[ \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0. \]  

(2.25e)

Similarly, when the Reynolds averaging is applied to the conservation equation of water vapor mixing ratio q, we obtain the Reynolds averaged equations for

\[ \frac{\bar{dq}}{dt} = -\left[ \frac{\partial \bar{u}'q'}{\partial x} + \frac{\partial \bar{v}'q'}{\partial y} + \frac{\partial \bar{w}'q'}{\partial z} \right] + \bar{S}_{q} \]

(2.25f)

and the covariances in the square brackets are turbulent fluxes of water vapor.

2.2.3. Reynolds fluxes and their physical interpretation

Equation set (2.25) is the **Reynolds averaged equations for the mean state variables**.

The various covariances in the square brackets in the equations represent **turbulent fluxes** (a flux is always a quantity multiplied by a velocity).

**For example**, \( \bar{w'u'} \) is the vertical flux of momentum in the x direction or the flux of vertical velocity w in x direction. \( \bar{w'\theta'} \) is the vertical turbulent flux (flux due to turbulent motion) of heat.

The heat flux is **positive** when \( w' \) and \( \theta' \) are **positively correlated**, i.e., when \( w' \) and \( \theta' \) tend to have the same sign. The end result is that warmer air gets transported by the turbulent velocity upward, and cold air gets transported downward, resulting in net positive (upward) heat flux.
The net (turbulent) heat flux into a unit volume of air causes the temperature of this volume to change. This net flux is the difference between the flux going into a, for example, cubed volume on one side and that going out of the volume on the other side, it is therefore the flux divergence that causes the change in the quantity being transported (by turbulence), which is $\theta$ in (2.25e) and $q$ in (2.25f).

For momentum, the flux of the momentum parallel to a volume face (e.g., the flux of $u$ through lower boundary of a cubed volume by $w$) causes the parallel momentum ($u$ in this case) to change at that face. Therefore the momentum flux is the stress (force/unit area) applied at this face. We therefore also call the momentum fluxes in the above momentum equations the Reynolds stresses. We often symbol $\tau$ is used to denote the stress.

Let's look at the physical meaning of the fluxes in some more details. Let's look at the heat flux for example.

Suppose we have an idealized turbulent eddies near the ground on a hot summer day. If we start with a particular profile of $\bar{\theta}$, how will it change with time? Due to the surface heating, typically $\bar{\theta}$ is super-adiabatic near the ground ($d\bar{\theta}/dz < 0$), as shown in the following figure.

![Diagram showing the change in temperature profile due to surface heating.]

Assume that we have two parcels, A and B. The A moves downward, and B upward. When A moves downward, it becomes colder than its environment,
it therefore carries a negative $\theta'$. For parcel B, it's the opposite, it carries a positive $\theta'$, therefore

For parcel A, $w' < 0$ and $\theta' < 0 \Rightarrow w'\theta' > 0$ therefore the heat flux is positive.

For parcel B, $w' > 0$ and $\theta' > 0 \Rightarrow w'\theta' > 0$ therefore the heat flux is also positive.

Even though both fluxes are positive, the physical process is different – positive heat flux can be caused by either transporting (by turbulent eddies) colder air downward or transporting warmer air upward.

Considering only the vertical turbulent heat flux (which tend to dominate in convective boundary layer), and assuming the mean velocity is zero (i.e., there is no mean wind advection) the heat energy equation (i.e., the equation for $\theta$) becomes

$$\frac{\partial \theta}{\partial t} = -\frac{\partial w'\theta'}{\partial z}.$$  \hspace{1cm} (2.26)

We see that temperature changes as a result of heat flux divergence, not the heat flux itself. Temperature changes only when the net heat flux into an air parcel (or volume) is non-zero.

In the above example, the flux at both level $z_1$ and $z_2$ are positive. But because the vertical gradient of $\theta$ is larger at $z_1$, the flux there has a larger magnitude (this is a turbulent flux problem – one commonly used closure is to assume that the turbulent flux is proportional to the gradient of the mean quantity), therefore

$$\frac{\partial \theta}{\partial t} = -\frac{\partial w'\theta'}{\partial z} = -\frac{(w'\theta')_{z_2} - (w'\theta')_{z_1}}{z_2 - z_1} > 0$$

so that the mean potential temperature $\theta$ in the layer between $z_1$ and $z_2$ increases with time! If the heat flux is positive at both level but increases with height, then the temperature will decrease with time (even though the
heat flux is positive), because more heat is leaving this layer at the top boundary than the heat coming in from the bottom of this layer.

Similar concept can be applied to turbulent fluxes of other material quantities such as water vapor mixing ratio, and to momentum.