The importance of instrumental tilt on measurements of atmospheric turbulence

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SUMMARY

This note discusses the equations that govern the dependence of observed turbulence characteristics on instrumental tilt and then considers the magnitudes of the effects within the constant stress layer.

Notation

\( b \) constant of proportionality between \( \sigma_u \) and \( u_* \)
\( E_x \) the percentage error in the slope of the high frequency part of the spectral curve
\( f \) the dimensionless frequency, \( n z/V \)
\( f_m, f_m' \) \( n z/V \) at the peak of the spectral curve
\( n \) frequency, Hz
\( S_x (n) \) spectral density function of \( x \)
\( T' \) fluctuations of temperature
\( u', v', w' \) fluctuations of the horizontal, lateral and vertical velocity respectively
\( V \) total wind
\( y \) \( 900 \ n/V \)
\( z \) height
\( \alpha \) angle between frames of reference
\( u_* \) friction velocity
\( \sigma_x^2 \) variance of \( x \)
\( \sigma_u \) stress component
\( (\cdot)_\alpha \) value of a quantity when the axes are tilted by an angle \( \alpha \).

1. Introduction

When measurements of turbulent characteristics, from different sites and instruments, are compared it becomes evident that while certain parameters are compatible others are not. For example the relation \( \sigma_w/\sigma u_* \approx 1.3 \) is generally accepted but wide variations occur in the ratio \( \sigma_u/\sigma_* \). Similarly the \( \sigma u^2 \) spectra are better established than the corresponding \( \sigma u^2 \) (Busch and Panofsky 1968).

By extending the work of previous writers such as Pond (1968), this note purports to show that in part some of these anomalies could have arisen if the frames of reference used by the different workers had been tilted with respect to each other – the effect of using non-orthogonal axes is not considered, nor is the clear importance of compatible record lengths and sampling frequencies, quite apart from real atmospheric variations. The note also discusses the use of rotationally invariant quantities which remove the effect of tilt.

2. The transformation equations

For the two-dimensional case consider two sets of orthogonal axes with a common origin but inclined at an angle \( \alpha \) to each other. Variables measured with respect to one set of axes are
related to those measured with respect to the other set of axes by the following formulae,

\[(\sigma_w^2)_{\alpha} = \sigma_w^2 \cos^2 \alpha + \sigma_u^2 \sin^2 \alpha - \bar{u}' \bar{w}' \sin 2\alpha \]  
(1a)

\[(\sigma_u^2)_{\alpha} = \sigma_u^2 \cos^2 \alpha + \sigma_w^2 \sin^2 \alpha + \bar{u}' \bar{w}' \sin 2\alpha \]  
(1b)

\[(\bar{u}' \bar{w}')_{\alpha} = \bar{u}' \bar{w}' \cos 2\alpha + \frac{1}{2} (\sigma_w^2 - \sigma_u^2) \sin 2\alpha \]  
(1c)

and

\[(\bar{w}' T')_{\alpha} = \bar{w}' T' \cos \alpha - \bar{u}' T' \sin \alpha \]  
(1d)

assuming that the positive sense of rotation is anticlockwise (these are an extension of Pond's (1968) equations for a constant angle of tilt). Thus if the values of \(\sigma_w, \sigma_u, \bar{u}' \bar{w}', \bar{w}' T'\) and \(\bar{u}' T'\) are known for one set of axes their apparent values in another orthogonal frame of reference inclined at an angle \(\alpha\) to the original one may be predicted.

The Appendix extends these equations to three dimensions and shows how the rather complex variations of the turbulent quantities may be visualized with the aid of the Mohr Circle.

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Figure 1. The variation of the turbulent characteristics with \(\alpha\).

- \(\sigma_w = 2.5 u_*\)
- \(\sigma_u = 2.4 u_*\)
- \(\sigma_u = 2.6 u\)
3. THE RELATIONS BETWEEN $\sigma_u$, $\sigma_v$, AND $u_e$

Over land, within the constant stress layer, it is usually accepted that $\sigma_u^2 = 1.7 u_e^2$ (Busch and Panofsky 1968) but the corresponding relation between $\sigma_u$ and $u_e$ (i.e. $\sigma_u^2 = b^2 u_e^2$) varies considerably from site to site and worker to worker (Lumley and Panofsky 1964). However, if a median value of 2.5 is used for $b$ then Eqs. (1a) to (1c) may be used to prescribe the dependence of the turbulent parameters on $a$ as shown by the solid lines in Figs. 1 and 2.

It can be seen from Fig. 1 that if $|a| < 5^\circ$, each degree of tilt corresponds to a change in $(\overline{u'w'})_a$ of about 0.1 $u_e^2$ as has been pointed out by Pond (1968) and Deacon (1968). Most of this variation is reflected by an enhanced (or conversely reduced) low frequency contribution to the $(u'w')_a$ spectrum (see later). With the other two parameters the situation is rather more satisfactory since the corresponding figure for $\sigma_u^2$ and $\sigma_u$ is only about 0.02 $u_e^2$ per degree.

Due to $\sigma_u$ and $u_e$ having similar magnitudes and varying in the same sense with $a$, the ratio $(\sigma_u/\sqrt{\overline{u'w'}}_a$ is far less dependent on $a$ than is the corresponding relation between $(\sigma_u)_a$ and $(\sqrt{\overline{u'w'}}_a$ (see Fig. 2). Thus if the dependence of this ratio on parameters such as stability and height was known, $u_e$ could be determined from the measured value of $\sigma_u$ thereby drastically reducing any dependence on $a$. Furthermore it is easier to establish the value of $\sigma_u$ as it is less dependent on low frequency components (viz. Miyake, Stewart and Burling 1970). (That these conclusions are not affected much by using different values of $b$ can be seen from the dotted lines in Figs. 1 and 2.)

![Figure 2. The effect of instrumental tilt on the ratios of certain turbulent quantities.](image)

- $\sigma_u = 2.5 u_e$
- $\sigma_u = 2.4 u_e$
- $\sigma_u = 2.6 u_e$
- $C = 2.5$
- $C = 2.4$
- $C = 2.6$
Higher up in the atmosphere (or near the sea surface) the relative magnitudes of these quantities may differ from those encountered in the constant stress layer over land. Thus Deacon (1968) has argued that \((u'w)'_a\) is less dependent on \(\alpha\) over the sea than over the land. However, this gain will probably be offset somewhat by the increased instrumental difficulty of minimizing the tilt. Furthermore, Kraus (1968) pointed out that the use of the principal stress axes will not improve the situation because the orientation of the vertical is presumed. A possible way to overcome this dependence would be to use the following two quantities which are invariant with respect to rotation,

\[
R = \sqrt{\frac{1}{2} \left( (\sigma_{w}^{2})_a - (\sigma_{u}^{2})_a \right)^2 + (u'w)'_a^2}
\]

and

\[
C = \frac{1}{2} \left( (\sigma_{w}^{2})_a + (\sigma_{u}^{2})_a \right).
\]

Thus the comparison of different probes at different sites would be facilitated to a certain extent. Undoubtedly, however, the main use of these invariants is for a comparison of turbulence measuring probes in close proximity.

4. THE EFFECT OF TILT ON THE SPECTRAL DISTRIBUTIONS

It can be seen that the transformation equations (i.e. Eqs. (1)) are purely geometric in form and will hold at all frequencies as can be confirmed by expanding the variables in terms of complex Fourier series (this was in fact implied by Weiler and Burling 1967). Thus, for example, Eq. (1a) will reduce to

\[
\frac{(nS_{w}(n))_a}{u_*^2} = \frac{nS_{u}(n)}{u_*^2} \cos^2 \alpha + \frac{nS_{u}(n)}{u_*^2} \sin^2 \alpha - \frac{nS_{uw}(n)}{u_*^2} \sin 2\alpha.
\]

This means that by substituting suitable spectral expressions in the equations it is possible to investigate the dependence of the spectral forms on \(\alpha\).

Several writers have prescribed formulae which roughly fit measured spectral curves and for the purpose of the ensuing analysis it will be assumed that they hold for \(\alpha = 0\). The expressions used were as follows,

\[
\frac{nS_{u}(n)}{u_*^2} = \frac{4y}{1 + y^{5/3}}
\]

(Lumley and Panofsky 1964)

where \(y = 900 n/V = 900 f/z\). This equation integrates to give \(\sigma_u^2 = 2.82 u_*^2\) which differs from the value used in Section 3, but only marginally affects the following discussion.

\[
\frac{nS_{uw}(n)}{u_*^2} = \frac{-1.075 f/f_m}{1 + 1.5 (f/f_m)^{5/3}}
\]

(Busch and Panofsky 1968)

and

\[
\frac{nS_{uw}(n)}{u_*^2} = \frac{-f/f_m'}{(1 + 0.6 f/f_m')^{5/3}}
\]

(Panofsky and Mares 1968).

In using these expressions it was assumed that \(z = 10\ m, f_m = 0.32\) and \(f_m' = 0.08\). The results of the analysis are illustrated in Fig. 3 in which the solid lines denote the spectral forms for \(\alpha = 0\) and all other cases are represented by pecked lines.

Provided \(|\alpha| \leq 5^\circ\), \((nS_{u}(n))_a\) shows little dependence on the degree of tilt, as can be seen from Fig. 3 (a). This is due to the spectral contributions of the vertical component and the momentum flux being multiplied by rather small angular terms.

The slope in the inertial sub-range is given by

\[
\delta \frac{\ln (nS_{u}(n)/u_*^2)}{\delta \ln f} = -2/3
\]

but with tilt this becomes

\[
\delta \frac{\ln \{(nS_{u}(n))_a/u_*^2\}}{\delta \ln f} = -2/3 + 1/[f (1.72 \cot \alpha + 2.89 \tan \alpha - 1/f)].
\]
So that for tilts of $< 10^\circ$ the percentage error in the slope, $E_u$, can be represented, fairly closely, by

$$E_u = -1.5 \alpha / f$$

e.g. for $\alpha = 5^\circ$ and $f = 1$ the slope is less steep and $E_u = -7.5$ per cent.

Similar arguments can be applied to the $(nS_w(n))_a$ spectra. Here also the ordinate discrepancy and the changes in the peak frequency (which is sometimes used to calculate other turbulent quantities) are rather small (see Fig. 3 (b)). However, the differences caused by tilt are more pronounced in the low frequency region due to the intrusion of the $nS_w(n)$ spectral maximum. In the inertial sub-range the percentage error in the slope is given by

$$E_w = 0.91 \alpha / f$$

i.e. less than for $S_u(n)$.

The $(nS_{uw}(n))_a$ spectrum is the most sensitive to tilt errors as can be seen in Fig. 3 (c). At low frequencies and negative tilts the $nS_w(n)$ spectrum can enter as a large negative factor causing regions of positive $(nS_{uw}(n))_a$ to occur. A similar effect can happen at high frequencies where $nS_w(n)$ exceeds $nS_u(n)$. These effects are much reduced if $|\alpha| < 2^\circ$ (see Fig. 3 (d)) and meaningful spectra of momentum flux can only be produced if $\alpha$ is, at least this small.

The corresponding expression for the percentage error in the slope is

$$E_{uw} = -60/(1 - 24.41/f\alpha)$$

e.g. for $\alpha = 5^\circ$, $f = 1$ the slope is steeper and $E_{uw} = +15.5$ per cent.

It should be noted that all these curves have been standardized with respect to $u^3$ while in practice $(u'w')_a$ may be used. This would tend to reduce the ordinate value for positive tilts but increase it for negative ones, so that agreement would be worsened for the $(nS_w(n))_a$ spectrum but improved for the $(nS_{uw}(n))_a$ spectrum at all frequencies, while for the $(nS_{uw}(n))_a$ spectrum there is an improvement at low but a worsening at high frequencies.

Spectral values for $(nS_{uw}(n))_a$ would not be expected to show much dependence on the degree of tilt since $u' T'$, which can be of the order of $u' T'$, is multiplied by a small angular term.

Figure 3. The effect of tilt on spectra.
5. Conclusions

Careful alignment of sensors so that $\alpha < 2^0$ is probably only possible where short, rigidly fixed towers or masts can be used. Inevitably in many circumstances it will prove impossible to reduce these errors to within acceptable limits (i.e. $|\alpha| < 5^0$ at the very most) which means that the results will have a large inherent uncertainty. As this could arise whenever measurements are made from a moving platform, these comments apply to work done over the sea, from tall towers, from tethered balloons or from aircraft.

Even over land within the constant stress layer, the effect must be taken into account when identifying inertial sub-ranges. Furthermore, $\overline{S_{uu}}(\alpha)$ and $\overline{u'w'}$ show a marked dependence on $\alpha$ which may be rather difficult to allow for when results of different workers are compared, although the use of the two invariants may be of assistance here. The use of $\sigma_w^2$ to estimate $\overline{u'w'}$ has been discussed and, provided the constant of proportionality can be relied on, it should give a reasonable estimate of the mean flux if not its instantaneous value. The suggestion that $\sigma_u^2$ be used to normalize spectra instead of $\sigma_w^2$ seems appropriate provided of course that it is independently available.

In other environmental situations the relative magnitudes of the parameters will probably change making it difficult to predict how they will vary with $\alpha$. Thus, for example, over the sea Deacon (1968) argues that the dependence will be reduced. However, this gain could be offset by increased instrumental difficulties which incidently would probably also affect results obtained higher up in the boundary layer where assumptions such as $\overline{w} = 0$ cannot be made.

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References


Appendix

A symmetric tensor of the second order is formed by the six turbulent quantities $\sigma_u^2$, $\sigma_v^2$, $\sigma_w^2$, $\overline{u'w'}$, $\overline{u'v'}$, and $\overline{v'w'}$. A change from one set of mutually perpendicular axes to another set with the same origin will form a new group of turbulent quantities related to the original group by the following equations

$$(\sigma_u)^{\tau} = a_{u} \sigma_u a_{u} \sigma_u$$  \hspace{2cm} (A1)
where the new quantity is denoted by the subscript $T$, $a_T$ is the cosine of the angle between $(x)$ and $x_i$, $\sigma_u^2 = \sigma_3^2$, $\sigma_v^2 = \sigma_{22}^2$, $\sigma_{uv}^2 = \sigma_{11}$, $\bar{u}'w' = \sigma_{13}$, $\bar{u}'v' = \sigma_{12}$ and $\bar{v}'w' = \sigma_{23}$. In two dimensions these equations reduce to Eqs. (1).

As it is always possible to choose a set of mutually perpendicular axes so that $a_{ij} = 0$ unless $i = j$, Eq. (A1) may be rewritten as

$$\sigma_{ij} = a_{jm} a_{jn} S_m$$  \hspace{1cm} \text{(A2)}

where $S_m$ are the so-called principal components. Thus in two dimensions Eq. (A1) reduce to

$$(\sigma_{11})_T = \frac{1}{2} (S_1 + S_2) + \frac{1}{2} (S_2 - S_1) \cos 2\theta$$

$$(\sigma_{33})_T = \frac{1}{2} (S_1 + S_2) - \frac{1}{2} (S_2 - S_1) \cos 2\theta$$  \hspace{1cm} \text{(A3)}

and

$$(\sigma_{13})_T = \frac{1}{2} (S_2 - S_1) \sin 2\theta$$

where $\theta$ is the angle between the $x_i$ and principal axes. Eqs. (A3) may be represented by the plane diagram, called the Mohr Circle, shown in Fig. 4. The co-ordinate origin is somewhere on the line $S_1 S_2$ such that e.g. $(\sigma_{33})_T$ represents the distance from the origin. The following relations emerge directly from the diagram:

$$R^2 = \frac{1}{4} (S_2 - S_1)^2 = \sqrt{\frac{1}{4} ((\sigma_{33})_T - (\sigma_{11})_T)^2 + (\sigma_{13})_T^2}$$

$$\tan 2\theta = 2 (\sigma_{13})_T / [(\sigma_{33})_T - (\sigma_{11})_T]$$

$$C = \frac{1}{2} (S_1 + S_2) = \frac{1}{2} ((\sigma_{33})_T + (\sigma_{11})_T).$$

The Mohr Circle is a useful concept as not only does it clarify the manner in which the turbulent quantities change with rotation of axes but it also provides two rotationally invariant quantities (i.e. $R$ and $C$). (See Nye 1957.)