An operational linear lee wave model for arbitrary basic flow
and two-dimensional topography

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SUMMARY

A partial survey of recent theoretical work on the flow over obstacles is presented. A model for linear, stationary mountain waves with arbitrary basic flow and two-dimensional topography is described in detail. It contains the option of a nonlinear lower boundary condition. The wave drag is computed by three methods and the complete flow field is obtained. The sensitivity of the solutions to small changes of the upstream flow is demonstrated as a critical factor limiting the applicability of the model. The strong downslope wind storms of the Boulder area are briefly discussed in the general context of foehn phenomena. Computations indicate that resonance lee waves do not seem to be part of the mechanism of these winds.

NOTATION

\begin{align*}
x, y, z & \quad \text{Cartesian co-ordinates, } x \text{ pointing along basic wind } \mathbf{u} \\
\nabla & \quad (\partial/\partial x, \partial/\partial z) \\
\nabla^2 & \quad \partial^2/\partial x^2 + \partial^2/\partial z^2 \\
\frac{d}{dt} & \quad \partial/\partial t + u \partial/\partial x + v \partial/\partial y + w \partial/\partial z \\
\frac{D}{Dt} & \quad \partial/\partial t + \bar{u}(z) \partial/\partial x + \bar{v}(z) \partial/\partial y \\
\bar{u}'(z) & \quad u_z = d\mathbf{u}/dz \\
p & \quad \text{pressure} \\
p* & \quad \text{density} \\
T & \quad \text{temperature} \\
\Theta & \quad \text{potential temperature} = T \left( \frac{1000}{p} \right)^{6.266} \text{, } p \text{ in mb} \\
u & \quad \bar{u}(z) + u' \quad \text{wind velocities in } x, z \text{ directions} \\
w & \quad u' \quad \text{(See Eq. (3))} \\
\bar{w} & \quad \text{density-modified vertical velocity} = \sqrt{\frac{\rho_0}{\rho}} w' \\
\mathbf{u}, \bar{u}, \rho(z), \ldots & \quad \text{mean quantities (basic flow)} \\
z' & \quad \text{streamline displacement} \\
g & \quad \text{acceleration of gravity} \\
N(z) & \quad \text{Brunt-Vaisala frequency } N^2 = g \frac{d\bar{\Theta}}{dz} \\
C_p & \quad \text{specific heat of air at constant pressure} \\
C_v & \quad \text{specific heat of air at constant volume} \\
R & \quad C_p - C_v \quad \text{gas constant for air} \\
I^2(z) & \quad \text{Scorer parameter, see Eq. (6)} \\
I_o & \quad \text{value of } I \text{ or } I_o \text{ (Eq. 10) for the uppermost layer} \\
b, \Theta, u_o & \quad \text{mean density, potential temperature, wind speed at surface} \\
h(x) & \quad \text{mountain profile} \\
k & \quad \text{horizontal wave number} 
\end{align*}

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AN OPERATIONAL LINEAR LEE WAVE MODEL

A (k)  Fourier transform of $\tilde{\omega} (z = 0)$ (see Eq. (17))
$\omega$  radian frequency
$\tilde{\omega} = \omega - k\tilde{\alpha}$  intrinsic frequency
Ri  Richardson number $= N^2/(dt/dz)^2$
Re ( )  real part of ( )
Im ( )  imaginary part of ( )
$\|\|$  magnitude of a complex number
$H_\infty (z)$  a scale height defined by $\frac{1}{H_\infty} = -\frac{d}{dz} \ln \sqrt{\gamma}$
numerical value $H_\infty \approx 25$ km
$D$  drag on the mountain (see Eq. (31))
$C_D$  drag coefficient (see Eq. (38)).

1. Introduction

The study of air flow over mountains continues to be a challenging subject for both experimental and theoretical work. While scientific curiosity is a legitimate motivation, related problems of an applied nature are numerous: strong surface winds (chinook), waves, rotors and turbulence as they affect flight operations in the troposphere and stratosphere, radiation of wave energy to the stratosphere and higher levels and the mechanisms of its dissipation at these levels, interaction between the mesoscale mountain effects and larger scales (drag).

The experimental and theoretical work by Kuettner, Lyra, Queney, Scorer, Wallington, Corby, Worlote, Holmboe, Kieffer, Palm, Foldvik, Sawyer and many others up to 1960 is well summarized by Queney, Corby, Gerbier, Koschmieder and Zierep (1960). More recently fluid dynamicists have turned their attention to the flow over obstacles (Miles 1968a, b; Huppert and Miles 1969; Miles and Huppert 1969; Pao 1968, 1969; Pao and Timm 1966; Drazin and Moore 1967; Kao 1965 and many others). Miles (1969) has given an authoritative and comprehensive survey of much of this work in a geophysical context, including many historical references and a sample of references to recent work in the USSR. Pao’s model consists of vortex pairs and doublets arranged at the lower boundary. As already noted by Long (1953), such a linear model can be exact for the nonlinear case if the basic wind speed is uniform and the stability constant. In general the shape of the obstacle depends on the basic flow and cannot be prescribed (see, however, Miles 1968a and the three following parts). Because of that and the severe restrictions on the basic flow, these so-called ‘Long’s models’ are of limited usefulness in the present context. The quasi-analytic solutions of Huppert and Miles (1969) for elliptical obstacles are, however, very helpful for comparative purposes.

Some attempts have been made at solving the pertinent steady-state equations (Helmholtz wave equation) by a forward marching numerical scheme. This would seem to be a natural way of solving the problem, as a prescribed upstream flow is propagating downstream. A peculiar difficulty arises, however, because the eigenmodes in a channel of finite vertical depth (and consequently also in a numerical model with a finite number of grid points) contain solutions which decay exponentially upstream and downstream away from the obstacle (see Long 1955). For these components the effective boundary conditions, i.e. finiteness, have to be prescribed upstream and downstream, and a marching scheme cannot do this. Pekelis (1966, 1969) and Zeytounian (1968) have discussed this problem in detail. Krishnamurti (1964) did not seem to be aware of it. He apparently obtained some solutions, but states that in other cases the scheme did not converge. Onishi (1969) seems to have solved the problem by introducing an artificial viscosity coefficient. This approach is also applicable for three-dimensional waves and seems very promising, although more analysis on the peculiar boundary conditions and experiments with real data are needed.
The possibility of three-dimensional flow should always be kept in mind (flow around rather than above the mountain), but it was not explored here since the region of main interest, the Colorado Rocky Mountains near Boulder, are very closely two-dimensional.

The phenomenon of the 'hydraulic jump,' long familiar to engineers, was brought to the attention of meteorologists by Long (1954), Kuettner (1958) and others. In the Sierra Wave Project the jump was sometimes identified as the leading edge of a rotor cloud. A detailed theoretical analysis of the one-layer hydraulic jump model was published by Houghton and Kasahara (1968). Houghton and Isaacson (1970) went on to investigate a two-layer jump model. These models are hydrostatic and cannot therefore produce lee waves. They are suspected to be applicable, however, for strong downslope winds. Even in the absence of strong chinook winds, data gathered by the Mountain Wave Program at NCAR (Vergeiner and Lilly 1970) show a variety of flow patterns which can be explained by hydraulic jump models: (almost) jumps, supercritical and subcritical flow.

Time-dependent numerical calculations have been demonstrated by Foldvik and Wurtele (1967), Hovermale (1965), Fromm (private communication) and Pao (private communication) and are no doubt the final answer to the problem. So far, however, a number of problems remain to be solved. Computer time requirements are a nuisance rather than a serious obstacle, but they do limit the possibilities for experimenting. Computations appear to have been limited so far to rather simple upstream profiles of wind and stability. Moreover, it is not clear how the peculiar asymmetric boundary condition downstream should be formulated. It is customary to terminate the computations before reflection from either boundary becomes significant.

Laboratory simulation of mountain waves and hydraulic jumps has greatly improved our understanding of these phenomena, following Long's (1954, 1955) original experiments. His and more recent laboratory studies by Timm and Pao (1966) and Lin and Binder (1967) frequently yield flow patterns suggestive of strong downslope winds on the lee side of the obstacle.

Cumulus clouds, islands and other thermal inhomogeneities may be expected to excite gravity waves in a stable atmosphere much like mountains. This is the so-called 'heated island' effect (Malkus and Stern 1953, Stern and Malkus 1953, Trubnikov 1964). One would expect that conditions are more often three-dimensional than two-dimensional for this effect.

Satellite pictures have been used to document lee wave activity and determine the dominant wavelength and even to obtain wind information from the wavelengths (Conover 1964, Fritz 1965, Cohen and Doron 1967, Lovill 1969).

The model presented in this paper is somewhat similar to recent models by Wallington (1970) and Danielsen and Bleck (1970). Experiments were performed with both a linearized lower boundary condition and a nonlinear one in the sense that the kinematic lower boundary condition is satisfied on the actual slope. The assumption of steady state is not desirable, but necessary in the present context. Group velocities of gravity waves on the scale discussed here appear to be of the order of 5 m s$^{-1}$ (Houghton and Jones 1969a), so that it should take about an hour for the steady state to become established up to 10 km. The viewpoint adopted here is that both linear and nonlinear models should be consulted for a given case, each being capable of explaining different phenomena. It could be argued, for example, that weak or no resonant tendency of the atmosphere is a condition for the applicability of one- and two-layer hydraulic jump models, which exclude interaction with upper layers.

2. Basic equations

When the Earth's rotation is neglected (see Houghton 1969), the equations for frictionless, adiabatic, compressible flow are
\[ \frac{du}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \]
\[ \frac{dv}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0 \]
\[ \frac{dw}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial z} + g = 0 \]
\[ \frac{dp}{dt} + \frac{\rho \nu}{\rho} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \]
\[ \frac{d\theta}{dt} = 0 \]
\[ p = R\rho T \]

For the stationary case and two-dimensional flow \((\partial / \partial t) = (\partial / \partial y) = 0\) Long (1953), Yih (1958; 1965) and Claus (1964) have shown that Eqs. (1) may be condensed into a nonlinear vorticity equation which, in turn, may be integrated once to obtain a nonlinear Helmholtz-type equation. This end product is rather elegant, but not helpful for obtaining solutions.

Here only the linearized equations are considered. The most compact notation for linearization has been given by Bretherton (1966):

Let
\[
\begin{pmatrix}
  u \\
  v \\
  w \\
  \rho \\
  p
\end{pmatrix} = \begin{pmatrix}
  u(z) \\
  \psi(z) \\
  0 \\
  \rho(z) \\
  \bar{p}(z)
\end{pmatrix} + \begin{pmatrix}
  u' \\
  \psi' \\
  w' \\
  \rho' \\
  p'
\end{pmatrix} \tag{2}
\]

The notation
\[ \frac{D_0}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + \psi \frac{\partial}{\partial y} \]
is used with
\[ \frac{\partial}{\partial z} \frac{D_0}{Dt} = \frac{D_0}{Dt} \frac{\partial}{\partial z} + \frac{du}{dz} \frac{\partial}{\partial x} + \frac{d\psi}{dz} \frac{\partial}{\partial y} \]

and a scale height \( H_* (z) \) is introduced according to
\[ \frac{1}{H_*} = - \frac{d}{dz} \ln (\Theta \sqrt{\bar{p}}). \]

\( H_* \) is the negative of Bretherton's \( H \) and has a value of about \(+25 \) km in the troposphere.

With a further transformation of the perturbation variables
\[
\begin{pmatrix}
  u' \\
  \psi' \\
  w'
\end{pmatrix} \equiv \sqrt{\frac{\rho_0}{\bar{p}(z)}} \begin{pmatrix}
  \tilde{u} \\
  \tilde{\psi} \\
  \tilde{w}
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
  p'
\end{pmatrix} \equiv \sqrt{\frac{\bar{p}(z)}{\rho_0}} \begin{pmatrix}
  \bar{p}
\end{pmatrix} \tag{3}
\]

where \( \rho_0 \) is a reference (surface) density for dimensional consistency, Bretherton's Eq. (41) may be obtained by elimination. The term \((1/c^2) (D_0^2 \bar{p}/Dt^2)\) contains a factor of the order of Mach number squared relative to the other terms and is therefore negligible in the present context in Bretherton's Eq. (41) and in the equation following (40).
For stationary, two-dimensional flow \( ((\partial \psi)/\partial t) = (\partial \psi)/\partial y = 0 \), Bretherton’s Eq. (41) then becomes the familiar ‘mountain wave’ equation

\[
\frac{\partial^2 \tilde{\psi}}{\partial x^2} + \frac{\partial^2 \tilde{\psi}}{\partial z^2} + \left[ N^2 \left( \frac{\psi}{u^2} - \frac{\partial}{\partial \rho} \sqrt{\frac{\psi}{\rho}} \right) \frac{\partial}{\partial \rho} \sqrt{\frac{\psi}{\rho}} \right] \tilde{\psi} = 0 . \tag{4}
\]

where \( N^2 (z) \equiv (g/\partial) (d\rho/dz) \) is the square of the Brunt-Vaisala frequency. For a given horizontal wave number \( k \),

\[
\tilde{\psi} = e^{ikz} f (x, k)
\]

and

\[
f'' (z) + \left[ \frac{N^2}{u^2} - \cdots - k^2 \right] f (z) = 0 . \tag{5}
\]

This equation describes the propagation characteristics of internal gravity-shear waves of wavelength \( 2\pi/k \) in stratified shear flow.

The analog of the refractive index of the problem is the commonly called ‘Scorer-parameter’:

\[
l^2 (z) = \frac{N^2}{u^2} - \left( \frac{\psi}{\rho} \sqrt{\frac{\psi}{\rho}} \right)'' \tag{6}
\]

so that Eq. (5) becomes

\[
f'' (z) + [l^2 (z) - k^2] f (z) = 0 . \tag{7}
\]

Eq. (7) was solved by a multi-layer technique. We note that for the latter technique interface conditions are required; namely that streamline displacement \( z' \) be continuous across the interface, as well as pressure (total and perturbation).

If mean temperature \( T \) is assumed continuous (not \( dT/dz \)), the same follows for \( \rho \). We have

\[
\sqrt{\frac{\rho_0}{\rho} (z)} \tilde{\psi} \equiv u' \equiv \frac{dz'}{dt} = u \frac{\partial z'}{\partial x}
\]

for steady, two-dimensional flow. If \( z' \) is continuous,

\[
\frac{\tilde{\psi}}{u} (z) \quad \text{or} \quad \frac{f (z, k)}{u} \tag{8}
\]

must be continuous across the interface.

The second interface condition (\( \rho' \) continuous) follows from the equation between (40) and (41) in Bretherton (1966) with the term \( (1/c^2) \left( D_o^2 \tilde{\psi}/Dt^2 \right) \) neglected as discussed before viz. continuity of the expression

\[
\frac{D_o}{Dt} \left[ \frac{\partial \tilde{\psi}}{\partial z} - \tilde{\psi} \right] - u_x \frac{\partial \tilde{\psi}}{\partial x} - u_z \frac{\partial \tilde{\psi}}{\partial y} .
\]

For the steady two-dimensional state, this reduces to continuity of

\[
\frac{u^2}{\partial} \sqrt{\frac{\psi}{\rho}} \frac{\partial}{\partial z} \left( \frac{\psi}{\rho} \sqrt{\frac{\psi}{\rho}} \tilde{\psi} \right) \tag{9}
\]

or the corresponding condition for \( f (z, k) \) in place of \( \tilde{\psi} \). If, and only if, \( \psi \) and \( (d/dz) \left( \psi/\rho \sqrt{\psi/\rho} \right) \) are continuous, it follows that \( \tilde{\psi} \) and \( \partial \tilde{\psi}/\partial z \) should be continuous across interfaces. This is to be compared, for example, with Scorer’s remark in Scorer (1967, p. 80).
3. Multi-layer model

For the computations discussed here the following multi-layer approach was considered to be realistic and convenient; enough significant levels are chosen so that the basic quantities do not vary too strongly within a layer -- typically 10 to 15 levels between the surface and about 50 mb were used:

\[ \tilde{u}(z) \text{ and } \tilde{T}(z) \text{ continuous at significant levels} \]

\[ N^2 = \text{const within each layer} \quad (10) \]

\[ l^2(z) \equiv l_0^2 - \frac{2}{(z + \alpha)^2} \text{ within each layer,} \]

where \( l_0 \) and \( \alpha \) are two disposable constants.

One could, of course, try to solve numerically for the precise shape of the wind profile in each layer from these assumptions and the definition of \( N^2 \) and \( l^2 \). It seemed sufficient, however, to assume that Eq. (10) will yield a more or less smooth, if not exactly linearly-varying profile \( \tilde{u}(z) \) such that \( (u/\tilde{\theta} \sqrt{\rho})'' \) is negligible in the definition of \( l^2(z) \) and \( (d/dz) \ln (u/\tilde{\theta} \sqrt{\rho}) \) can be considered a constant for each layer. Then the parameters \( l_0 \) and \( \alpha \) are computed from the two conditions at the top and bottom of each layer.

\[ l^2(z_{t+1}) = \frac{N_t^2}{\tilde{u}_{t+1}^2} = l_0^2 - \frac{2}{(z_{t+1} + \alpha)^2} \]

\[ l^2(z_t) = \frac{N_t^2}{\tilde{u}_t^2} = l_0^2 - \frac{2}{(z_t + \alpha)^2}. \]

This requires an iteration from a suitably close first guess, the details of which are not given here. When the Scorer parameter was constant within a layer, a large value of \( \alpha \) (10 m) was used arbitrarily to avoid \( \alpha = \infty \). The interface conditions are, for either \( \tilde{u} \) or \( f(z, k) \):

\[ \tilde{u} \text{ continuous} \]

\[ \frac{\partial \tilde{u}}{\partial z} - \tilde{u} \cdot \frac{d}{dz} \ln \left( \frac{\tilde{u}}{\tilde{\theta} \sqrt{\rho}} \right) \text{ continuous} \quad (11) \]

The profile (10) arises as a special case \( (m = 3/2) \) of the more general one \( l^2(z) = l_0^2 - (m^2 - \frac{1}{2})/(z + \alpha)^2 \), \( l_0, m, \alpha = \text{const.} \) The solution of the differential equation (7) for this profile is

\[ f(z, k) \sim \sqrt{z + \alpha} J_{\pm m} \left[ \sqrt{l_0^2 - k^2} (z + \alpha) \right]. \]

For \( m = 3/2 \), the Bessel functions reduce to elementary functions, so the profile (10) used here yields as basic solutions of Eq. (7)

\[ f_1(z, k) = e^{iy} \left( 1 + \frac{i}{y} \right) \quad \text{and} \]

\[ f_2(z, k) = e^{-iy} \left( 1 - \frac{i}{y} \right) \quad \text{ (12)} \]

where \( y \equiv \sqrt{l_0^2 - k^2} (z + \alpha) \).

We thus have as much flexibility (2 parameters) as, say, the Palm-Foldvik solution (Palm and Foldvik 1960, Döös 1961, Danielsen and Bleck 1970) without needing Bessel functions. The functional form (10) for the Scorer parameter is just as plausible as the exponential one.
4. The upper boundary condition

The common way to formulate the upper boundary condition has been to assume an upper layer extending from \( z = H \) to infinity with a constant value of the Scorer parameter \( l_{\infty}^2 \), and to choose

\[
 f(z) \sim \exp \left[ i \sqrt{l_{\infty}^2 - k^2} \ z \right] \text{ for } 0 \leq k \leq l_{\infty} (H \leq z < \infty) \quad \text{(radiation condition)}
\]

and

\[
 f(z) \sim \exp \left[ - k^2 - l_{\infty}^2 \right] \text{ for } l_{\infty} \leq k < \infty \quad \text{(evanescent solution)} \tag{13}
\]

The radiation condition results in positive energy flux (leaking) away from the source, while the choice of \( - i \sqrt{l_{\infty}^2 - k^2} \ z \) would lead to an unrealistic energy flux toward the source (mountain). Some models, notably the Palm-Foldvik model with \( l^2 \) decreasing exponentially to zero at large heights, circumvent the problem of energy leakage. For arbitrarily small \( l^2 \) Eq. (7) has two exponential-type solutions, of which only the evanescent is realistic. Theoretically there is no energy loss in this case.

The choice of \( l_{\infty}^2 \) is essentially arbitrary depending on the level \( H \) where one wishes to terminate the calculation. This does not matter much in the layers near the surface, but it will influence the solution in the stratosphere. The radiation condition implies that somewhere beyond our domain of computation, wave energy is absorbed without reflection. Two possible mechanisms are the 'critical layer' where the intrinsic frequency \( \omega = \omega - ku \) vanishes (Booker and Bretherton 1967, Hines and Reddy 1967, Jones 1967, Hazel 1967, Houghton and Jones 1969b and others), and a dissipative layer. In numerical studies the radiation condition can be simulated by an eddy diffusivity increasing smoothly from zero above the domain of interest to avoid spurious reflections (Lindzen, private communication) or even by a Newtonian friction term with the same behaviour (Houghton and Jones 1969a).

For stationary waves the relevant critical level would seem to be the first level where the basic wind \( \bar{u} \) reverses its direction. If in a layer below the critical level \( z = z_0 \) \( \bar{u} = (du/dz) (z_0 - z) \), Eq. (7) for that layer becomes

\[
 f''(z) + \left[ \frac{Ri}{(z_0 - z)^2} - k^2 \right] f(z) = 0
\]

where

\[
 Ri \equiv \frac{N^2}{(du/dz)^2}
\]

The solutions of Eq. (14) have been investigated extensively by Wurtele (1953). If the computations are terminated at a level \( z = H \) close to the critical level, \( (Ri/(H - z_0)^2) \) is larger than \( k^2 \) for all wavenumbers of interest. The solution of Eq. (14) for the layer between \( z = H \) and \( z = z_0 \) then takes the simplified form

\[
 f(z) \approx |z - z_0|^{1 - i \sqrt{Ri - 1}}
\]

and the boundary condition at \( z = H \) is consequently

\[
 \frac{df}{dz} \bigg|_{z=H} = -\frac{i}{2} + i \sqrt{Ri - \frac{1}{2}} \cdot \frac{1}{|z_0 - H|}.
\]

Another possibility would be to include constant eddy viscosity and conductivity in the uppermost layer and choose the outgoing wave solution, as formally described by Giwa (1967). Such a model, however, reflects waves downward from the discontinuity, where the uppermost layer begins.
All the possibilities mentioned were actually tried with only small changes resulting for the main low-level features. The most logical choice is still to use \( f_1(z, k) \) (see Eq. (12)) in the uppermost layer, i.e. the outgoing wave only, and this procedure was finally adopted. For all practical purposes \( l_0^2 \) for that layer is much larger than \( k^2 \), and we get

\[
\left[ \frac{df}{dz} \right]_{z=H} = i l_0 \frac{1 + \frac{i}{y} + \left( \frac{i}{y} \right)^2}{1 + \frac{i}{y}}.
\]

where \( y(z=H) = l_0 (H + z) \) (real) and \( l_0 \) and \( z \) are the two parameters for the uppermost layer, given by Eq. (10).

### 5. Formal solution

To specify the flow field completely we have to add the lower, kinematic boundary condition on \( \bar{v} \). Let us simply say for the moment that \( \bar{v}(x, z=0) \equiv w_0(x) \) is a prescribed, known function of \( x \) with a Fourier representation:

\[
w_0(x) = \text{Re} \int_0^\infty A(k) e^{ikx} dk
\]

where

\[
A(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} w_0(x) e^{-ikx} dx.
\]

The real part of \( A(k) \) represents the part of \( w_0(x) \) which is symmetric around \( x = 0 \) (the crest of the mountain, say) and the imaginary part the one which is asymmetric. The formal solution of our problem is now readily obtained as

\[
\bar{v}(x, z) = \text{Re} \int_0^\infty A(k) \frac{f(z, k)}{f(0, k)} e^{ikx} dk.
\]

This expression clearly satisfies the lower boundary condition for \( z = 0 \); Eq. (16) and the upper boundary condition as well as the differential equation (4) if \( f(z, k) \) satisfies Eq. (7). Eq. (7) for \( f(z, k) \) can be integrated from top to bottom, using \( f(z=H) = 1 \) (arbitrary), \( (df/dz)(z=H) \) from the radiation condition (15), and the two basic solutions (12) for each layer. Between layers the proper interface conditions are applied.

As is well known, the formal solution (17) cannot usually be integrated as it stands because the integrand may have poles close to the integration path, where \( f(0, k) \) becomes zero. The poles correspond to resonant excitation of the atmosphere at that particular wavelength. The zeroes of \( f(0, k) \) which have physical significance lie rather close to the real axis ("Gamov poles"), with a positive imaginary part equivalent to downstream damping. The lee waves are not completely trapped. R. Bleck kindly supplied his Fortran subroutine which finds all zeroes of a complex function by use of Rouche's theorem. In Scorer's model (1949) the magnitude of the Scorer parameter \( l_0^2 \) in the upper layer, a very ill-defined quantity, marks a cut-off point. No resonance waves are supposed to be possible for wavenumbers smaller than \( l_0^2 \). This is clearly unrealistic. Corby and Sawyer (1958) noted that there could be complex zeroes of \( f(0, k) \) reflecting the leakage of wave energy from the top. Most classical models get around this problem by letting \( l^2(z) \) go to zero at large height. It was found that the magnitude of the imaginary part of a resonance wavenumber gives already a good indication of the amplitude: significant modes with large amplitudes tend to show exceedingly small damping. This finding is corroborated by Berkshire and Warren (1970).
Let \( k_1, k_2, \ldots k_n \) be the zeroes of \( f (0, k) \) near the positive real \( k \)-axis. The strategy adopted here is to subtract out the poles of the integrand in Eq. (17) and treat them in a quasi-analytic manner, while the remaining function is well-behaved and can be integrated numerically. A new function \( \phi \) is introduced:

\[
\phi (z, k) \equiv \frac{f(z, k)}{f(0, k)} - \sum_n \frac{A(k_n)}{A(k)} \cdot \frac{f(z, k_n)}{\frac{df(0, k)}{dk}} \cdot \frac{4k_n^3}{k^4 - k_n^4 \sqrt{k_n}}. \tag{18}
\]

Here \( k_n \) is assumed to be sufficiently close to the real axis so that \( A(k_n) \) is defined. The simplest way of describing the behaviour of \( f(z, k)/f(0, k) \) near \( k = k_n \) is

\[
\frac{f(z, k)}{f(0, k)} \approx \frac{f(z, k_n)}{\frac{df(0, k)}{dk}} \cdot (k - k_n)
\]

but another form was chosen for no strong reason except that it converges much better. \( \phi (z, k) \) has the analytic behaviour suitable for numerical integration. We now substitute for \( f(z, k)/f(0, k) \) in Eq. (17) with the aid of Eq. (18) and interpret the integral involving the poles in the classical way by applying Cauchy’s theorem. For \( x > 0 \) (downstream) the path of integration (\( k = 0 \) to \( \infty \)) is deflected according to \( k = k_n e^{i\pi/4} \mu^2 \), \( \mu \) (real) = 0 to \( \infty \) (slightly different for each \( k_n \)), and the integral from \( k = 0 \) to \( \infty \) is transformed into an integral from \( \mu = 0 \) to \( \infty \) along a straight line in the first quadrant plus \( 2\pi i \) times the residue at each pole \( k_n \) (the lee waves). For \( x < 0 \) (upstream) the transformation is

\[
k = k_n e^{-i\pi/4} \mu^2, \quad \mu = \text{real from } 0 \text{ to } \infty
\]

but there are no residues enclosed by the contour. As a result Eq. (17) transforms to

\[
\tilde{\omega}(x, z) = \text{Re} \left\{ \sum_n A(k_n) \cdot \frac{f(z, k_n)}{\frac{df(0, k)}{dk}} \cdot \begin{cases} \frac{8i e^{-i\pi/4}}{\sqrt{\mu^4 + 1}} \int_0^{\infty} e^{-k_n|z| e^{i\pi/4} \mu^2} \frac{\mu^4 d\mu}{\mu^4 + 1} & \text{for } x < 0 \\ 2\pi i e^{i\pi/4} z - \frac{8i e^{-i\pi/4}}{\sqrt{\mu^4 + 1}} \int_0^{\infty} e^{-k_n|z| e^{i\pi/4} \mu^2} \frac{\mu^4 d\mu}{\mu^4 + 1} & \text{for } x > 0 \end{cases} \right\} \tag{19}
\]

Here the integrals along the deformed paths may be expressed in terms of the error function (see Appendix).

6. THE LOWER BOUNDARY CONDITION

Let \( \vec{u}(z) = u_0 + (du/dz) z \) be the variation of the basic wind with height near the surface (\( z = 0 \)). The classical linearized boundary condition is

\[
\tilde{\omega}(x, z = 0) = u_0 \frac{dh}{dx}. \tag{20}
\]

It is applied at \( z = 0 \) rather than on the mountain slope \( z = h(x) \). This form makes the complete solution proportional to the surface wind \( u_0 \), a quantity which is typically ill-defined. R. Bleck (Danielsen and Bleck 1970) made a significant improvement by using the actual wind \( \vec{u}(z) \) along the mountain slope:

\[
\tilde{\omega}(x, z = 0) = \left( u_0 + \frac{du}{dz} h(x) \right) \frac{dh}{dx}. \tag{21}
\]
To put it another way, Eq. (21) may still be used, provided the actual mountain \( h \) is replaced by a fictitious mountain:

\[
h^* = h + \frac{h^2}{2u_0} \frac{du}{dz}.
\]

The inclusion of the wind shear near the surface alleviates the difficulty of determining a representative \( u_0 \), but \( \bar{\omega} (x, h(x)) \) is still approximated by \( \bar{\omega} (x, 0) \). One possible approach is to formally consider \( \bar{\omega} (x, z = 0) \) an unknown function and to determine \( A (k) \) (see Eq. (16)) numerically in such a way that the exact boundary condition is satisfied; i.e., the total streamfunction is zero along the mountain slope. Taking into account the definition of \( \bar{u} \) and \( \bar{\omega} \) (Eq. (3)), a perturbation stream function \( \psi' \) may be defined as

\[
\frac{\partial \psi'}{\partial x} = -\sqrt{\frac{\rho (z)}{\rho_0}} \bar{\omega},
\]

\[
\frac{\partial \psi'}{\partial z} = \sqrt{\frac{\rho (z)}{\rho_0}} \bar{u}.
\]

The total stream function is \( \psi = \bar{\psi} (z) + \psi' \) with \( (d\bar{\psi}/dz) = (\rho (z)/\rho_0) \bar{u} \). Neglecting the variation of density near the surface we get from Eq. (20)

\[
\bar{\psi} (z) = u_0 z + \frac{du}{dz} \frac{z^2}{2}.
\]

Similarly, by integrating Eq. (23) it follows that

\[
\psi' (x, z) = - \int_{-\infty}^{z} \bar{\omega} (\xi, z) d\xi.
\]

Thus the correct boundary condition becomes

\[
u_0 h (x) + \frac{du}{dz} \frac{h^2}{2} = \int_{-\infty}^{z} \bar{\omega} (\xi, h(x)) d\xi.
\]

This is exactly the same as the condition

\[
w = (u (z) + u') \frac{dh}{dx} \text{ at } z = h(x).
\]

Substituting for \( \bar{\omega} \) from Eq. (19), Eq. (24) transforms to

\[
\text{LHS} (x) \equiv u_0 h + \frac{du}{dz} \frac{h^2}{2} = Re \int_{0}^{\infty} \frac{A (k)}{ik} \phi (h(x), k) e^{i k x} dk + \]

\[
+ \left. \begin{array}{ll}
8i e^{i k} \int_{0}^{\infty} e^{-k_n |x|} e^{i n/4 \mu^2} \frac{d\mu}{\mu^5 + 1} & \text{for } x < 0 \\
-
\end{array} \right]
\]

\[
- \left. \begin{array}{ll}
2\pi i e^{ik_n x} - & \\
8i e^{-i k} \int_{0}^{\infty} e^{k_n |x|} e^{-i n/4 \mu^2} \frac{d\mu}{\mu^5 + 1} & \text{for } x > 0 \\
\end{array} \right]
\]

(25)
\( \phi (z, k) \) is defined in Eq. (18). We have \( \phi (0, k) = 1 \). The linearized boundary condition is obtained by setting \( h (x) = 0 \) in Eq. (25). The corresponding solution for \( A (k) \) is

\[
\frac{A^{(0)} (k)}{ik} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left[ u_0 h + \frac{d}{dz} \cdot \frac{h^2}{2} \right] e^{-ikx} dx.
\]

There is no obvious iteration which would lead from \( A^{(0)} \) as a first guess to the nonlinear solution. We note first that the left-hand side of Eq. (25) is zero, or may be assumed to be zero, away from the mountain. With a suitably chosen origin of the \( x \)-axis we may say that \( LHS (x) \) is zero outside the strip \(-L < x < L\). We also have for the outside region \( \phi (h (x), k) = 1 \). These restrictions impose a certain form on \( A (k) \), which may be investigated by taking the inverse Fourier transform of Eq. (25). The result is that \( A (k) \) must have the form

\[
\frac{A (k)}{ik} = \sum_{j=1}^{\infty} C_j e^{ikL} - (-1)^j e^{-ikL} \frac{k^2 - \left( \frac{j\pi}{2L} \right)^2}{k^2} = \Sigma_{j=1}^{\infty} C_j \psi_j (k),
\]

if \( LHS (x) \) is expanded into a sine series in the interval \(-L < x < L\). Here the \( C_n \) are real constants. Let a set of real functions \( \Phi_j (x) \) be defined as the right-hand side of Eq. (25) with \( A (k)/ik \) replaced by \( \psi_j (k) \) and \( A (k_n)/ik_n \) replaced by \( \psi_j (k_n) \).

We call this definition Eq. (27).

\[
LHS (x) = \sum_{j=1}^{\infty} C_j \Phi_j (x).
\]

Eq. (25) then transforms to

\[
LHS (x) = \sum_{j=1}^{\infty} C_j \Phi_j (x).
\]

This is an expansion of the known function \( LHS (x) \) into a combination of the given functions \( \Phi_j (x) \). Only the interval \(-L < x < L\) need be considered. Outside of it Eq. (28) is identically zero. Eq. (28) may be solved either by applying the Gram-Schmidt orthonormalization process to the truncated set \( \Phi_j (x) \) or equivalently by computing the coefficients \( C_j \) by minimizing the expression

\[
\int_{-L}^{L} \left[ \sum_{j=1}^{J} C_j \Phi_j (x) - LHS (x) \right]^2 w (x) dx
\]

from which it follows that

\[
\int_{-L}^{L} \left[ \sum_{j=1}^{J} C_j \Phi_j (x) - LHS (x) \right] \Phi_l (x) w (x) dx = 0, \quad l = 1, 2, 3, \ldots J.
\]

\( w (x) \) is a suitable weighting function. For the computation of the integrals in Eq. (27) it is convenient to use an interpolation scheme for \( \phi (h (x), k) \), for example:

\[
\phi (h (x), k) = \frac{(q - 1)(q - 2)}{2} \phi (0, k) + q (2 - q) \phi (DZ, k) + \frac{q (q - 1)}{2} \phi (2DZ, k)
\]

where

\[
q (x) \equiv \frac{h (x)}{DZ}.
\]

With the coefficients \( C_j \) determined from Eq. (29), \( A (k) \) and \( A (k_n) \) are established from Eq. (26).
7. Momentum flux

The momentum flux (wave drag) associated with gravity waves is fundamentally different from other known momentum transport processes like surface frictional drag or the drag exerted on upper layers by penetrating \( C_u \) and \( C_b \), inasmuch as it may act across deep atmospheric layers. Typically the drag is generated by mountains and acts on turbulent layers in the stratosphere, as has been shown by Bretherton (1969).

Sawyer (1959) appears to be the first to have pointed out that in stratified flow the pressure is systematically higher on the upstream side, resulting in a drag force on the obstacle, and a corresponding drag of opposite sign on the airstream. Sawyer also recognized that this drag could be significant on the synoptic scale, and obtained estimates to the effect that frictional and wave drag were comparable in magnitude. The classic result of the constancy of momentum flux with height in the non-resonant, non-dissipative case is due to Eliassen and Palm (1961). Momentum flux due to random topography has been investigated by Blumen (1965) and Bretherton (1969).

For three-dimensional topography \( h(x, y) \) the total force exerted in the \( x \)-direction is

\[
F_x = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho \left( \frac{\partial h}{\partial x} \right) dx \, dy
\]

where the pressure, strictly speaking, should be evaluated on the slope: \( p(x, y, h(x, y)) \).

For two-dimensional topography it is natural to define the force per unit slab \( (\Delta y = 1) \) as

\[
D(\text{drag}) = \int_{-\infty}^{\infty} \rho \left( \frac{\partial h}{\partial x} \right) dx
\]

where, again, \( p \) should be understood as \( p(x, h(x)) \). If the equation for horizontal momentum

\[
\frac{\partial}{\partial x} (\rho uu) + \frac{\partial}{\partial z} (\rho uw) + \frac{\partial p}{\partial x} = 0,
\]

is integrated over a slab between the mountain and \( z = H \) as indicated above, the result is

\[
\int_{-L}^{L} p(x, h(x)) \frac{dh}{dx} dx = -\int_{-L}^{L} (\rho uu)_{z=H} dx - \int_{h(x)}^{H} \left( \int (p + \rho u^2) dz \right)_{x=-L}^{x=L}.
\]

Taking into account that at the slope \( u = u \left( \frac{dh}{dx} \right) \). If \( (p + \rho u^2) \) is the same far upstream and downstream \( (L \to \infty) \) and \( h(x) \to 0 \) on both sides, it follows that the momentum flux \( \int \rho uu \, dx \) is constant as a function of \( z \) and equal to the drag. In the linearized case with \( p(x, z = 0) \), the drag from Eq. (31) may be transformed by using the linearized equation for horizontal momentum and \( w' = u \left( \frac{dh}{dx} \right) \) \( z = 0 \) into

\[
D = -\rho \left( z \right) \int_{-\infty}^{\infty} u^' w^' dx = -\rho_0 \int_{-\infty}^{\infty} \bar{u} \tilde{w} dx
\]

at \( z = 0 \) or any other value, because

\[
\frac{d}{dz} \left( \rho_0 \int_{-\infty}^{\infty} \bar{u} \tilde{w} dx \right) = 0.
\]

The wave energy flux is given by

\[
\int_{-\infty}^{\infty} p' w' dx = \int_{-\infty}^{\infty} \tilde{p} \tilde{w} dx = -\bar{u} \left( z \right) \cdot \rho_0 \int_{-\infty}^{\infty} \bar{u} \tilde{w} dx.
\]
Eq. (34) is valid for the nonresonant case. There does not seem to be a unique way of separating the drag due to the continuous spectrum and the line spectrum, if resonance waves are present.* Following Bretherton (1969), the best one can do for obtaining the continuous part is to use Eq. (17) with the understanding that near the poles \( k = k_n \) the contribution of the singularity to the drag should be removed subjectively. With \( \bar{\omega} \) represented by Eq. (17) and \( \bar{u} \) computed from the continuity equation

\[
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{\omega}}{\partial z} - \bar{\omega} = 0 \quad \text{(35)}
\]

the convolution integral yields for the momentum flux

\[
\rho \int_{-\infty}^{\infty} \bar{u} \, \bar{\omega} \, dx = - \rho_0 \pi \cdot \int_{0}^{\infty} \frac{||A(k)||^2}{||f(0,k)||^2} \left( f_R \frac{df_R}{dz} - f_I \frac{df_I}{dz} \right) \frac{dk}{k} \quad \text{(36)}
\]

The upper limit of this integral could be made \( L \), because waves with \( k > L \) do not escape upwards. \( f_R(z,k) \) and \( f_I(z,k) \) are the real and imaginary parts of \( f(z,k) \). The expression \( (f_R \frac{df_R}{dz} - f_I \frac{df_I}{dz}) \) is constant with height for each real wave-number \( k \) separately, as can be proved easily by manipulating the real and imaginary parts of Eq. (7). The wave drag due to the continuous spectrum defined by Eq. (36) thus has the property of being constant with height. For each layer with \( f(z,k) = Af_1 + Bf_2 \) according to Eq. (12) we have

\[
f_R \frac{df_R}{dz} - f_I \frac{df_I}{dz} = \begin{cases} (||A||^2 - ||B||^2) \sqrt{l_0^2 - k^2}, & \text{if } k \leq l_0 \\ \frac{2}{2 \Re(A) \Im(B) - \Im(A) \Re(B)} \sqrt{k^2 - l_0^2}, & \text{if } k \geq l_0 \end{cases}
\]

and these expressions must come out to be the same for all layers.

The line spectrum ('lee waves') makes a quasi-periodic contribution to the momentum flux, as long as the resonance wavenumber \( k_n \) has a non-zero imaginary part, i.e. the lee wave is not completely trapped. The computation of this periodic momentum flux was demonstrated by Bretherton (1969), but its significance is not clear. The \( \bar{u} \bar{\omega} \) correlation from resonance waves should not yield a periodic contribution to the drag judging from Eq. (31). If the mountain is limited in extent, the contribution to the drag is zero beyond a certain \( x \) where \( dh/dx = 0 \), no matter how \( p \) fluctuates.

For each case three values of the 'drag' may be obtained:

(i) The pressure drag defined in Eq. (31). Pressure on the slope was computed utilizing the interpolation scheme, Eq. (30).

(ii) The momentum flux (Eq. (33)), computed numerically from the fields of \( \bar{\omega} \) and \( \bar{u} \).

(iii) The momentum flux from the convolution integral (36) as a check on (ii) above.

From the previous discussion it is clear that exact agreement between the three different methods should only be expected for the nonresonant case with either a complete nonlinear solution or with both dynamics and boundary condition linearized. Nevertheless, the three values are usually close. Discrepancies by a factor of 2 may occur. They show the effect of a strong resonance wave or the inconsistency of a 'mixed' model with linear dynamics and a nonlinear boundary condition.

Results of drag computations are presented as correlations \( \bar{u} \bar{\omega} \) in units \( m^2 \text{s}^{-2} \) for ease of visualization. The bar does not imply a periodic contribution. The notation only means that the total drag is

\[
D = - \rho_0 \bar{u} \bar{\omega} \cdot L \quad \text{(37)}
\]

* Berkshire and Warren (1970) discuss drag computations in the presence of Garnov poles, but they do not seem to suggest that these spikes should or could be eliminated.
where $L$ is the computational $x$-interval. $\rho_0$ equals 0.011 if pressure $p$ is in mb and velocities in m s$^{-1}$. Actually this drag is achieved by the obstacle, of course, and the total drag in the presence of many mountains would have to be computed by applying the drag $D$ to each mountain and adding up those values, ignoring the interaction between mountains. In this way a mean drag per unit $x$-distance (stress) can be defined.

---

**Figure 1.** (a) Observed flow pattern for 19 January 1967, flight number 4. From Vergeiner and Lilly (1970);
(b) Computed flow pattern for same date as Fig. 1 (a).

No blocking. $U$: cross-ridge (west-east) wind velocity in m s$^{-1}$
$N$: Brunt-Vaisala frequency of layers in s$^{-1}$ times 100.

* 100 km.
A most convenient measure of the drag is the drag coefficient, $C_D$. For a two-dimensional obstacle the definition is

$$\text{Force on a slab of width } \Delta y = D \cdot \Delta y = C_D \frac{1}{2} \bar{ho} \bar{u}^2 \cdot h \cdot \Delta y.$$  \hspace{1cm} (38)

where $\bar{\rho}$ and $\bar{u}$ are suitable mean values of density and wind speed and $h$ is the maximum height of the mountain.

---

**Figure 2.** (a) Observed flow pattern for indicated date; (b) Computed flow pattern for same date as Fig. 2 (a).

No blocking. $U$ and $N$ as in Fig. 1 (b).
8. Discussion of results

Computations were performed for a number of days in 1967, for which observations were available (see Figs. 1 to 4 and Table 1). The latter are taken from Vergeiner and Lilly (1970). A rather well-documented case from 1968 is reproduced from Lilly and Toutenhofd (1969) in Fig. 5 (a). Fig. 5 (b) shows the corresponding computation.

The agreement between observed and computed air flow is close, as it should be. This statement, however, needs some elaboration.

In 1967 only Denver soundings at 12Z, 18Z and 00Z were available, roughly 50 km downstream from the region of main interest. Interpolation between these times was performed to approximate the sounding at flight time. In addition, actual measured winds were substituted for interpolated ones whenever available. The amplitude of a linear solution is essentially proportional to the surface wind (see Eqs. (21) and (22)), a typically ill-defined quantity. Many times the response of the atmosphere at various wavelengths is sensitive to the wind structure near the surface.

The averaged topography near Boulder as shown in all figures consists roughly of a western plateau descending abruptly to the plains, plus a superimposed finite mountain. The semi-infinite plateau cannot be handled by linear theory, as it leads to infra-red catastrophe (see for example Bretherton 1969). Consequently only the 'linear' part can be used, very nearly a bell-shaped mountain with height \( H = 1.2 \) km and half width \( B = 5.4 \) km.* From the Fourier-transform of a bell-shaped mountain:

\[
F \left( H \frac{B^2}{B^2 + x^2} \right) = HB e^{-\kappa B}
\]

it is evident that a smooth mountain (large \( B \)) cuts down the amplitude of short waves very effectively. By comparison of observed with computed lee wave amplitudes, it was

![Figure 2](image-url)
found necessary to use a mountain considerably steeper ($B = 3$ km) than the averaged topography. This is not entirely unreasonable, because individual cross-sections are steeper than the average slope.

A further uncertainty arises with the possibility that a low-level inversion upstream

---

**Figure 3.** (a) Observed flow pattern for 28 January 1967, flight numbers 2 to 5. The time interval between successive flights is roughly 1½ hr. From Vergeiner and Lilly (1970); (b) Computed flow pattern for same date as Fig. 3 (a).

No blocking. Higher wind speeds used from balloon trajectories. $U$ and $N$ as in Fig. 1 (b).
(cold air in the valleys) may prevent the air flow from penetrating to the surface. The present model allows the effective base to be above the upstream valley floor. The corresponding lowest parts of the sounding and the mountain are cut off.

This last procedure is yet another test of the sensitivity of the computed resonance modes and flow field to slight changes in the upstream conditions. The crucial point to come out of this work is that a solution obtained with a given upstream sounding means nothing until its sensitivity to possible changes of wind, temperature, base height, etc. has been established. Pearce and White (1967) and Berkshire and Warren (1970) have recently explored the sensitivity of related models.

Examples of extremely sensitive solutions are shown in Figs. 7 and 8. These are pre-chinook soundings to be discussed later. Two solutions are presented for each date: one assuming the upstream flow to penetrate to the surface (740 mb), the other assuming that the lowest 600 meters are blocked (effective base at 686 mb, effective mountain height 600 m). Note the very strong waves which vanish completely in the partially blocked case. If the upstream conditions could be maintained exactly over a sufficiently long time, the indicated stationary solution would presumably be obtained. However, the stationary solution has apparently no meaning in this case, because the upstream flow is never free of fluctuations.

Less extreme, but still evident, is the sensitivity for other cases (Fig. 4, see also Table 1). Fig. 4 applies to a day with strong surface winds in the plains. For these cases computed long-wave resonance modes are typical, but their sensitivity suggests that flow patterns obtained from hydrostatic (hydraulic jump) theory would probably be more realistic. The observations show waviness with considerable time variation, as expected.

Finally, the most stable observed mountain waves (Figs. 1 (a) and 2(a), 3 (a) to a lesser extent) correspond to stable computed waves (Figs. 1(b), 2(b), 2(c) and 3(b)) in the sense that the resonance modes are quite insensitive to small changes in the upstream flow. These are the cases for which the model is most applicable. Generally speaking, the basic requirement for stable waves seems to be moderate winds across the ridge with a wind maximum somewhere at upper levels, but no extreme wind shears or odd profiles of any kind. The tropospheric stability may be concentrated in a thin layer or spread out.

Fig. 6(a) was selected as an example in which the linearized lower boundary condition (Eq. (22)) yields no sensible solution. Application of the nonlinear boundary condition (Eq. (24)) gives a very reasonable solution resembling sub-critical flow (Fig. 6(b)). In many cases the difference is far less striking. In the presence of a strong resonance wave the nonlinear boundary condition frequently yields too high amplitudes, which contradict the assumption of linear dynamics. In a number of computations no reasonable solution was obtained (i.e., Eq. (28) could not be satisfied) when applying the nonlinear lower boundary condition. This may well be a realistic result, indicating that the mountain is too high and/or the stability too high (surface wind too low) for a steady state to exist without a change of the basic flow upstream. Quantitative conclusions may, however, only be possible with a completely nonlinear model.

The case of 15 February 1968 (Fig. 5(a) and (b)) is rather unique because little wave activity occurred in the troposphere, whereas waves and turbulence were apparent in the stratosphere. The computed flow pattern is stable and agrees strikingly with the observations. In this case turbulence appears to be convincingly explained by a large amplitude standing wave pattern. More often, however, alternative explanations of CAT are likely to be true: critical layer absorption, Kelvin-Helmholtz instability (see Ludlam 1967 and Dutton and Panofsky 1970) or degradation of nonstationary wave packets by weak interactions (Bretherton, personal communication).

Table 1 summarizes the results of flow computations, including many runs for which the streamline pattern is not shown in a Figure. The emphasis is on demonstrating the degree of sensitivity of the computed solution to the indicated changes in upstream flow. The most frequently shown variation was to block the lowest 600 m of the upstream flow, as discussed previously (mountain height 600 m instead of the full 1,200 m). Many more runs were made, but only a representative sample is shown in Table 1.
9. Some remarks on the chinook winds and their relation to lee waves

'Chinook' in its original meaning is a synoptic condition of large-scale subsidence on the eastern slopes of the Rocky Mountains, with unseasonably warm temperature, low humidity and excellent visibility, not necessarily associated with gusty downslope winds.

Figure 4. (a) Observed flow pattern for indicated date. Strong surface winds in the plains; (b) Computed flow pattern for same date as Fig. 4 (a).

No blocking.  $U$ and $N$ as in Fig. 1 (b).
Sometimes the chinook is more an air mass than an orographic phenomenon, when a series of migratory lows alternately draws west-south-westerly winds northward and pushes cold Arctic air southward (Glenn 1961; Cook and Topil 1952; McClain 1952, 1959). The chinook is just one variety of foehn winds, whose climatological influence is felt in the lee of mountains everywhere, especially during the cold season.

Embedded in this synoptic situation are the occasionally severe 'chinook' winds in the Boulder area, a local phenomenon extending no more than about 20 miles into the plains. A recent description of these winds has been given by Kuettnner and Lilly (1968) and L. and P. Julian (1969). Other regions have their own local foehn wind effects (see Defant 1951), each controlled by a peculiar combination of synoptic and orographic influences. Following the European classification of foehn intensity (0 = no appreciable surface wind, 1 = light surface wind, 2 = strong, gusty surface wind, see Reiter 1958) the chinook winds of the Boulder area could probably be classified as 'Stage 2' in the evolution of various foehn phases.

The lee of the Rocky Mountains is a preferred cyclogenetic region. On a large scale there is a tendency to form a lee trough explained by conservation of vorticity. On a smaller scale the adiabatically-warmed downslope winds produce horizontal temperature gradients and pressure falls at the surface, as explained in Palmen and Newton (1969). Thus we have the somewhat confusing picture that at the outset a suitable surface pressure gradient must normally exist to force airflow across the mountain, whereas further pressure falls in the lee are a consequence of the downslope flow. Typically, but not always, the strong local chinook winds occur before the passage of an upper trough, in advance of a surface cold front moving in from the north-west.

The synoptic setting (upper winds, surface pressure gradient across the mountain) apparently does not determine the occurrence or absence of strong surface winds completely. For this reason a variety of mechanisms has been suggested which might, in

![Figure 4.](image-url)
**TABLE 1. Results of selected lee wave and drag computations**

When available, observed wavelengths and amplitudes (approximate mean values) are shown for comparison (from Vergeiner and Lilly 1970). The emphasis is on illustrating the degrees of sensitivity to changes in the upstream flow. The variation most frequently applied was to block off the lowest 600 m of the sounding, as explained in the text. Note that the mountain used for the computations was very nearly bell-shaped with a height of 1.2 km, similar to the average Rocky Mountain profile shown on the figures, but steeper with a half-width of 3 km instead of 5.4 km.

<table>
<thead>
<tr>
<th>Location, Date, Remarks</th>
<th>Observed</th>
<th>Computed</th>
<th>Resonance (km)</th>
<th>Wavenumber (km⁻¹)</th>
<th>Real part</th>
<th>Imaginary part</th>
<th>Mean wind speed u (m s⁻¹)</th>
<th>Pressure drag</th>
<th>Convolution integral</th>
<th>Drag coefficient a (m² s⁻²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Denver, 16 January 1967, 02Z Pre-chinook sounding</td>
<td>16.9</td>
<td>1.05</td>
<td>LT</td>
<td>0.44</td>
<td>1.26</td>
<td>0.11</td>
<td>15.6</td>
<td>-2.41</td>
<td>-1.51</td>
<td>1.67</td>
</tr>
<tr>
<td>See Fig. 7 (a)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Denver, 16 January 1967, 02Z Pre-chinook sounding</td>
<td>14.3</td>
<td>0.2</td>
<td>S</td>
<td>0.44</td>
<td>1.1</td>
<td>0.11</td>
<td>12.5</td>
<td>-0.65</td>
<td>-0.56</td>
<td>0.83</td>
</tr>
<tr>
<td>See Fig. 7 (b)</td>
<td>X</td>
<td></td>
<td></td>
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<td></td>
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<td></td>
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<tr>
<td>Denver, 19 January 1967, 21Z Wave flight. See Fig. 1 (b)</td>
<td>20.5</td>
<td>0.8</td>
<td>UT</td>
<td>0.31</td>
<td>0.10</td>
<td>0.00</td>
<td>15.9</td>
<td>-0.67</td>
<td>-0.56</td>
<td>0.83</td>
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<td>Denver, 19 January 1967, 21Z Wave flight</td>
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<td>MT</td>
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<td>0.00</td>
<td>14.7</td>
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<td>-0.53</td>
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<tr>
<td>Denver, 28 January 1967, 21Z Higher wind speeds used from observation. See Fig. 3 (b)</td>
<td>14.7</td>
<td>0.1</td>
<td>S</td>
<td>0.43</td>
<td>0.09</td>
<td>0.00</td>
<td>12.8</td>
<td>-0.51</td>
<td>-0.42</td>
<td>1.08</td>
</tr>
<tr>
<td>Denver, 28 January 1967, 21Z Higher wind speeds used from observation</td>
<td>15.0</td>
<td>0.05</td>
<td>LT</td>
<td>0.49</td>
<td>0.00</td>
<td>0.00</td>
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<tr>
<td>Denver, 29 January 1967, 00Z Linear lower boundary condition. See Fig. 6 (a)</td>
<td>5.0</td>
<td>0.15</td>
<td>S</td>
<td>0.43</td>
<td>0.08</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Denver, 29 January 1967, 00Z Nonlinear lower boundary condition. See Fig. 6 (b)</td>
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<td>-</td>
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<td></td>
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</tr>
<tr>
<td>Denver, 30 January 1967, 12Z Pre-chinook sounding</td>
<td>5.2</td>
<td>0.3</td>
<td>LT</td>
<td>1.21</td>
<td>0.00</td>
<td>0.00</td>
<td>0.9</td>
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<td>1.03</td>
</tr>
<tr>
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<td>MT, S</td>
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<td>MT, S</td>
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<td>Location</td>
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<td>X</td>
<td>long</td>
<td>0-70</td>
<td>53:1</td>
<td>1:3</td>
<td>MT</td>
<td>0:12</td>
<td>0:07</td>
<td>15:9</td>
</tr>
<tr>
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<td>Wave flight</td>
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<td>long</td>
<td>0-70</td>
<td>53:1</td>
<td>1:3</td>
<td>MT</td>
<td>0:12</td>
<td>0:07</td>
<td>15:9</td>
</tr>
<tr>
<td>Denver, 14 February 1967, 2230Z</td>
<td>Wave flight. See Fig. 4 (b)</td>
<td>34:4</td>
<td>1:2</td>
<td>MT, S</td>
<td>0:18</td>
<td>0:02</td>
<td>20:8</td>
<td>-2:18</td>
<td>-2:20</td>
<td>-2:38</td>
</tr>
<tr>
<td>Denver, 14 February 1967, 2230Z</td>
<td>Wave flight. See Fig. 4 (c)</td>
<td>41:7</td>
<td>0:6</td>
<td>MT, S</td>
<td>0:15</td>
<td>0:03</td>
<td>23:3</td>
<td>-0:32</td>
<td>-0:35</td>
<td>-0:36</td>
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<td>16</td>
<td>0-80</td>
<td>14:0</td>
<td>0:8</td>
<td>MT</td>
<td>0:45</td>
<td>0:00</td>
<td>11:3</td>
<td>-1:63</td>
</tr>
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<td>Wave flight. See Fig. 2 (c)</td>
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<td>0:5</td>
<td>MT</td>
<td>0:37</td>
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<td>13:4</td>
<td>-0:54</td>
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<td>Wave flight</td>
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<td>long</td>
<td>0-70</td>
<td>31:7</td>
<td>1:0</td>
<td>MT, S</td>
<td>0:20</td>
<td>0:05</td>
<td>22:9</td>
</tr>
<tr>
<td>Denver, 2 March 1967, 18Z</td>
<td>Wave flight</td>
<td>25</td>
<td>0-30</td>
<td>18:5</td>
<td>1:0</td>
<td>MT</td>
<td>0:34</td>
<td>0:02</td>
<td>9:9</td>
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<tr>
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<td>1:1</td>
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<td>Wave flight</td>
<td>X</td>
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<td>S, MT</td>
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<td>Wave flight</td>
<td>X</td>
<td>16</td>
<td>0-25</td>
<td>14:8</td>
<td>0:2</td>
<td>S, MT</td>
<td>0:42</td>
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<td>14:1</td>
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<tr>
<td>Denver, 7 January 1969, 22Z</td>
<td>Pre-chinook sounding</td>
<td>21:5</td>
<td>1:0</td>
<td>MT</td>
<td>0:29</td>
<td>0:01</td>
<td>15:1</td>
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<td>Pre-chinook sounding</td>
<td>X</td>
<td>28:3</td>
<td>1:2</td>
<td>MT</td>
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<td>0:01</td>
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<td>-1:21</td>
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<td>Pre-chinook sounding</td>
<td>X</td>
<td>21:0</td>
<td>2:6</td>
<td>T, S</td>
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<td>Pre-chinook sounding</td>
<td>X</td>
<td>15:7</td>
<td>0:4</td>
<td>S, T</td>
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<td>0:08</td>
<td>20:6</td>
<td>-0:26</td>
<td>-0:18</td>
</tr>
</tbody>
</table>

(1) If not marked: sounding starts at 740 mb, mountain height 1,200 m.
If marked X: sounding starts at 686 mb, mountain height 600 m (partial blocking, see text).
(2) Total displacement is twice the amplitude, or even more in the primary downslope. Resonance waves with negligible amplitudes are not shown.
(3) Maximum amplitude in troposphere (T), lower troposphere (LT), middle troposphere (MT), upper troposphere (UT), stratosphere (S).
(4) The imaginary part of the resonance wavenumber is very small for significant modes. Imaginary part equal to real part would correspond to downstream damping by a factor of $e^{x-p}$ = 1/535 per wavelength.
(5) All drag terms are expressed as $\bar{u}\bar{w}$ correlations, as explained in text. The total drag is $D = -p_0 \bar{u}\bar{w} L$, where $L$ is the computational x- interval (100 km) and $p_0$ = 0.011 mb s$^{-2}$. If we assume only one identical mountain every 100 km downstream, the mean stress would be $= -p_0 \bar{u}\bar{w}$ or $= 11$ dyn/cm$^2$ for $\bar{u}\bar{w} = -1$ m$^2$ s$^{-2}$. For definition of the three drag computation methods see text.
seemingly similar synoptic situations, sway the delicate balance of forces one or the other way. E. Reiter and some of his students have investigated selected cases of chinook (Beran 1967, Lovill 1969) and presented considerable synoptic evidence. Beran in particular

Figure 5. (a) Observed flow pattern for indicated date. From Lilly and Toutenhoofd (1969); (b) Computed flow pattern for same date as Fig. 5 (a). No blocking. $U$ and $N$ as in Fig. 1 (b).
attempts to demonstrate the validity of the upstream blocking criterion derived by Scorer and Klieforth (1959). The distinct preference of violent local chinook winds for the evening and night hours (Julian 1969) is still not fully explained. Possibly this time of day is a compromise: the layer up to mountaintop level is still more or less adiabatic—a necessary condition for penetration of streamlines to the surface on the eastern plains. At the same time, nightly cooling has started to form a thin layer of drainage flow, which

Figure 6. (a) Computed flow pattern for indicated date using linear lower boundary condition. Unreasonable solution; (b) Computed flow pattern for same date as in Fig. 6 (a) using nonlinear lower boundary condition (see text). No blocking. $U$ and $N$ as in Fig. 1 (b).
acts in the same sense as the general westerly flow. Stabilization of the flow near the slope would also encourage the penetration of chinook winds by preventing separation of the turbulent boundary layer at the crest if, indeed, this effect is significant (Soma 1969).

From the foregoing discussion it should be obvious that a simplified model such as the one presented in this paper can at best answer a very limited question: What, if any,

Figure 7. (a) Computed flow pattern for indicated date, corresponding to onset of strong chinook in Boulder. Sounding is interpolated between standard times. No blocking. U and N as in Fig. 1 (b); (b) Computed flow pattern for same date as Fig. 7 (a). Partial blocking assumed (see text). U and N as in Fig. 1 (b). Note extreme sensitivity of long-wave amplitude in comparison with Fig. 7 (a).
is the role of lee waves in explaining the chinook? It has frequently been implied in the literature that lee waves have a lot to do with the chinook (for example by mixing high-speed air from the upper troposphere downward) or even that the chinook is a manifestation of large-amplitude lee waves. We have not, unfortunately, been able to obtain upper-air data during a chinook, but the present model should be adequate to test

![Diagram](image)

Figure 8. (a) Computed flow pattern for indicated date, corresponding to onset of light chinook in Boulder. No blocking. $U$ and $N$ as in Fig. 1 (b); (b) Computed flow pattern for same date as Fig. 8 (a). Partial blocking assumed (see text). $U$ and $N$ as in Fig. 1 (b). Note extreme sensitivity of long wave in comparison with Fig. 8 (a).
whether resonance waves are important to the problem. Figs. 7 and 8 as well as Table 1 illustrate that pre-chinook soundings (Denver soundings, interpolated in time) produce either very long or extremely sensitive wave modes, or no response at all, certainly not the classical mountain waves. This result does not contradict visual evidence that clouds during a chinook (if there are any) resemble a single hydraulic jump rather than periodic waves. Moreover, the surface wind was never strong on days with observed strong lee waves (see Figs. 1 (a), 2 (a) and 3 (a)).

Since short resonance modes (wavelength 15 km or smaller) do not seem to be relevant for explaining the chinook, it would probably be advantageous to exclude them altogether by using a hydrostatic model. Consequently, there is a good chance that models similar to recent hydraulic jump models (Houghton and Kasahara 1968; Houghton and Isaacson 1970), but with more realistic vertical structure, will explain the details of the chinook. It should be remembered, however, that the action of mesoscale synoptic systems, which are not resolvable by the present radiosonde network, cannot be ruled out. Some possibilities are mesoscale advection of cold air aloft as a destabilizing mechanism (Lovill 1969, p. 49) and active drainage of cold surface air towards the east by mesoscale pressure waves.

10. CONCLUSIONS

The model presented in this paper yields very close agreement with observed stable mountain waves, for which it is most applicable. When the observed waves are not so stationary, the computed flow field is sensitive to small changes of the upstream sounding. It is shown that the flow associated with downslope ('chinook') windstorms in Boulder is probably nearly hydrostatic, with computed resonance waves either absent or extremely sensitive.

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AN OPERATIONAL LINEAR LEE WAVE MODEL


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APPENDIX

Let

\[ I (a) \equiv \int_0^\infty e^{-ax^2} \frac{dx}{x^8 + 1} \cdot \]

and

\[ \frac{d^2 I}{da^2} \equiv \int_0^\infty e^{-ax^2} x^4 \frac{dx}{x^8 + 1} \cdot \]

\[ I (a) \text{ can easily be shown to satisfy the differential equation :} \]

\[ \frac{d^{(4)} I}{da^4} + I = \frac{1}{2} \cdot \sqrt{\frac{\pi}{a}} \cdot \]

By standard methods the solution may be written in terms of complementary error functions

\[ j = 1 \quad j = 2 \quad j = 3 \quad j = 4 \]

\[ I (a) = \frac{\pi}{8} \sum_{j=1}^{4} e^{a_{j}} \cdot \text{erfc} (\sqrt{a_{j}}) \cdot \left[ e^{t_{j}^{\frac{a}{8}}} \cdot i e^{-t_{j}^{\frac{a}{8}}} \cdot -i e^{t_{j}^{\frac{a}{8}}} \cdot e^{-t_{j}^{\frac{a}{8}}} \right] \]

and

\[ \frac{d^2 I}{da^2} = \frac{\pi}{8} \sum_{j=1}^{4} e^{a_{j}} \cdot \text{erfc} (\sqrt{a_{j}}) \cdot \left[ i e^{t_{j}^{\frac{a}{8}}} \cdot e^{-t_{j}^{\frac{a}{8}}} \cdot e^{t_{j}^{\frac{a}{8}}} \cdot -i e^{-t_{j}^{\frac{a}{8}}} \right] \]

where

\[ a_{j} (j = 1, 2, 3, 4) = \left( e^{t_{j}^{\frac{a}{8}}} \cdot -e^{-t_{j}^{\frac{a}{8}}} \cdot -e^{t_{j}^{\frac{a}{8}}} \cdot e^{-t_{j}^{\frac{a}{8}}} \right) \]

and

\[ \sqrt{a_{j}} = \left( e^{t_{j}^{\frac{a}{8}}} \cdot i e^{-t_{j}^{\frac{a}{8}}} \cdot -i e^{t_{j}^{\frac{a}{8}}} \cdot e^{-t_{j}^{\frac{a}{8}}} \right) \].
The handbook of mathematical functions (Abramowitz and Stegun 1964 pp. 297 and 299), contains definitions and approximations for the error function. For $|a| > 20$ the asymptotic expansions

$$\frac{dl}{da} = \frac{\sqrt{\pi}}{4} a^{-3/2}$$

$$\frac{d^2 I}{da^2} = \frac{3 \cdot \sqrt{\pi}}{8} a^{-3/2}$$

were considered adequate.