Atmospheric frontogenesis models: some solutions

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(Manuscript received 28 September 1970)

SUMMARY

In this paper are presented some solutions of analytic models in which large-scale velocity fields act on initially large-scale distributions of temperature. Phenomena very similar to atmospheric fronts both surface and upper tropospheric are described. At the surface front there is a tendency to form discontinuities in velocity and temperature in a finite time. The upper tropospheric front and its jet stream are associated with the descent of a tongue of stratospheric air. The main assumption underlying the models is the justifiable one of geostrophic balance across but not along the front.

1. Introduction

Observational studies of fronts have shown (e.g. Reed and Sanders 1953) that there are two basic types. Surface fronts may have extremely intense gradients of physical quantities near the Earth's surface. Upper and mid-tropospheric fronts, as the names imply, are not surface phenomena. They have large gradients which have been shown to be connected with the subsidence of mid and upper tropospheric and stratospheric air. Associated with them are the high velocity winds of the jet stream core.

Numerical work by Edelmann (1963), Williams (1967) and others has shown that very large gradients almost indistinguishable from discontinuities can develop within solutions of the so-called primitive equations. This is true when the complicating effects of friction, latent heat, orography and variation of the Coriolis parameter with latitude are all absent.

In this paper we attempt to clarify the essential dynamics of frontogenesis according to the primitive equations by considering certain simplified wind and temperature fields. We exhibit the process of front formation in analytical models in which geostrophic velocity fields characteristic of the large-scale weather systems act on large-scale distributions of temperature. These fronts are shown to have many of the features which have been observed in the atmosphere. Here the emphasis is on results and discussion of these. The mathematical basis of these models is sketched only very briefly. It will be given in detail in Hoskins and Bretherton (1971) (later referred to as 'HB').

2. Co-ordinates and equations

We use the equations of motion with the traditional quasistatic approximations of ignoring the vertical acceleration and the horizontal component of the Earth's rotation. Thus hydrostatic balance in the vertical is assumed. We use as vertical co-ordinate a known function of pressure ($p$):
\[ z = \frac{H_s}{\kappa} \left[ 1 - \left( \frac{p}{p_{00}} \right)^\kappa \right] \]  \hspace{1cm} (1)

where \( H_s \) = a scale height = \( R\theta_a / g \), say,
\( \kappa = (\gamma - 1) / \gamma \)
\( p_{00} \) = standard pressure = 1,000 mb, say.

For an adiabatic atmosphere of potential temperature \( \theta_a \), \( z \) is equal to the physical height. This is always true to a good approximation below 600 mb. For the ICAO standard atmosphere the connection between \( z \), pressure and height is shown in Fig. 1.

\[ \text{Figure 1. The relation between physical height (h) and } z \text{ for the ICAO standard atmosphere and for isothermal and adiabatic atmospheres. Pressure is also marked on the } z \text{ axis (every 100 mb below 300 mb and 50 mb between 300 mb and 100 mb).} \]

The boundary condition at the top of the atmosphere (\( p = 0 \), \( z = H_s / \kappa \sim 28 \text{ km} \)) is \( w = Dz / Dt = 0 \), and we shall assume that this condition holds also at the surface \( p = p_{00}, z = 0 \). The latter is a commonly used approximation to the true condition (in the absence of orography) of zero vertical velocity at a geopotential surface. In our calculations the upper boundary condition \( w = 0 \) will be imposed at an artificial lid \( z = H \). Mostly, the Boussinesq approximation is made. This is valid if material particles do not move a vertical distance comparable with \( H_s \). The lid and this approximation are not really fundamental to the dynamics of the models, and are invoked primarily for computational convenience. In one case the induced errors will be shown to be small.

We shall consider situations in which velocity and temperature fields develop large gradients in one horizontal direction but are otherwise everywhere consistent with quasi-geostrophic theory. The \( x \) axis is chosen to be in this direction of developing large gradients. Observation and scale analysis suggest that accelerations in the \( Ox \) direction may be neglected compared to the corresponding Coriolis and pressure forces, but the \( Oy \) accelerations along the incipient front play a crucial role. Thus geostrophic balance across but not along the front is retained. This approximation is the major factor that enables us to
make progress with the study of our models. It is not as restrictive as those underlying the quasi-geostrophic theory used in the study of deformation models by Stone (1966), Williams and Plotkin (1968), and Williams (1968). Unlike the quasi-geostrophic assumption, it should be valid even in the presence of strong frontal gradients. The consistency of any solution obtained using it may be verified a posteriori by comparing the neglected component of acceleration with the same component of the Coriolis force. The situations described below are, indeed, consistent.

The variation of the Coriolis parameter with latitude is neglected since on the length scales characteristic of fronts its effect must be small.

With these qualifications, the full primitive equations for an inviscid, adiabatic, compressible fluid above a plane surface are used. Subsequently, a brief qualitative discussion of the effects of the release of latent heat of condensation and of surface friction will be given.

3. A horizontal deformation model

(a) The general model

A horizontal deformation field with stretching in one horizontal dimension balanced by contraction in another is the classical frontogenetic mechanism postulated by Bergeron (1928). It has been investigated using quasi-geostrophic theory by Stone (1966), Williams and Plotkin (1968), and Williams (1968). Taking Cartesian co-ordinates with appropriate orientation, the simplest deformation field is

\[ u = -\alpha x, \quad v = \alpha y. \]  

(2)

Ignoring other induced velocities, this velocity field will tend to concentrate the distribution of potential temperature \( \theta \) in the direction of the axis of contraction (x-axis) and weaken those in the direction of the axis of dilatation (y-axis) — see Fig. 2. The most efficient production of potential temperature gradient should be when the potential temperature does not vary along the dilatation axis (\( \partial \theta / \partial y = 0 \)). For simplicity we consider a model in which this is true for all time. Other components of velocity besides the deformation field of Eq. (2) arise during the frontogenesis but, consistently, these will also be independent of \( y \).

![Figure 2](image.png)

Figure 2. The simplest deformation field and a potential temperature distribution (contours indicated by broken lines). The deformation field tends to strengthen x gradients and weaken y gradients of potential temperature.
We pose an initial state $\theta_0(x, z)$ in which the temperature distribution is so wide-spread in the horizontal that the thermal winds $\partial u/\partial z$ associated with the gradients $\partial \theta_0/\partial x$ are so small as to be negligible, and Eq. (2) describes the complete velocity distribution. As the horizontal temperature contrasts are compressed, thermal winds $\partial u/\partial z$ develop, together with vertical velocities $w$ and significant ageostrophic components of $u$. These modify the thermal pattern. However, at large distances from the incipient front ($x \to -\infty$) $\partial \theta/\partial x$ remains small and Eq. (2) continues to hold. It should be noted that, to be dynamically consistent, the deformation field must extend though the whole depth of the fluid. The deformation rate may vary in time provided it is uniform in space and is small compared to the Coriolis parameter $f$.

This problem is dynamically complex and extremely non-linear. However, it will be shown in HB that via a Lagrangian approach and the introduction of a new independent variable in the $x$ direction the mathematical problem may be greatly simplified. In particular, the velocity $u$ and the potential temperature $\theta$ at any time $T$ may be determined from the initial distribution of $\theta$ without calculating them for all intermediate times, or solving for $u$ and $v$. Only the total geostrophic deformation $\exp \int \alpha(t) dt$ need be specified at each stage, not the rate $\alpha$. Particle motions in the $x$, $z$ plane normal to the axis of the front may be determined either from a circulation equation as done by Sawyer (1956) and Eliassen (1962), or by identifying fluid particles in the solution for two times a short interval apart.

(b) Uniform potential vorticity under the Boussinesq approximation

In this section attention is restricted to a model in which initially the apparent static stability $C = \frac{\partial \theta}{\partial z}$ and Brunt-Vaisala frequency $N = \left( \frac{g}{\theta_0} \frac{\partial \theta}{\partial z} \right)^{1/2}$ are uniform, and the fluid is bounded above by a lid at $z = H$. Making the Boussinesq approximation, there is an exact analytic solution for $u$ and $\theta$ at any time for any initial temperature distribution of the form $\theta_0(x, z) = Cz + F(x)$. We describe solutions for particular $C$ and $F(x)$, chosen to be representative of atmospheric conditions, using $H = 8 \text{ km} \sim 300 \text{ mb}$, $f = 10^{-4} \text{ s}^{-1}$, $g/\theta_0 = 1/30 \text{ m s}^{-2} \text{ (K)}^{-1}$, $C = 3 \text{ K km}^{-1}$ ($N = 10^{-2} \text{ s}^{-1}$).

We first consider a typical initial temperature distribution

$$F(x) = \Delta \theta \frac{2}{\pi} \tan^{-1} \frac{x}{L_o}$$

corresponding to a state of cold air at $x = -\infty$ and warm air at $x = \infty$ with a gradual change in temperature between the two. The scale $L_o$ is arbitrary, provided it is large compared to the Rossby radius of deformation $NH/f \sim 800 \text{ km}$. The horizontal contrast in potential temperature $(2\Delta \theta)$ is taken to be equal to the initial difference $(24 \text{ K})$ between the bounding surfaces $z = 0$ and $z = H$. The solution for this model always has odd symmetry about the planes $x = 0$, $z = H/2$. The upper boundary is not realistic and we are more interested in the picture near the 'surface'. Therefore we only display the solution below $z = H/2$ and, for ease of interpretation, $z$ may be considered as vertical height.

If there were no induced ageostrophic velocities, the geostrophic deformation would reduce the length scale such that at a later time the temperature distribution would be

$$\theta(x, z, t) = Cz + (2\Delta \theta/\pi) \tan^{-1} x/L,$$

where $L = L_o \exp \left( -\int \alpha dt \right)$. This 'geostrophic length scale' provides a convenient measure of the total geostrophic deformation due to the imposed velocity field. In fact, considerable distortion occurs. When $L$ is $148 \text{ km}$, contours of equal potential temperature $\theta$ and velocity $u$ are shown in Fig. 3. Isolines of these quantities are drawn every $2.4 \text{ K}$ and
4 m s\(^{-1}\) respectively. A horizontal length 200 km is shown to give the scale. The centre of the geostrophic contraction and position of initial maximum temperature gradient, \(x = 0\), is also marked. It is seen that large gradients in both \(v\) and \(\theta\) have formed. These gradients are a maximum at the ‘ground’ approximately 400 km on the warm side of \(x = 0\). At each level, gradients are a maximum in a region whose slope is approximately one in one hundred. These features are clearly very similar to those of surface fronts observed by e.g. Sanders (1955). He describes a sloping frontal region with intense gradients at the surface but a rapid decrease in gradients away from the surface. This is in opposition to the classical picture of a discontinuity line extending from the surface through the atmosphere. The slope of our frontal region is in the middle of the range of frontal slopes observed. The frontal region has warmer air above than below. It has formed in the warmer air, and on the warm side there is very little baroclinicity. However, on the cold side there is some temperature gradient. The tendency for this to be true has been observed. The static stability near the surface on the warm side of the front is small and this could well correspond with the lack of stability denoted by a squall line which sometimes precedes a cold front. The maximum stability in the front is six times the basic stability, and the maximum temperature gradient is equivalent to 1°C in 7 kilometers.

The velocities are in the negative direction, with a maximum \(\sim 38\) m s\(^{-1}\) at the surface. At the front, the horizontal and vertical shears are large and positive. The maximum vertical component of absolute vorticity occurs at the surface point marked by an arrow and is approximately 4.9f. The maximum vertical wind shear is equivalent to 5 m s\(^{-1}\) in 100 meters. The minimum vertical component of absolute vorticity is 0.56f.

Particle motions from the previous time \(L = 200\) km to \(L = 148\) km are exhibited in Fig. 4 by vectors superimposed on the potential temperature distribution at \(L = 148\) km. For comparison the basic geostrophic deformation is shown below the lower surface. There is intense convergence at the surface front and consequent strong upward motion above it. In the warmer air there is a tendency to up-glide along surfaces of constant potential temperature, while the cold air tends to subside, particularly a few kilometers from the surface. The strength of the convergence is responsible for the large growth in vorticity there, and the whole picture of convergence and strong upward motion is as was observed.
by Sanders (1955). The up-gliding of warm air has been a noted feature of fronts right from the first observational studies and models of Bjerknæs (1919). We may expect that this rising air will become saturated and lead to the formation of cloud and rain characteristic of fronts. Downward motion of the cold air which warms the air and makes it appear dry has also frequently been commented on e.g. Miles (1962).

It is found that the maximum vorticity at the surface occurs at the same fluid particle in the two cases and that in our frame of reference this particle has barely moved in physical space. This is consistent with the particle motions in Fig. 4. Thus if the basic geostrophic contraction occurs in a wind blowing from warm to cold air such that the front is a warm front, its motion is subgeostrophic. That this is observed to be true has been mentioned by Bjerknæs (1951). Vice versa, if the front is a cold front we predict that its motion is supergeostrophic.

To find characteristic velocities and times we need to specify a rate of contraction. Setting $\alpha = 10^{-3} \text{ s}^{-1}$, the time required to reduce $L$ from 200 km to 148 km is approximately 8.4 hours. Thus, despite the small geostrophic deformation rate chosen, the vertical component of relative vorticity is tripled from $1.3f$ to $3.9f$ in only 8.4 hours. The largest geostrophic velocity drawn under the surface is, as indicated, equivalent to an average speed of 5.2 m s$^{-1}$. The maximum upward motions are approximately 1.1 cm s$^{-1}$, or 5 mb hr$^{-1}$.

It should be emphasized that the solutions described depend only on the total geostrophic deformation at that instant, and do not require $\alpha$ to be constant. In fact the model could be applied to the frontolytic situation in which the large-scale geostrophic motion is such as to weaken a front. Then $\alpha < 0$ and the particle motions would be in the reverse directions of the arrows in Fig. 4, with descending motion in the warmer air and ascending motion in the colder air. Also, the motion of a warm front would be supergeostrophic and that of a cold front subgeostrophic. The independence of the instantaneous value of $\alpha$, to some extent, indicates a deficiency of the model due to dropping the acceleration term in the $x$ equation of motion.

In our solution with $L = 148$ km, the minimum Richardson number is 21 and the contours of Richardson number 5 and 1.0 are drawn in Fig. 3. The region of largest gradients is also one of lowest Richardson numbers. In the atmosphere, in a region like this, there would be a strong possibility of Kelvin-Helmholtz instability and consequent mixing, thus inhibiting the formation of stronger gradients there.

The non-linear process involved in the frontogenetic mechanism is so dominant, that, given the same small geostrophic deformation rate, it is found that the model predicts infinite vorticity and hence frontal discontinuities in a further five hours. Clearly, as a model of the atmosphere, there must be a breakdown in the model before this time. However, the tendency to ‘collapse’ in a finite time should not be relevant to the atmosphere and the extremely intense surface fronts sometimes observed.

Our next model is the rather less important case of an even initial temperature
distribution corresponding to air at \( x = \infty, x = -\infty \) at the same temperature but a mass of air of a different temperature at the origin and a very gradual transition to the temperature at infinity. We choose an initial potential temperature distribution:

\[
F(x) = \Delta \theta/[1 + x^2/L^2]
\]

and as before take a temperature contrast of \( \Delta \theta = 12^\circ \text{K} \) between the origin and infinity. For this model there is no simple symmetry about \( z = H/2 \). The distribution of \( \nu \) and \( \theta \) when the geostrophic length scale \( L \) has been reduced to 416 km are displayed in Fig. 5. The velocity contours show that there is large positive relative vorticity at the surface at \( x = 0 \). Here the vertical component of absolute vorticity \( \zeta \sim 48f \). The minimum velocity is approximately \(-15 \text{ m s}^{-1}\) at the cross on \( x \) negative and the maximum \(+15 \text{ m s}^{-1}\) at the cross on \( x \) positive. At the lid, the negative relative vorticity at \( x = 0 \) has not increased in the same manner and there \( \zeta \sim 56f \). There are maxima and minima in velocity at the crosses on \( x \) negative and positive respectively, each with magnitude \( 15 \text{ m s}^{-1} \).

The potential temperature contours show that at the surface there are large gradients on either side of \( x = 0 \). The warm air appears to have been squeezed upwards by the cold air which has moved in at greater than the geostrophic velocity from either side. The solution appears to be analogous to an occluded front as it is observed in the atmosphere. The static stability at \( x = 0 \) on the surface is \(-21 \) of the basic value which is indicative of the upward motion near there. There is clearly a similarity between the frontal situation in the region \( x > 0 \), and in the region \( x < 0 \), and that produced in the previous odd temperature model. At the lid it is noticeable that there has been no equivalent development, and indeed the cold air has moved in with less than the geostrophic velocity.

Solving the model with smaller \( L \) again indicates that there is a tendency to form infinite vorticity and associated discontinuities in velocity and temperature at the ground within five hours.

A model with a cold mass of air at the origin may be obtained by using

\[
F(x) = -\Delta \theta/[1 + x^2/L^2].
\]

The solution when \( L \) is 416 km is Fig. 5 inverted with the sign of \( \nu \) changed and \( \theta \) replaced by \( 36^\circ \text{K} - \theta \). The cold air at the surface contracts at less than the geostrophic deformation rate and that at the lid at a larger rate. There is a tendency to produce discontinuities at the lid in a finite time.

![Figure 5](image-url)  
**Figure 5.** The uniform potential vorticity model with warm air at the origin. Continuous lines are isolines of potential temperature drawn every 2.4 K (1 in nondimensional units). Broken lines are isolines of long front velocity drawn every 4 m s⁻¹ (0.5 in nondimensional units).
(c) Modifications to the tan$^{-1}$ model with $L = 148$ km

(i) When the Boussinesq approximation is not made, the instantaneous solution for $v$ and $\theta$ may be found numerically in a straightforward manner. It is found that frontogenesis proceeds rather more slowly at the ground, but more quickly at the lid. If this non-Boussinesq problem is solved for smaller $L$, it is found that gradients increase considerably both at the surface and at the lid. There is the same tendency to 'collapse' as before, however, this process is more advanced at the lid than at the surface.

(ii) Smooth changes in potential vorticity may be simply included in the numerical problem. In particular, a solution has been obtained when $N^2$ increases by a factor of 6 from the surface to the lid. In this model, the effect of the lid on the dynamics near the ground is of very secondary importance. As might be expected, the frontogenesis at the ground proceeds a little more slowly than before but that at the lid is retarded greatly.

(iii) The effects of moisture and latent heat of condensation may be incorporated crudely into this model by including a small diabatic heating where the greatest upward vertical displacements have occurred. The potential vorticity then changes following a fluid particle. This change can be estimated assuming that the solution is little altered. Using the new distribution of potential vorticity, the problem may be solved numerically. This is not a rigorous calculation but should give the correct qualitative result. The surface frontogenesis is found to proceed more quickly than before. Clearly this is because of the increased upward motion associated with the release of latent heat.

(iv) Representing the effect of surface friction by a simple Ekman layer suction at $z = 0$, from the solutions at $L = 200$ km and $L = 148$ km, we may estimate the upward displacement of a fluid particle at the top of the surface friction layer. If a small correction for this is incorporated into the numerical problem, it is found that surface frontogenesis is somewhat inhibited.

It has often been considered that surface friction helps the frontogenetic process and, indeed, that it is a basic part of the process Eliassen (1959). Observations of fronts indicate that the frontal cloud and rain increases when a front passes from sea to land with the associated strong increase in surface friction. Clearly, the Ekman layer suction produces increased vertical motion and this leads to increased cloud. However, above the Ekman layer there must be a horizontal divergence to balance the convergence in the layer. This divergence tends to weaken frontal gradients.

The inclusion of latent heat release in the increased vertical motion must counteract a proportion of the frontoltyic property of the Ekman divergence. The precise effect of including both phenomena is unclear, and further study is needed. However, one central thesis of the present paper is that neither is necessary for the formation of apparently realistic surface fronts. The dynamics included in the inviscid, adiabatic primitive equations will under certain circumstances give rise to near discontinuities after only a finite development time.

(d) A troposphere, stratosphere model

We now consider a deformation field independent of height acting on a fluid with regions of high and low potential vorticity separated by a surface of discontinuity. This is a model of the stratosphere separated by the tropopause. We take the 'pseudo Brunt-Väisälä frequency' $N$ to be uniform in both regions, but differing by a large factor, and make the Boussinesq approximation. We impose a lid at 135 mb and choose the initial height of the tropopause in such a manner that the potential temperature on the lid is uniform. The details of the mathematical problem and its numerical solution will be given in HB. However, we remark here that at any time the position of the tropopause is unknown but that the potential temperature is known on fluid particles on this internal boundary and velocity is continuous across it.

Before introducing our solutions, we exhibit in Fig. 6 two results of the observational studies by Reed (1955) and Reed and Daniels (1959). The vertical co-ordinates as used in these two studies are, respectively, approximately and exactly equivalent to our $z$. Reed's
picture is the middle one of three which he gives showing various stages of the formation of a very strong upper air front. The final diagram he shows is very similar to the composite cross-section of five fronts given by Danielsen and him. These studies provide the motivation for our experiments. In particular, they inspire our choice of initial conditions.

In Experiment 1 we choose an inverse tangent form for the initial surface potential temperature:

$$g_1 = (2\Delta \theta/\pi) \tan^{-1} x/L_0$$

with a total contrast of $2\Delta \theta = 39^\circ K$. In the lower region $N = 10^{-2} \text{s}^{-1}$, and in the upper region $N = 3.10^{-2} \text{s}^{-1}$. The compensating tropopause has an inverse tangent profile in our co-ordinate system —  from 290 mb on the cold side to 220 mb on the warm. We seek the distributions of potential temperature and long front velocity when the geostrophic length scale has contracted to 262 km. In Fig. 7 we show the tropopause position and surface temperature distribution that would be obtained if there was no induced ageostrophic motion.

Figure 7. The geostrophic contractions of the surface temperature distribution ($\theta_1$) and tropopause height ($h$).
The solution is shown in Fig. 8. At the surface, frontogenesis is occurring in a manner extremely similar to that in the one fluid deformation model. The maximum vertical component of absolute vorticity ($\zeta$) in the lowest grid is $2.6f$. The ageostrophic motions have deformed the tropopause so that its appearance is more that of a step. The maximum subsidence of the tropopause from its basic state is 40 mb. A realistic jet stream pattern has formed with maximum velocity $36 \text{ m s}^{-1}$ on the tropopause. The maximum $\zeta$ on the stratospheric side of the tropopause $\sim \frac{2}{3}f$, and on the tropospheric side $\sim 1.6f$. As there is zero temperature gradient on the lid, there is no tendency to front formation there.

It is clearly of interest to study whether the situation at the tropopause becomes more like an upper tropospheric front as the geostrophic contraction continues. Experiment 2 is identical to Experiment 1 except that we choose $L = 170$ km. The geostrophic contraction of the tropopause and of the surface temperature distribution is shown in Fig. 7.

The solution of this model is exhibited in Fig. 9. The surface front has developed and,
according to this model there is infinite vorticity at the surface. This is clearly not realistic. In the atmosphere, mixing would have set in and stopped gradients becoming too sharp. However, our interest in this Section is in the motion away from the surface. The step-like character of the tropopause has been accentuated. It has become almost vertical at one point and there is the beginning of the development of a tongue of stratospheric air. The tropopause at this point has descended to 335 mb, with a maximum subsidence from the basic state \( \sim 65 \) mb. The jet core velocity now \( \sim 41 \) m s\(^{-1}\). Gradients in \( v \) and \( \theta \) near the tropopause have increased. The maximum \( \xi \) on the stratospheric side of the tropopause is now 2.8f, and on the tropospheric side 2.2f.

An extra feature in Fig. 9 is the particle motions from \( L = 262 \) km. For comparison, the geostrophic deformation field is shown below the surface. In the stratosphere, motion is generally along potential temperature surfaces. However, below the jet core and on the cold side there is downward motion, tending to push the tongue of stratospheric air down into the troposphere. In the troposphere, there is rising motion in the air on the warm side of the jet core. There is strong descending motion below the tongue. This motion is mostly in a surface of constant potential temperature. Taking the same small value of the geostrophic deformation rate as was used before, \( \alpha = 10^{-5} \) s\(^{-1}\), this descending is equivalent to an average of \( \sim 8 \) mb hr\(^{-1}\). We shall compare these motions with observations below.

Experiment 3 is a re-run of Experiment 1 using a larger contrast in potential vorticity. In the upper region we choose \( N = 4 \times 10^{-5} \) s\(^{-1}\). The initial contrast in tropopause height is now approximately 40 mb. The solution is similar to that of Experiment 1. The largest \( \xi \) at the ground \( \sim 2.3f \), compared with \( 2.6f \) in 1. The tropopause moves less, a maximum subsidence \( \sim 27 \) mb. However, more vorticity is generated there: on the stratospheric side \( \xi_{\text{max}} \sim 2.3f \) and on the tropospheric side \( 1.8f \). The jet core speed takes the larger value \( \sim 42 \) m s\(^{-1}\).

The particle motions exhibited above do not show the same strong ageostrophic convergence as did those at the surface front (Fig. 4), and the above experiments do not suggest a tendency to form infinite vorticity at the tropopause. It will be shown in \( HB \) that this tendency is possible only if there is on the tropopause a maximum or minimum in potential temperature. A minimum on the ‘cold tropopause’ is the most attractive choice. A feature of the synoptic situation before the development of the phenomenon described by Reed (1955) was a mass of extremely cold air on the cold side of the temperature transition region in the upper troposphere. A potential temperature minimum on the tropopause towards the cold side of the tongue is evident in his diagrams. This is a necessity if the air on the cold side of the tongue is to have very small baroclinicity as is usually observed to be true.

In Experiment 4 we specify an initial surface temperature distribution and compensating tropopause height whose geostrophic contraction is shown in Fig. 7. The potential temperature contrast \( \sim 49^\circ \)K and there is a weak minimum on the cold side of the transition zone. As in Experiments 1 and 2, we use a potential vorticity contrast of 9 to 1.

The solution is shown in Fig. 10. The surface front has developed so that in the lowest grid \( \xi_{\text{max}} \sim 3f \). The tropopause has started to fold and a tongue of stratospheric air has descended to 410 mb. The maximum subsidence of the tropopause from its initial position is 118 mb. Heightening the contrast, the warm tropopause has risen a little — maximum 15 mb. The minimum of potential temperature is almost at the lowest point of the tongue. The tropopause maxima of \( \xi \) are 3.4f and 2.5f, on the stratospheric and tropospheric sides respectively. An aircraft flying through the tongue of stratospheric air would observe strong shears and large temperature gradients characteristic of fronts. The jet core velocity is 45 m s\(^{-1}\). In the mid-troposphere, the colder air has little baroclinicity. There is what may be termed a weak frontal region extending from the stratospheric tongue. The warmer air has some baroclinicity.

Other experiments with minima in potential temperature on the tropopause have shown that a minimum in tropopause height also tends to occur there. As shown in \( HB \) this indicates no tendency to form infinite vorticity in upper air fronts.
(e) Discussion and comparison with observation

Reed and Danielsen (1959) remarked on the discontinuity in potential vorticity between the stratospheric air and the tropospheric air. The ratio of 9 to 1 which we have used was suggested by their data and that of Reed (1955). The large temperature contrast we have used was also suggested by their data. We have assumed a geostrophic deformation independent of height between 1,000 mb and 135 mb. Clearly this will never occur, but the entrance region of jet streams on the west side of upper air troughs is a region of geostrophic convergence over a depth including the tropopause. If there is no surface convergence we may expect no front there, but we expect that the dynamics at the tropopause will be little altered. The curvature of jet streams is not included in our model. However, our aim here is to show how most of the observed characteristics of upper air fronts and their associated jet streams may be produced when a large-scale geostrophic deformation field acts on a two potential vorticity system.

It is easily seen that our Experiment 4, in particular, has produced a solution very similar to Reed’s observation of a developing front. We now show that most of the specific features pointed out by Reed and Danielsen have their counterpart in our solution. The potential temperature lines show a warm hollow in the stratosphere and a cold peak along the tropopause boundary. The hollow and peak converge downwards to a ‘triple point’ at the tropopause where it is vertical. They also commented on the barotropy of the cold air, strong gradients in the frontal zone, and a moderate degree of baroclinicity in the warm air. The jet core is located at 270 mb above the intersection of the frontal region with the 500 mb level, in agreement also with Palmén and Newton (1948). The jet core tends to follow the tropopause, but the peak wind in our solutions is only 45 m s⁻¹, compared with observed values of ~75 m s⁻¹. This may be partly explained by our imposing zero temperature gradient at 135 mb. In observational studies it may be noticed that the reversed temperature gradient (\(\partial \theta / \partial x < 0\)) is not negligible there. The thermal wind equation then suggests that we are underestimating the jet speed. The Boussinesq approximation we have made implies that we have underestimated the vorticity production at the tropopause and thus also the jet core speed.

The folding and descent of stratospheric air we have produced are similar to the developing front of Reed and are probably quite typical of what occurs. However, we have failed to produce the final knife down to 700 mb shown by Reed in his exceptional case. Further geostrophic contraction, ignoring the collapse of the surface front, will clearly give further subsidence. It should be remarked that in the numerical calculation, the tropopause is fixed at its end points at finite x. This may slightly inhibit motion of the tropopause. We have not so far commented on the Richardson numbers produced away from the surface. In all the experiments there is a tendency to produce small Richardson numbers in the troposphere below the tongue of stratospheric air, and in the lowest part of the tongue. The contour Ri = 1 is indicated in Fig. 10 of Experiment 4. With the small-scale shears that occur in the atmosphere, we may expect a strong possibility of Kelvin-Holmholz instability in this region. That this is a preferred region of clear air turbulence has been noted by many authors e.g. Endlich and McLean (1957). The mixing in this region clearly provides another process by which air previously in the stratosphere may push deeper in to the troposphere. The change of potential vorticity and other tracers following such air has been noted by Briggs and Roach (1963).

The particle motions we exhibited in Fig. 9 are clearly typical of those that must be associated with the pictures produced by our model. The strong downward motion in the mid and upper troposphere and low stratosphere on the cold side of, and underneath the jet, and the upward motion in the upper troposphere on the warm side of the jet are in excellent agreement with the observational studies. Observations of jet contrails by Endlich and McLean (1957) and of dew point by Vuorela (1953) and many others show the existence of very dry air precisely where we indicate downward motion. Shaefer and Hubert (1955) and others have commented on the bands of cirrus and cirrostratus on the warm side of the jet, indicative of the upward motion there.
FRONTogenesis models

The frontolysis of upper air fronts in the delta region of jets which has been examined by Newton (1954) may also be represented by our model. Here we postulate a large-scale deformation field tending to weaken $\alpha$ gradients. Then $\alpha < 0$ and the motion in the $x, z$ plane is the reverse of that indicated in Fig. 9.

4. A Horizontal Shear Model

Williams (1967) has published a numerical study of the finite amplitude development of a baroclinic wave. He found that the distortion of the wave produced a quite realistic surface cold front and his numerical calculations suggested a tendency to form discontinuities in physical variables in a finite time. It will be shown in HB that this problem, when approached by a method very similar to that used in the deformation problem, has an analytic solution which confirms Williams' calculations. (For a description of the solution see his paper.) In this model, the initial frontal temperature gradients are produced by the "horizontal shear effect" of warm southerly winds and cold northerly winds. However, the tightening and production of frontal structure is very similar to that found in the deformation model. In this horizontal shear model rather extreme conditions are required before the tendency to a discontinuity becomes dominant: for the most unstable wave, there is a contrast in surface wind of 102 m s$^{-1}$ across the front by the time the maximum vertical component of absolute vorticity in the front $\sim 5f$. At this stage the minimum value of the Richardson number at the surface front is approximately 0.26.

5. The Breakdown of the Models

The deformation model and the horizontal shear model both predict the possible formation of discontinuities at the surface in a finite time. But true discontinuities do not form in the atmosphere. The approximation of geostrophic balance across the front would break down in the final stages. However, for the atmosphere, the real restriction on the validity of these models is the neglect of mixing. Before the maximum vertical component of absolute vorticity is $10f$, the Richardson number in the frontal zone becomes so small ($<2$) that we expect instability, e.g. Kelvin-Helmholtz, and turbulence. The precise effect of the consequent mixing is unclear. It will stop the formation of larger gradients at
the surface, but away from the surface where the front is less intense and Richardson numbers larger, the frontogenetic mechanism may well continue thus extending the strong frontal zone through more of the troposphere.

6. Conclusion

We have described solutions of the inviscid, adiabatic primitive equations which exhibit the formation of phenomena very similar to atmospheric fronts. A deformation model with simple but realistic initial temperature distributions has produced surface and upper tropospheric fronts and a horizontal shear model has produced a surface front. The surface fronts show a tendency to collapse to a discontinuity in a finite time, while the upper tropospheric front is associated with the descent of a tongue of stratospheric air.

The emphasis in this paper has been on producing simple models. The fronts formed are straight and have trivial variation in physical quantities along them. Latent heat release and surface friction when included have been considered separately and the modelling of the former, in particular, is crude. Further, we have neglected mixing. The author, in the near future, hopes to study more complicated models including these effects.

Acknowledgments

The author is deeply indebted to Dr. F. P. Bretherton for his constant optimism, guidance and enthusiasm during this work. For two years at the Department of Applied Mathematics and Theoretical Physics in Cambridge the author was in receipt of a research studentship given by the National Environment Research Council. His one year at the Johns Hopkins University was supported by the National Science Foundation under Grant GA-16603.

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