Generalizations of the Richardson criterion for the onset of atmospheric turbulence

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Summary

The energetics of air-parcel interchange are examined for the case in which the atmospheric potential temperature $\theta$ and background wind $U$ are continuous but otherwise arbitrarily varying in space, and in which the direction of $U$ is free from constraint. Interchange along a single s-axis, inclined at angle $\alpha$ to the vertical, is shown to be energetically feasible if

$$(g/\theta)(\partial \theta/\partial z) \cos \alpha / \langle \partial U / \partial s \rangle^2 \ll 1/4,$$

where $g$ is the gravitational acceleration. This condition, which is in general far less stringent than that imposed by the standard Richardson criterion, is suggested as a condition portending the onset of anisotropic turbulence. An isotropic interchange throughout a sphere is shown to be energetically feasible if

$$(g/\theta)(\partial \theta/\partial z)/\langle \nabla U \rangle^2 + \langle \nabla U \rangle^2 + \langle \nabla U \rangle^2 \ll 1/4,$$

where the $z$ co-ordinate of the Cartesian $x, y, z$ system is directed upward. This condition, which is in general somewhat less stringent than that imposed by the elementary Richardson criterion, may well portend the occurrence of isotropic turbulence. An outer time scale for the turbulence is estimated in each case, and its relevance to the occurrence of turbulence is discussed for the case when the background parameters of the system change with time.

1. Introduction

The onset of atmospheric turbulence is often discussed with the aid of the (gradient) Richardson number,

$$Ri \equiv \omega_{BV}^2 / \langle \partial U / \partial z \rangle^2,$$  \hspace{1cm} (1)

and the elementary Richardson criterion

$$Ri \leq 1/4.$$  \hspace{1cm} (2)

Here, $\omega_{BV}$ is the Brunt-Väisälä frequency:

$$\omega_{BV}^2 \equiv (g/\theta)(\partial \theta/\partial z)$$  \hspace{1cm} (3)

where $g$ is the gravitational acceleration, $z$ is a Cartesian co-ordinate increasing upward, $\theta$ is the 'potential temperature', proportional to $p^{\gamma p} / \rho$ where $p$, $\rho$ and $\gamma$ are the background atmospheric pressure, density and specific heat ratio, respectively. $U$, which is taken to be horizontal and often is confined to a single horizontal direction, is the background wind velocity upon which turbulent motions might come to be superimposed.

The numerator of $Ri$, namely $\omega_{BV}^2$, is a measure of the static stability of the atmosphere. The denominator, $\langle \partial U / \partial z \rangle^2$, is a measure of the destabilizing effect of wind shear. Any condition that $Ri$ be small, such as Eq. (2), is therefore a condition that the stabilizing influence shall in some sense be small relative to the destabilizing influence, and hence is a condition that might well portend the occurrence of turbulence.

The only rigorous derivations of the specific condition of Eq. (2) are based upon appropriate stability analyses. Most such analyses to date have been concerned with incompressible fluids, for which $\omega_{BV}^2$ in Eq. (1) must be defined somewhat differently, but they have confirmed Eq. (2) as a necessary condition for instability in the circumstances assumed (Miles 1961; Howard 1961) and in some cases have established Eq. (2) or a more stringent variant thereof as a sufficient condition for instability (Drazin 1958; Menkes 1961). The extension of the 'Miles-Howard theorem' to compressible fluids has been accomplished.
recently by Chimonas (1970), confirming Eq. (2) as a necessary condition for instability in such fluids. By implication, Eq. (2) becomes a necessary condition for the onset of turbulence in a previously laminar flow, though it is by no means guaranteed as a sufficient condition.

This rigorous result, and its implication, can of course be upheld only when the fluid under consideration satisfies the assumptions on which the mathematical analyses were based. These have included invariably an assumption of horizontal stratification: the 'background' parameters of the fluid have been taken to vary in the vertical direction only, the partial derivatives in Eqs. (1) and (3) being then equivalent to total derivatives. And, as already noted, U has been assumed to be horizontal. In the real atmosphere, these assumptions are satisfied only approximately at best, and many situations arise in which they are violated completely. It seems unlikely, in consequence, that Eq. (2) can be maintained as a necessary condition for instability in practice. On the contrary, it has been argued elsewhere (Hines 1963) that obliquely shearing oblique winds should be more effective than vertically shearing horizontal winds, in producing instability; and that the oblique winds caused by atmospheric gravity waves are a likely source of turbulence, at least at high altitudes.

Analysis of this argument by rigorous stability techniques appears to be difficult if not impossible. It is appropriate then to consider whether some less rigorous approach might not provide at least a provisional generalization, and perhaps point the way for more reliable methods of analysis in the future.

There is indeed such an approach available, based upon the energetics of a virtual interchange of air parcels in an initially laminar flow. This approach is exemplified by Chandrasekhar (1961) and by Ludlam (1967) for the circumstances envisaged in earlier stability analyses, and it recovers Eq. (2) as the criterion for instability in those circumstances. It is an approach that can be generalized without difficulty. The present paper takes this step, and extends it into a more speculative discussion of the nature of the turbulence that might result from instability. The conclusions are to be employed and extended further in a subsequent paper, in application to the case where turbulence is generated by gravity waves.

In Section 2 are discussed the energetics of two air parcels, interchanged along some arbitrary axis, in an initially nonturbulent region of the atmosphere whose variations of \( p, \rho \) and U are continuous but otherwise unrestricted. The argument follows classical meteorological lines as represented by Ludlam (1967), but with a generality in the background parameters and in the treatment of available kinetic energy that appears to be new in this context. The development is given in more detail than is usual, both to ensure that the generality does not by itself introduce any unexpected considerations (or to incorporate them properly if it does), and to provide a self-contained argument for those workers, notably aeronomers, who may be unfamiliar with the standard meteorological approach but may have a special interest in the new results.

The results of Section 2 are discussed in application to anisotropic turbulence in Section 3, and are developed in a more stringent form suggestive of isotropic turbulence in Section 4. In Section 5, it is shown that the energetics of the interchange process impose a restriction on the outer time scale of turbulence, such that the largest-scale eddies, whose energy is taken to be derived from the local background régime, must have a time scale greater than a certain characteristic value defined from the parameters of that régime. The effects of this restriction, when the background itself is variable in time, are discussed in Section 6 together with other restrictions on the turbulence and on the validity of the whole analysis. The principal conclusions are summarized in Section 7.

It should be recognized at the outset that the path to be followed here is not rigorous: that decisive steps are to be taken on the grounds of plausibility, not of physical or mathematical necessity. Other writers might well adopt other criteria of plausibility, and might thereby derive other conclusions. Businger (1969), for example, has already challenged Ludlam's (1967) analysis, and has reached somewhat discrepant conclusions on the basis
of somewhat different arguments. As it happens, Businger's approach seems to be more relevant to the maintenance of turbulence already in existence, than to the onset of turbulence from a non-turbulent state. He claims to recover Richardson's original flux criterion (Richardson 1920), which was designed for the maintenance of turbulence, as would seem to be appropriate to his model since he assumes that the turbulent interchanges occur throughout an extended volume. Even then, he recovers the flux criterion only if the eddy-diffusion coefficients for heat and momentum are identical, which is perhaps not surprising since he tacitly assumes that the turbulent interchanges of momentum proceed as efficiently as do those of potential temperature. In any event, for our immediate purposes it is sufficient to note once again that Ludlam's analysis yields the same criterion as that which emerges from rigorous stability analyses, as a necessary criterion for inherent instability in a laminar flow of a certain type. Herein lies the principal justification for seeking an extension of Ludlam's analysis and of the arguments that underlie it, to deal with laminar flows of more general type.

2. Virtual interchange of two air parcels

We shall consider first the virtual interchange of two parcels of air, initially separated by some small distance $S$ along some chosen Cartesian $s$-axis inclined at an angle $\alpha$ to the vertical (see Fig. 1). Let the first parcel have unit volume initially, and let it move adiabatically to the position of the second, maintaining pressure equilibrium as it goes. Upon arrival at its new position it will have volume $(p_1/p_2)^{1/\gamma}$, where $p_1$ and $p_2$ are the ambient pressures at the first and second locations respectively. Let the second parcel have this same volume initially, $(p_1/p_2)^{1/\gamma}$, and let it move under the same conditions to the initial position of the first. There it will have volume $(p_1/p_2)^{1/\gamma}$ $(p_2/p_1)^{1/\gamma} = \text{unit volume}$, and each parcel will have replaced the other without net deformation of the surrounding medium. The energetics of the interchange may then be computed from the energetics of the two parcels alone.

It is clear that the change of internal energy for the one air parcel, $\int p \, dv$ where $dv$

![Figure 1. Illustrating the interchange of two parcels of air, the first having unit volume, pressure $p_1$, density $\rho_1$, velocity $U_1$, and the second having volume $(p_1/p_2)^{1/\gamma}$, pressure $p_2$, density $\rho_2$, velocity $U_2$; through a distance $S$ along an $s$-axis inclined at angle $\alpha$ to the vertical (z) axis; in the presence of gravity $g$; the gradient of potential temperature ($\nabla \theta$) being inclined at angle $\beta$ to the vertical.](image-url)
is an element of volume change, will be just offset by the corresponding change for the other. Likewise, the energy expended on the first against the pressure-gradient force, \(\int (p_1/p)^{1/\gamma} (-\nabla p) ds\), will be just offset by the corresponding gain from the second. If the density of the first is initially \(\rho_1\), however, and that of the second \(\rho_2\), then the interchange of the two parcels will have resulted in a net increase of gravitational potential energy per unit mass equal to

\[
G_A \equiv [\rho_1 - (p_1/p_2)^{1/\gamma}\rho_2] gS \cos \alpha [\rho_1 + (p_1/p_2)^{1/\gamma}\rho_2] = \rho \left[ \frac{1}{\gamma p} \frac{\partial p}{\partial s} S + \ldots \right] \left( 1 + \frac{1}{\rho} \frac{\partial \rho}{\partial s} S + \ldots \right) gS \cos \alpha [1 + 1 + \ldots] \equiv \frac{1}{2} \left. \frac{\partial \theta}{\partial s} \right| S^2 \cos \alpha \tag{4}
\]

where the dots indicate series expansions in \(S\) (of order \(S^2\) and higher in the numerator, \(S\) and higher in the denominator) and the final equality is correct only to lowest non-vanishing order in \(S\); \(p, \rho, \theta\) and their derivatives are evaluated at the initial position of the first parcel, say, or at the mid-point between the two positions without alteration to the final (lowest-order) result. If the background flow is accelerating, both \([g]\) and the 'vertical' direction must be defined as they would appear to be in a similarly accelerating frame of reference, a fact pointed out to the author by Dr. G. Chimonas.

If \(G_A\) is negative, as it must be for some family of \(s\)-axes unless \(\nabla \theta\) is purely vertical and directed upwards, then the system is inherently unstable for interchanges along those axes. This situation produces the so-called 'slant-wise instability' or 'baroclinic instability' in the large-scale dynamics of the atmosphere, and has been discussed by Green (1960) in a manner analogous to that adopted here. Its relevance to the formation of small-scale turbulence does not appear to have been investigated until now, but is examined automatically in the present study without separate discussion.

If \(G_A\) is positive, then the virtual interchange will be a possible interchange only if an amount of energy at least equal to \(G_A\) is available for conversion. We therefore enquire into the maximum amount of energy that might be gained from the background flow.

The kinetic energy of the first parcel is initially \(\rho_1 u_1^2/2\), and it may be changed to \(\rho_1 (U_1 + u_1)^2/2\), say, where \(u_1\) is a change of velocity as yet unspecified. Likewise the kinetic energy of the second may be changed from \(F_1 u_2^2/2\) to \(F_1 (U_2 + u_2)^2/2\), where \(F \equiv (p_1/p_2)^{1/\gamma}(\rho_2/\rho_1)\). The net loss of kinetic energy per unit mass will then be

\[
K_A \equiv \rho_1 [U_1^2 - (U_1 + u_1)^2 + FU_2^2 - F(U_2 + u_2)^2]/2\rho_1[1 + F] \equiv -(U_1 - u_1 + u_1/2 + FU_2 - u_2 + Fu_2^2)/[1 + F]. \tag{5}
\]

We may require that the total momentum of the two parcels be conserved, whence

\[
u_1 + Fu_2 = 0 \tag{6}
\]

and then

\[
K_A = [(U_2 - U_1) - (1 - F^{-1})U_1^2]/[1 + F]. \tag{7}
\]

It is evident that this loss will maximize, for a given \(|u_1|\), if \(u_1\) is taken to lie in the direction of \(U_2 - U_1\); say

\[
u_1 = f(U_2 - U_1) \tag{8}
\]

and then

\[
K_A = [f - (1 + F^{-1})f^2]/2(U_2 - U_1)^2[/1 + F]. \tag{9}
\]

This in turn will maximize under variations of \(f\) if

\[
f = (1 + F^{-1})^{-1} \tag{10}
\]

and then

\[
K_A^\text{max} \equiv (1 + F^{-1})^{-1}(U_2 - U_1)^2/2(1 + F) \equiv (1 + 1 + \ldots)^{-1}(U_2 - U_1)^2/2(1 + 1 + \ldots) = (\partial U/\partial s)^2 S^2/8. \tag{11}
\]
where the dots represent terms (of first and higher order in $S$) in series expansions of $F^{-1}$ and $F$, and the final line is given only to lowest order in $S$; $\partial U/\partial s$ is evaluated at the initial position of the first parcel or, as before, at the mid-point.

By application of Eq. (10) to Eqs. (8) and (6), it may be seen that the condition for maximum release of kinetic energy is equally the condition for equipartition of momentum. The latter condition was adopted by Ludlam (1967) without explanation, though its interpretation in terms of maximum energy release is an essential part of the derivation of the critical Richardson number by the present approach: unless equipartition is achieved, the conversion of kinetic energy will be less than optimum and the de facto critical Richardson number will be decreased.

Even this statement must be qualified, in line with previous comments, because rigour has already been abandoned. For example, we have demanded that the total momentum of the interchanging air parcels be conserved, whereas the relevant conservative law applies to the fluid as a whole. Again, we have introduced pressure jumps (because of the different weights of the two parcels) and velocity jumps, at the boundaries of the air parcels after their interchange — jumps that would not occur in reality, but that we can do little to avoid if the present line of argument is to be followed at its own level of sophistication. For justification, we must return to the empirical fact that this line of argument does recover Eq. (2) in those circumstances where Eq. (2) is known to be relevant. In effect, we must hope that the imposition of plausible demands (such as conservation of the momentum of the interchanging air only) just compensates for the loss of the full set of fluid-dynamic equations that would enter a rigorous analysis, and we note that this hope is justified in certain tractable cases. We cannot guarantee continued success in more general cases, but we can at least explore the consequences in these cases and employ or reject them as judgment dictates. This appears to be the only path available, short of a rigorous stability analysis, and it may provide some clues as to the manner by which such an analysis might be achieved in due course.

3. Anisotropic Turbulence

For the virtual interchange of Section 2 to be a possible interchange energetically, in the absence of further energy sources, it is clear that we must impose the condition $K_A^{\text{max}} \geq G_A$, or

$$\frac{(g/\theta)(\partial \theta/\partial s) \cos \alpha}{(\partial U/\partial s)^2} \leq \frac{1}{4}.$$  \hspace{1cm} (12)

In the particular case where $\theta$ and $U$ vary only with height, $\partial/\partial s$ may be converted to $\cos \alpha \partial/\partial z$, and then Eq. (12) converts directly to Eq. (2), with or without the additional restriction that $U$ be horizontal. Thus Eq. (12) may be considered to be a generalization of the Richardson criterion, to systems in which $\theta$ and $U$ need not vary only in the vertical (and $U$ need not be horizontal), at least as a criterion for the energetic feasibility of effecting an interchange on some chosen $s$-axis.

It will be apparent that Eq. (12) can be a less stringent criterion than Eq. (2). When the $s$-axis is horizontal, for example, and so $\cos \alpha = 0$, Eq. (12) is satisfied automatically in any system that contains horizontal variations in wind velocity. Indeed, whenever $\nabla \theta$ is other than directly upward, $G_A$ and the numerator of Eq. (12) are automatically negative for some $s$-axes as already noted. These considerations suggest that the systems involved may be inherently unstable. Whether this is true or not, there is a clear indication that the Richardson criterion as currently employed is much too stringent in principle as a criterion for the onset of small-scale turbulence in the real atmosphere.

On the other hand, the new criterion of Eq. (12) may well be too lax. It is certainly established only as a necessary condition for the energetic feasibility of a virtual interchange, with internal conservation of momentum. Whether a real interchange could in fact proceed if Eq. (12) were satisfied is quite a different matter (Chandrasekhar 1961); and whether it would proceed spontaneously in the sense required of an instability, let
alone of turbulence (however ‘turbulence’ may be defined), is different again. Such questions depend on the full fluid dynamics of the real interchange process, and clearly require that changes should occur in the surrounding medium additional to those in the pair of air parcels that have been considered up to this point. Indeed, it is somewhat irrelevant to consider the interchange of air parcels along a single axis, in isolation from changes along other axes (or more generally, in three dimensions), since a real single-axis interchange is inherently impossible.

These considerations point up the necessity of limiting any claims to relevancy that might be made for Eq. (12); but they do not undermine the general inference already drawn from Eq. (12), namely, that something less stringent than Eq. (2) is likely to be relevant. Indeed, in the absence of any more satisfactory analysis, we may treat Eq. (12) and the formulae that led to Eq. (12) as relations appropriate to the generation of turbulence in an initially non-turbulent régime, whose variations of $\theta$ and $U$ are continuous but otherwise arbitrary. At the very least, it may be suggested that Eq. (12) must be satisfied for some $s$-axis as a necessary condition for the onset of turbulence, even though such a requirement may fall far short of being a sufficient condition for turbulence.

It should be evident that $G_A$, $K_A$, and their ratio are in general variable with the $s$-axis chosen for the interchange, the only exceptions to this arising for constant $\theta$ and/or $U$, or in the case of the ratio when $\theta$ and $U$ vary with height only. Accordingly, any turbulence discussed with the aid of these quantities is likely to be inherently anisotropic, at least in the larger-scale ‘eddies’ that derive their energy directly from the background régime. The subscripts on $G_A$ and $K_A$ are employed to suggest this inherent anisotropy.

We may usefully consider corresponding quantities from which anisotropy has been eliminated by integration over all directions. This is done primarily to overcome the inadequacy inherent in the treatment of single-axis interchanges, by treating an interchange that is at least a possible interchange. The results will be suggested as results that contain a more-than-adequate provision for the fluid-dynamic limitations (as distinct from the energetic limitations) on the spontaneous generation of turbulence, and hence as results that contain a more-than-adequate compensation for the possibility that Eq. (12) is too lax. The isotropic interchanges of the new analysis may be related conceptually to isotropic turbulence, and indeed are discussed most conveniently in terms of such turbulence.

4. ISOTROPIC TURBULENCE

We shall adopt as our model of isotropic turbulence an interchange of air parcels within a sphere of diameter $S$, such that each parcel of air within the sphere undergoes an interchange with a corresponding parcel almost diametrically opposite it. The word ‘almost’ is needed in the preceding sentence because of the variation of the volume of each air parcel in the course of its interchange, as represented by the factor $(p_1/p_2)^{1/4}$ at an earlier stage. One might first contemplate the interchange of air parcels in the outermost spherical shell, for example, but that shell would have to be thicker on one side than on the other in proportion to $p^{-1/4}$. The next spherical shell would then have its centre offset slightly, in the direction of $\nabla p$; and so on with the next and the next, until interchanges of all the air parcels within the sphere had been taken into account. The successive offsets introduce corrections that are of higher order in $S$, however, and so may be ignored for present purposes. Accordingly, we may proceed by considering a succession of spherical shells, of radius $s$ and thickness $ds$, whose energetics are treated as in Section 2 but with integration over the solid angle $2\pi$ that defines half of the interchanging partners, and with subsequent integration over $s$ to a radius $S/2$.

It may be noted that other models of ‘isotropic turbulence’ might well be designed, each according to its author’s own taste. But it seems reasonable to impose, as a test for legitimacy in the present context, the condition that the resultant criterion for the onset of turbulence should be compatible with the Miles-Howard-Chimonas theorem in the circumstances for which that theorem was designed. The present model meets this test.
For the gravitational potential energy we adopt Eq. (4), with \( S \) replaced by \( 2s \), and adopt spherical polar co-ordinates \((s, \alpha, \phi)\). The net increase of gravitational potential energy per unit mass is then seen to be

\[
G_I = \frac{\iiint \frac{g}{2\theta} |\nabla \theta| (2s)^2 \cos \alpha \cos \psi \cdot s^2 \sin \alpha \, ds \, d\alpha \, d\phi}{\iiint s^2 \sin \alpha \, ds \, d\alpha \, d\phi},
\]

(13)

where \( \psi \) is the angle between \( \nabla \theta \) and the co-ordinate direction \((\alpha, \phi)\); the integrations are extended over the range \( 0 \leq s \leq S/2, 0 \leq \alpha \leq \pi/2, 0 \leq \phi \leq 2\pi \). If the \( \phi = 0 \) plane is taken to contain the direction of \( \nabla \theta \), then \( \cos \psi = \sin \beta \sin \alpha \cos \phi + \cos \beta \cos \alpha \), where \( \beta \) is the angle between \( \nabla \theta \) and the vertical. The integrations may be performed without complication, to yield the result

\[
G_I = \frac{1}{10} \frac{g}{\theta} |\nabla \theta| \cos \beta S^2
\]

(14)

\[
= \frac{1}{10} \frac{g}{\theta} \nabla_\theta \theta S^2
\]

\[
= \frac{1}{10} \omega_{BV}^2 S^2
\]

where \( \omega_{BV}^2 \) is defined as in Eq. (3), the partial \( \partial \theta/\partial x \) in it now being needed.

For the release of maximum kinetic energy, it is advisable to rewrite Eq. (11) as

\[
K_{\alpha}^{\max} = \left[ (\nabla U_x \cdot S)^2 + (\nabla U_y \cdot S)^2 + (\nabla U_z \cdot S)^2 \right] / 8
\]

(15)

in an obvious notation. From this it is apparent that the necessary integration may be performed on each of the three terms separately, with each term introducing a factor of the form \( \cos^2 \psi_4 \) \((i = x, y, z)\); and that it is convenient to employ different polar co-ordinates, \((s, \psi_4, \phi_4)\) say, for each of the terms in turn. The maximum release of kinetic energy per unit mass in the isotropic case is then seen to be

\[
K_I^{\max} = \frac{\sum \iiint (\nabla U_i)^2 (2s)^2 \cos^2 \psi_4 \cdot s^2 \sin \psi_4 \, ds \, d\psi_4 \, d\phi_4}{8 \iiint s^2 \sin \psi_4 \, ds \, d\psi_4 \, d\phi_4}
\]

(16)

where the summation extends over all \( i = x, y, z \) and the integration extends over the ranges \( 0 \leq s \leq S/2, 0 \leq \psi_4 \leq \pi/2, 0 \leq \phi_4 \leq 2\pi \). The result is

\[
K_I^{\max} = \left[ (\nabla U_x)^2 + (\nabla U_y)^2 + (\nabla U_z)^2 \right] S^2 / 40
\]

(17)

\[
= (\nabla U)^2 S^2 / 40
\]

where \( (\nabla U)^2 \) is defined by the equation itself, as a symbolic form for the sum in the preceding line.

It is perhaps not evident, a priori, that Eq. (17) is indeed the maximum available kinetic energy, since its method of derivation tacitly requires momentum to be shared between parcels of air taken two at a time, whereas some more favourable partitioning of momentum might now be contemplated. The earlier result is but a special case of a general mechanical law, however, that a set of particles (or air parcels) with a given total momentum will have minimum kinetic energy if all have the same velocity, the (mass-weighted) mean. This situation is indeed the one produced in the present case, to lowest order in \( S \), even though it is produced in a fashion that appears at first sight to be less general. Thus, Eq. (17) is indeed the maximum available kinetic energy.

It will now be apparent that, for isotropic turbulence to be energetically feasible in the eddies that derive their energy directly from the background régime, we may expect \( K_I^{\max} \geq G_I \), whence

\[
\omega_{BV}^2 (\nabla U)^2 \leq 1/4.
\]

(18)
This is a generalization of the usual Richardson criterion, Eq. (2), only in that it allows for general orientations and variations of \( \mathbf{U} \). It must be remembered, however, that Eq. (18) is designed as a criterion for wholly isotropic turbulence. It does not supersede the less stringent Eq. (12) as a criterion for anisotropic turbulence, though the proper criterion for such turbulence might well lie between the two since Eq. (12) might well be too lax.

It may also be noted that the present derivation of Eq. (18) has justified the standard criterion of Eq. (2), under the standard circumstances of \( \theta \) and \( \mathbf{U} \) varying only in the vertical, for application as a criterion that wholly isotropic turbulence might be formed; standard derivations of Eq. (2) that are based on the energetics of the interchange do not deal with the inherent anisotropy of the derivations themselves, and so provide no commentary on the degree of isotropy or anisotropy that might be produced in the case of small-scale turbulence.

5. Outer time scales of turbulence

The present development is intended for application to systems in which the background régime may be variable in time, as would be the case, for example, if gravity waves were present. But the analysis has ignored temporal changes in the background parameters, and so can be justified only if those changes proceed sufficiently slowly. In order to assess this aspect, we must first derive a relevant time scale for the turbulent changes. This we do with the aid of elementary concepts of turbulence.

The concept of 'eddies' permeates much of the elementary discussion of isotropic homogeneous turbulence, these eddies being characterized by some spatial scale \( L \) and some temporal scale \( T \). Energy is pictured as being extracted from the background régime by eddies of some large or 'outer' scale \( L_1, T_1 \), and as cascading without loss through eddies of intermediate scale to eddies of 'inner' scale \( L_2, T_2 \), and thence to eddies of still smaller scale in which dissipation becomes severe. The rate of transfer of energy per unit mass through the intermediate spectrum is a constant, given by

\[
\epsilon = L^2 / T^3. \tag{19}
\]

This picture gains some support from the universal equilibrium theory of turbulence (Batchelor 1953), the equality of Eq. (19) being representative of its 'inertial subrange'.

Though the earlier discussion centred on the stability (or instability) of an initially non-turbulent régime, its results may be applied to good effect in the discussion of fully developed turbulence. Or, conversely, Eq. (19) may be applied to good effect in the discussion of incipient turbulence. The two sets of concepts are somewhat at variance with one another, but perhaps not so much as is often supposed. In any event, our immediate objective may be served and some clues for further progress may be gained if we combine the two sets of results.

We do this by identifying the spherical interchange of Section 4 as a model for the isotropic eddy of outer scale, since it derives its energy directly from the background régime. We therefore replace \( S \) in Eqs. (14) and (17) by \( L_1 \), and, since the energy changes occur while parcels of air in the eddy are moving through maximum distances equal to \( L_1 \), we identify them as the changes that occur in time \( T_1 \). Factors of the order of \( \pi \), associated perhaps with motion around rather than across the sphere, are ignored in conformity with the general uncertainties of elementary turbulence theory. We may then derive from a combination of Eqs. (14) and (17) the maximum possible rate of energy extraction from the background flow, per unit mass, and hence the maximum possible value of \( \epsilon \) for insertion in Eq. (19). We may go so far as to apply Eq. (19) to the eddies of outer scale, even though it is not designed for accurate application to them. Thus, with \( S = L = L_1 \) and \( T = T_1 \),

\[
L_1^3 / T_1^3 = \epsilon \leq [K_{\text{max}} - G] / T_1 = L_1^3 / \tau_1^\gamma T_1, \tag{20}
\]

where

\[
\tau_1^\gamma \equiv 40 / [ (\nabla \mathbf{U})^2 - 4 \omega^2 ] , \tag{21}
\]

from which, for \( \tau_1^\gamma > 0 \) which is the only relevant case,
The \( \tau_f \) defined by Eq. (21) is suggested, then, as the minimum possible value for the outer time scale of isotropic turbulence derived directly from the background régime. An isotropic sub-range of turbulence could, of course, be derived from a larger-scale range of anisotropic eddies, in which case its minimum outer time scale would be dependent on the parameters of the anisotropic system.

No general law corresponding to Eq. (19) is available for anisotropic turbulence, except perhaps in the small-scale eddies which may tend toward isotropy. But it is tempting to employ Eq. (19) once again, with \( L = L_1 = S \) and \( T = T_1 \) as before, in conjunction with Eqs. (4) and (11) to determine a corresponding limit on \( \tau_f \) even in the anisotropic case. For comparison with Eq. (21), however, it is perhaps more appropriate to integrate Eqs. (4) and (11) first, in a fashion to ensure that a full column of air parcels is interchanged, of total length \( L_1 \), rather than just the end parcels. Each of Eqs. (4) and (11) would then be reduced by a factor of three (a factor which, like \( \tau \), might well be ignored), with the result

\[
L_1^2/T_1^2 = \epsilon \leq \frac{[K_{A}^{\text{max}} - G_A]}{3T_1} = L_1^2/\tau_4^2T_1,
\]

where

\[
\tau_4^2 \equiv 24[[(\partial U/\partial s)^2 - 4(\partial U/\partial s)(\partial V/\partial s) \cos \alpha],
\]

from which

\[
T_1^2 \geq \tau_4^2.
\]

This \( \tau_4 \), ostensibly the minimum possible value for the outer time scale of anisotropic turbulence, is of course a function of the \( s \)-axis chosen; the ultimate minimum will be found only after a further evaluation under variations of the \( s \)-axis.

6. Additional restrictions

As previously noted, the present development is intended for application to systems in which the background régime may be variable in time. It cannot be justified for such systems, however, unless the time scale of the variations, say \( \tau \), greatly exceeds that of the turbulent interchanges. This requires \( \tau > T_1 \) in particular, a requirement that cannot be checked directly in the absence of firm knowledge of \( T_1 \). Eqs. (21) and (24) nevertheless permit us to conclude that the requirement cannot be met unless

\[
\tau > \tau_f \quad \text{or} \quad \tau > \tau_A
\]

as appropriate, and these are conditions on the background régime which may be checked purely by reference to the parameters of that régime. Thus we may state categorically that the present analysis is not applicable unless Eq. (26) is satisfied, and we may suppose but cannot prove that it is likely to be at least semi-quantitatively applicable if Eq. (26) is satisfied.

Even when Eq. (26) is satisfied, however, a further restriction on time scales must be imposed. By the very nature of a time-varying system, the time scale \( \tau_f \) is itself varying with time. If the background régime includes a periodically varying wave, for example, \( \tau_f \) will vary periodically; it may well be small over some portion of the period, rise to infinity at some phase of the cycle, become imaginary (corresponding to stability), and then reduce from infinity to small real values once again as the cycle is completed. In such circumstances we might expect turbulence to be formed at a given location during the interval when \( \tau_f \) is small there, and to begin a decay with the onset of stability or with an increase of the 'flux Richardson number' above unity, only to intensify or begin once again when \( \tau_f \) becomes small. In the case of a single propagating wave, the region(s) of most intense turbulence would propagate through the atmosphere with the phase speed of the wave. But even this expectation would be thwarted if \( \tau_f \) did not remain sufficiently small for a sufficiently long interval. Symbolically, we could state as an extreme that \( \tau_f \) must be less than some (any real) value \( \tau^* \) for an interval at least as long as \( \tau^* \), or turbulence cannot be expected to develop with significant intensity at any stage throughout the cycle.
This conclusion seems valid, at least, for isotropic turbulence generated directly from the background régime. One might argue similarly, that $\tau_A$ must be less than some $\tau^*$ for an interval at least as long as $\tau^*$, or anisotropic turbulence cannot be expected. Such a conclusion might be overly restrictive, however, for $\tau_A$ depends on the $s$-axis chosen and the time variation of $\tau_A$ will be different for different $s$-axes. It seems possible that anisotropic turbulence might begin to form with one axis favoured by a low value of $\tau_A$, only to convert its anisotropy as some other axis becomes favoured. As already noted, there will generally be some family of $s$-axes that can support interchange at any given time so the anisotropic turbulence need never pass through a stable phase during which it is subject to decay: it may merely change its anisotropy as the cycle advances.

The present analysis has ignored the effects of viscosity. For this neglect to be justified, it is necessary that

$$L_i^2 \gg \eta_M T_i$$

(27)

(where $\eta_M$ is the molecular kinematic viscosity), which is a condition on the turbulence once again, not on the background régime. On the other hand, the analysis has also ignored all but the lowest-order terms in the expansions Eqs. (4) and (11), a step which can be justified only if $L_1 < \lambda$ where $\lambda$ is a characteristic scale for the spatial variations (or more properly, for the curvatures) in the background régime. This fact combines with Eqs. (21) and (24) to imply that the analysis cannot be valid unless

$$\lambda^2 > \eta_M \tau_i$$

(28)

or

$$\lambda_i^2 > \eta_M \tau_A$$

(29)

as appropriate, $\lambda_i$ being the value of $\lambda$ that characterizes variations in the $s$ direction chosen for $\tau_A$. (In the case of a single wave, it is the 'wavelength along the $s$-axis', an oblique projection of the wavelength $\lambda$ itself, and so greater than $\lambda$.) Conditions Eqs. (28) and (29) are, once again, conditions on the background régime; they must be satisfied by that régime if the analysis is to be valid, but the validity of the analysis cannot be guaranteed even if they are satisfied.

Finally, for the turbulence to be maintained indefinitely it is necessary that the background régime be capable of replenishing the energy that is lost to turbulence as rapidly as it is lost. This requirement depends for its exploitation on the nature of the background régime itself, and is best discussed separately for each such régime. It, together with the other restrictions, will be exemplified in a subsequent paper which will treat the case of atmospheric gravity waves.

7. Conclusion

The principal qualitative conclusion of the present paper is that the Richardson criterion of Eq. (2) is generally too stringent, as a necessary condition for the generation of turbulence in an initially nonturbulent régime. It might be applied equally well as a criterion for the maintenance of turbulence from local energy sources, though of course turbulence, once generated, may itself transfer energy from more distant sources.

Quantitatively, Eq. (2) may be replaced by the slightly generalized form of Eq. (18) when $U$ has general orientation and variation, but only if one demands that the turbulence produced directly from the background régime shall be isotropic.

More generally, it seems necessary to admit that a criterion less stringent than Eq. (18), and perhaps as lax as Eq. (12), might be appropriate to the generation of turbulence. Such turbulence would necessarily be anisotropic in its eddies of larger scale, except when Eqs. (12) and (18) are identical (i.e., except when $\theta$ and $U$ vary only in the vertical). It seems reasonable to suppose that the true criterion for the onset of turbulence lies somewhere between Eqs. (12) and (18).

The conditions implied by criteria Eqs. (12) and (18), or by the suspected but still unknown intermediate criterion, are by no means established as sufficient for the generation
of turbulence. On the contrary, and as already stressed, they were designed only as necessary conditions. And even if they were 'sufficient' in the usual sense, other considerations were seen to be relevant in practice. In particular, it was found that purely isotropic turbulence can be generated directly from the background régime only if \( \tau_e \) as defined by Eq. (21) exceeds some (any real) value \( \tau^* \) for a time at least as long as \( \tau^* \), which may well be impossible if the background régime is itself varying in time. A similar restriction may well apply to anisotropic turbulence. Moreover, the validity of Eqs. (12) and (18) depends upon the spatial and temporal variations of the background régime being large in comparison with those of the turbulence, which requires as a minimum that conditions such as Eqs. (26) and (28) or (29) be satisfied.

The list of restrictions on the utility of Eqs. (12) and (18) is rather formidable, and rather poorly defined. This is unfortunate, but it cannot be circumvented except by way of a rigorous stability analysis that manages to take all the complications into account. For the moment, it seems we must accept Eqs. (12) and (18) together with their limitations, while hoping for something better to emerge in the future.

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