Hysteresis effects in a differentially heated rotating fluid annulus

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SUMMARY

This paper is concerned with the motion of an annulus of fluid which is contained between two cylinders rotating about their common axis. The fluid is differentially heated in the horizontal. Examination of the stability of the wave régime flow is carried out by perturbing an averaged form of a non-linear wave-régime solution. This leads to a linear system describing the disturbance and is solved approximately using a truncated Fourier series. By this means theoretical stability criteria are obtained in terms of the Rossby number and the wave number. These results are compared with experimental wave change criteria and qualitative agreement is found.

1. Introduction

This paper is concerned with a theoretical investigation of certain experimental results obtained by Fultz, Long, Owens, Bohan, Kaylor and Weil (1959). These experiments were designed to investigate the possibility of modelling natural phenomena, such as large-scale atmospheric motions, in the laboratory. Related experiments have been carried out by Hide (1958).

In a typical experiment, liquid contained in an annulus is subject to horizontal differential heating and simultaneously rotated about its vertical axis. For certain ranges of the Rossby number $S$ defined below, large amplitude waves travelling without change of shape appear.

We define $S$ to be

$$ S = \frac{g\alpha(\Delta T)_{\text{H}}h}{4\omega^2(b-a)^2\rho_o} $$

(1)

where $g$ is the gravitational acceleration, $\alpha$ is the coefficient of cubical expansion of the liquid, $(\Delta T)_{\text{H}}$ is the temperature contrast between the boundary cylinders of radii $a$ and $b$, $h$ is the fluid annulus height, $\omega$ is the rotation rate and $\rho_o$ is a reference density.

It has been observed that a wave pattern with wave number $m$ may give way to a new wave pattern with wave number $n$ if the heating contrast is varied during an experiment. For example, if conditions favourable to a two-wave pattern hold and the heating contrast is continuously lowered there may come a point in time when the flow gives way to a three-wave pattern which becomes dominant. The three-wave pattern may give way to a four-wave pattern as the heating contrast is reduced still further, and so on. The process continues until the flow suddenly regains a symmetric character.

There exist therefore critical values of the Rossby number at which transitions between adjacent wave numbers within the wave régime occur when the heating contrast is steadily lowered. However, the transition values obtained when the heating contrast is slowly increased are not the same as before. Each transition point is shifted to a higher value of $S$.

This phenomenon, first studied in detail by Fultz et al. (1959) is termed hysteresis. Their experimental results are summarized in Fig. 1, which gives transition values for both slowly decreasing and increasing thermal gradient on an $S$-wave number diagram. Experimentally we find a wave number overlap in which two or more adjacent wave numbers can exist at the same Rossby number although only one can be dominant at any time.

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In this study we seek to predict the wave overlap phenomena from simple theoretical considerations. Associated problems of describing the transition between the wave and symmetric regimes have been investigated theoretically by Eady (1949), Kuo (1957), Davies (1956) and others. The Eady baroclinic instability model employs a number of simplifying features, yet produces apparently good agreement between theory and experiment (Fowlis and Hide 1963). In particular, Eady assumed that viscosity could be neglected, that linear perturbation theory held, that the steady basic motion could be characterized by constant temperature gradients in the horizontal and vertical, and that the narrow gap approximation held.

A number of Eady's assumptions are made also in this model but whereas Eady perturbed a basic axially symmetric régime, we perturb a basic wave régime model.

Kuo (1957) considered wave/wave transitions in addition to wave/symmetric transitions. His theoretical results for transitions within the wave régime agree with the Fultz experimental transitions for increasing temperature contrast \((\Delta T)_H\) which are seen in Fig. 1. His approach was to find that wave with wave number \(m\) for which the wave disturbance has the highest amplification rate. This requirement yields a relation between wave mode \(m\) and temperature contrast (and hence \(S\) in view of Eq. (1)).

![Experimental transition values of Rossby number \(S\) against wave number \(m\) for both increasing and decreasing thermal gradient.](image)

Conditions: \(h = 130\) mm, \(b = 49.5\) mm, \(a = 25\) mm, \(\omega = 1.5\) s\(^{-1}\).

The hysteresis phenomenon has received less attention. Lorenz (1962) found a stability criterion in terms of the Rossby number in which the onset of a wave number differed from the disappearance of the same wave number. The philosophy behind his method was to reduce the governing equations to their bare essentials consistent with retaining important non-linear features of the flow. Due to the non-linearity of the system the mean part of the flow depends upon the wave number. When the mean flow associated with wave number \(m\) is perturbed with respect to wave number \(n\), a stability criterion arises which differs from that obtained by reversing the roles of \(m\) and \(n\).

The approach adopted here exhibits similarities to the approaches of Eady and Lorenz. To make headway some severe simplifications are introduced.

Fowlis and Hide (1965) report that the Eady criterion for transition between axially symmetric and wave régime flow is in fair agreement with experimental evidence if the viscosity is not too high. In addition, the wave régime experiments indicate negligible diffusion of the concentrated jet stream in the interior flow. This suggests that an inviscid treatment of the steady wave régime is a reasonable first step. However, the model cannot distinguish between a rigid or a free top surface and cannot deal with the sidewall boundary layers. Considerable mathematical simplification is thus gained at the expense of the ability to closely relate theory with experiment and anything more than qualitative agreement cannot be expected.

We consider, the stability of a steady inviscid non-linear wave solution with wave number \(m\) due to Davies (1959), hereafter denoted by \(I\), by inserting a set of wave distur-
bances into the equation and truncating the result. The steady main flow is suitably
averaged and contains a built-in lower criterion for the existence of the wave régime which
depends upon S and m.

Examination of the linearized perturbation equation leads to further stability criteria
which again depend upon S and m. The theoretical stability curves are shown in Fig. 3. Each
wave number m possesses a range of S over which the steady flow is stable. Ranges of S exist
over which two or more adjacent wave numbers are stable as found in the experiments.
The form of the curve S = S_S(m) bears qualitative resemblance to the experimental
transitions in Fig. 1 for S increasing. The position of this critical curve however depends
strongly on steady flow details near the walls. The solution in I does not supply these
details, although a boundary layer extension of this work due to Rogers (1962) enables some
unknown constants to be evaluated.

2. Formulation of the problem

Our mathematical approach will be governed by two notions. First, the finite amplitude
waves that characterize jet stream motion require non-linear treatment. Second, an
analytic solution of relatively simple form will be sought. This latter requirement demands
drastic simplification of the governing equations whilst retaining important features which
the theory is attempting to model.

(a) Basic equations

Fluid is contained in the annulus defined by

\[ a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq h, \]

where \((r, \theta, z)\) refer to cylindrical polar co-ordinates. The axis common to the bounding
cylinders is vertical. Fluid velocity components are \((U', V', W')\) in the directions \(r, \theta, z\) —
increasing respectively, \(T'\) is the temperature and \(P'\) the pressure. Relative to axes which
rotate with constant angular velocity \(\omega\) about the \(z\)-axis, the equations are

\[
\begin{align*}
\frac{DU'}{Dt} - 2\omega V' &= -\frac{1}{\rho_0} \frac{\partial P'}{\partial r}, \\
\frac{DV'}{Dt} + 2\omega U' &= -\frac{1}{\rho_0} \frac{\partial P'}{\partial \theta}, \\
\frac{DW'}{Dt} &= -\frac{\partial P'}{\partial z} + \rho_0 \alpha T', \\
\frac{DT'}{Dt} &= K \left( \frac{\partial^2 T'}{\partial r^2} + \frac{1}{r} \frac{\partial T'}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T'}{\partial \theta^2} + \frac{\partial^2 T'}{\partial z^2} \right), \\
\frac{\partial (rU')}{\partial r} + \frac{\partial (rV')}{\partial \theta} + r^2 \frac{\partial W'}{\partial z} &= 0,
\end{align*}
\]

where the operator \(D\) stands for

\[
\frac{\partial}{\partial t} + U' \frac{\partial}{\partial r} + V' \frac{\partial}{\partial \theta} + W' \frac{\partial}{\partial z}.
\]

Here \(K\) is the thermal conductivity. The Boussinesq approximation is employed and the
usual linear relation is assumed between density and temperature. Molecular viscosity has
been ignored in the momentum equations, following the procedure in I which assumed a
primary balance between the Coriolis terms and the pressure gradient. The geostrophic
approximation together with the thermal wind equation give rise to a velocity field which
is a good approximation in the interior flow (Davies 1953).

Eq. (2) are highly non-linear. A feature of the steady Rossby régime is the regular
precession around the annulus of the wave pattern with an angular velocity close to that of the cylinders. Referred to axes which rotate with the steady wave pattern, the basic motion is independent of time. We seek a perturbation solution to this steady motion and write

\[
\begin{align*}
\tau U' &= \frac{1}{4} \omega (b + a)^2 [U(\tau, \theta, z) + u(\tau, \theta, z, t)], \\
\tau V' &= \frac{1}{4} \omega (b + a)^2 [V(\tau, \theta, z) + v(\tau, \theta, z, t)], \\
W' &= 2 \omega h [W(\tau, \theta, z) + w(\tau, \theta, z, t)], \\
P' &= \rho \omega^2 (b + a)^2 [P(\tau, \theta, z) + p(\tau, \theta, z, t)], \\
T' &= \frac{(\Delta T)_{m}}{S} \left( \frac{b}{b - a} \right)^2 [T(\tau, \theta, z) + \tau(\tau, \theta, z, t)]. 
\end{align*}
\]

where the time independent terms are those due to Davies in I and are given in the Appendix. The number of lobes in the primary wave pattern is denoted by \( m \).

(b) The primary flow

Equations which describe the primary flow may be found in I. They are derived from the Eq. (2) set on assuming a geostrophic balance. The solution is briefly recapitulated in the Appendix. We note in particular that the axially symmetric part of \( U \) is zero and that \( V \) and \( \partial T / \partial z \) may be written in the form

\[
V = V_o(r, z) + V_m(r, z) \cos (m\theta - \phi) + V_{-m}(r, z) \sin (m\theta - \phi),
\]

and

\[
\frac{\partial T}{\partial z} = -\frac{\partial T_o}{\partial z} (r, z) + 2 \frac{\partial T_m}{\partial z} (r, z) \cos (m\theta - \phi),
\]

where \( \phi \) is defined in the Appendix and assumed constant.

The broad picture that emerges is of an interior inviscid flow in which vertical motions are negligible whilst some non-linearity of the wave régime is captured in the equation of heat transfer. A lower criterion for the existence of the steady flow was found in the form

\[
S \geq S_o(m) = \frac{K}{\omega (b - a)^2} \left[ mR_o + \frac{\pi^2}{mR_o} \right],
\]

where \( R_o = \ln (b/a) \).

(c) The perturbation flow

In their full generality the equations governing the perturbation flow field are quite intractable. They require simplification before a simple analytic solution can be found. The steady flow is composed of a symmetric part independent of \( \theta \) and a part which corresponds to a wave solution with wave number \( m \). A wave disturbance of the form \( \exp (i\theta) \) induces ' sideband' wave numbers \( n \pm qm \) \( (q = 1, 2, 3, \ldots) \) due to the interaction of primary and perturbation waves.

It would appear that the simplest form of wave interaction which ultimately permits a normal mode analysis of the disturbance is achieved by making the following simplifications.

(i) Replace the basic velocity components in the momentum equations by their values averaged with respect to \( r, \theta, z \). Denote the mean of a function \( f(r, \theta, z) \) with respect to \( r, \theta, z \) by \( \bar{f} \).

(ii) Allow the basic terms \( V \) and \( \partial T / \partial z \) averaged with respect to \( r \) and \( z \) only, in the equation of heat transfer. Denote the mean of a function \( f(r, \theta, z) \) with respect to \( r, z \) only, by \( \bar{f} \).

It is convenient to express the mean values of Eqs. (4) and (5) with respect to \( r \) and \( z \) only in the form

\[
\begin{align*}
[V] &= V_o + V_m \cos (m\theta - \phi) + V_{-m} \sin (m\theta - \phi), \\
\left[ \frac{\partial T}{\partial z} \right] &= \bar{T}_o \{ 1 + 2 \mathcal{U}_m \cos (m\theta - \phi) \}.
\end{align*}
\]
where \( z = \zeta h \). We note that basic quantities \( V_o, V_{\pm m}, \Theta_o, \Theta_m \) are all functions of \( S \) and \( m \), and independent of \( \nu, \Theta, z \).

(iii) Make the narrow gap assumption \((b - a) < \frac{1}{4}(b + a)\) and replace \( \tau^2 \) by \((b + a)^2/4\) where appropriate.

Neglecting conductivity effects and noting from the Appendix that \( V_{-m} = V_m = 0 \), the perturbation equations may now be written in the form

\[
\begin{aligned}
\frac{1}{2} \left( \frac{\partial}{\partial t} + V_o \frac{\partial}{\partial \theta} \right) u - v &= - \frac{\partial p}{\partial R}, \\
\frac{1}{2} \left( \frac{\partial}{\partial t} + V_o \frac{\partial}{\partial \theta} \right) v + u &= - \frac{\partial p}{\partial \theta}, \\
\left( \frac{2h}{b + a} \right)^2 \left( \frac{1}{2} \frac{\partial}{\partial t} + V_o \frac{\partial}{\partial \theta} \right) w - \left( \frac{2h}{b + a} \right)^2 \tau &= - \frac{\partial p}{\partial \zeta}, \\
\frac{1}{2} \frac{\partial}{\partial t} + V_o \frac{\partial}{\partial \theta} \right) r + w \left[ \frac{\partial \tau}{\partial \zeta} \right] &= 0, \\
\frac{\partial u}{\partial R} + \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial \zeta} &= 0,
\end{aligned}
\]  

(8)

where \( \tau = b \omega t \).

A solution will be sought in the form

\[
u, v, w, p, \tau = \sum_{n=-\infty}^{\infty} (u_n, v_n, w_n, p_n, \tau_n)(R, \zeta) \exp(i2\omega ft + n\theta), \]

(9)

where \( f \) is the non-dimensional frequency.

The Eq. (8) set now reduces to

\[
\begin{aligned}
i(f + nV_o)u_n - v_n &= - \frac{\partial p_n}{\partial R}, \\
i(f + nV_o)v_n + u_n &= - i n p_n, \\
\left( \frac{2h}{b + a} \right)^2 [i(f + nV_o)w_n - \tau_n] &= - \frac{\partial p_n}{\partial \theta^2}, \\
i(f + nV_o)\tau_n + \Theta_o \omega w_n + \Theta_o \Theta_m w_{m-n} e^{-i\phi} + \Theta_o \Theta_m w_{n+m} e^{i\phi} &= 0, \\
\frac{\partial u_n}{\partial R} + i v_n + \frac{\partial w_n}{\partial \zeta} &= 0,
\end{aligned}
\]  

(10)

(d) Boundary conditions

Boundary conditions involving zero slip (or zero stress in the case of a top surface) and zero heat transfer cannot be applied because the order of the equations has been reduced. The requirement of zero normal velocity at the sidewall, top and bottom boundaries leads to the conditions

\[
\begin{aligned}
\nu_n(0, \zeta) &= u_n(-R_o, \zeta) = 0, \\
\nu_n(R, \zeta) &= w_n(R, 0) = w_n(R, 1) = 0, \\
n &= 0, \pm 1, \ldots.
\end{aligned}
\]  

(11)

A separable solution for \( p_n \) exists in the form

\[
p_n \propto \cos(\lambda \pi \zeta) \left[ n \sin \left( \frac{\mu \pi R}{R_o} \right) - \frac{\mu \pi}{R_o} (f + nV_o) \cos \left( \frac{\mu \pi R}{R_o} \right) \right],
\]

where \( \lambda \) and \( \mu \) are integers.

Superposition of normal modes does not provide a complete solution to the problem since they correspond to a discrete spectrum of eigenvalues for \( f \). Pedlosky (1964) has considered the complete spectrum of eigenvalues for the Eady problem and concluded
that the standard normal mode analysis does yield the correct stability criteria for the flow even though these normal modes solutions do not form a complete set. This suggests that the normal mode analysis is worth presenting as a first step here.

Elimination of \( u_n, v_n, u_n, \tau_n \) from Eq. (8) leads to the system

\[
X_{n-m} p_{n-m} + Y_{n} p_{n} + Z_{n+m} p_{n+m} = 0, \quad n = 0, 1, \ldots
\]  

(12)

where the coefficients are defined by

\[
[1 - (f + nV_0)^2] X_n
\]

\[
= - \Theta_0 \Theta_m (1 - \beta_n) (f + nV_0) e^{-i\theta} \left[ \left( \frac{\mu \pi}{R_0} \right)^2 + n^2 + (\lambda \pi)^2 \left( \frac{b + a}{2h} \right)^2 \right],
\]

\[
[1 - (f + nV_0)^2] Y_n
\]

\[
= (f + nV_0)^2 - \Theta_0 (1 - \beta_n) (f + nV_0) \left[ \left( \frac{\mu \pi}{R_0} \right)^2 + n^2 + (\lambda \pi)^2 \left( \frac{b + a}{2h} \right)^2 \right],
\]

\[
[1 - (f + nV_0)^2] Z_n
\]

\[
= - \Theta_0 \Theta_m (1 - \beta_n) (f + nV_0) e^{-i\theta} \left[ \left( \frac{\mu \pi}{R_0} \right)^2 + n^2 + (\lambda \pi)^2 \left( \frac{b + a}{2h} \right)^2 \right],
\]

and where

\[
\beta_n = (\lambda \pi)^2 \left\{ (\lambda \pi)^2 + \left[ n^2 + \left( \frac{\mu \pi}{R_0} \right)^2 \left( \frac{2h}{b + a} \right)^2 \right]^{-1} \right\}.
\]

(13)

(14)

We note at this point that \( m \) is a given quantity in Eqs. (12) and (13).

\( e \) The frequency equation

The term \( p_n \) is associated with a travelling wave whose azimuthal wave number is \( n \).

The experimental evidence in Fig. 1 indicates that a steady wave régime with wave number \( m \) usually gives way to a wave régime with wave number close to \( m \) including for example adjacent wave numbers \( m \pm 1 \). It is reasonable to suppose that the most important values for \( n \) to assume are the integers from 1 to some value which is of the order of \( m \). It is proposed to retain perturbation wave numbers such that

\[
| n | \leq \frac{3}{2} m,
\]

(15)

and we write \( n = m(\eta \pm 1) \) where \( \eta \) lies between \(-\frac{1}{2} \) and \(+\frac{1}{2}\).

Making this type of approximation has the effect of modelling the simplest kind of wave interaction between a large-scale large amplitude primary wave and a finite number of large-scale small amplitude perturbation wave in which a single pair of sidebands \( n \pm m \) is induced to accompany wave number \( m \). System (12) becomes

\[
\begin{bmatrix}
Y_{m(\eta-1)} & Z_{m_0} & 0 \\
0 & X_{m(\eta+1)} & Y_{m(\eta+1)} \\
& & P_{m(\eta+1)}
\end{bmatrix}
\begin{bmatrix}
P_{m(\eta-1)} \\
\end{bmatrix}
= 0
\]

(16)

The determinant condition derived from Eq. (16) may be regarded as a frequency equation for \( f \) which is of ninth order in general. We seek to establish criteria which ensure the stability of the basic motion. The frequency equation factorizes if the quantity \( \beta_n \) is replaced by \( \beta \) where

\[
\beta = (\lambda \pi)^2 \left\{ (\lambda \pi)^2 + \left( \frac{\mu \pi}{R_0} \right)^2 \left( \frac{2h}{b + a} \right)^2 \right\}^{-1}.
\]

(17)

This simplification is equivalent to the assumption \((nR_0)^2 < (\mu \pi)^2\), valid when \( m \) is not too large and justified as follows. An experimental result of Hide (1958) gives the maximum number \( m^* \) of waves in the annulus in the form \( 3m^*(b - a)/2\pi(b + a) \simeq 1 \).
For a narrow gap annulus \( R_o \approx 2(b - a)/(b + a) \) and the maximum wave formula becomes \( 3m^* R_o/4 \pi \approx 1 \). It follows that

\[
\left( \frac{n R_o}{\mu \pi} \right)^2 \approx \left( \frac{1}{\mu} \right)^2 \left( \frac{2n}{3m^*} \right)^2 \left( \frac{2m^*}{m^*} \right)^2.
\]  

(18)

Now \( 1 \leq \mu \) by definition and \( |n| \leq 3m/2 \) by assumption (15). We may therefore neglect \( (n R_o)^2 \) compared with \( (\mu n^2)^2 \) provided that the wave number ratio \( m/m^* \) is sufficiently small. Our attention is thus restricted to wave change criteria at relatively low wave numbers. We note that both \( \beta \) and \( \beta_n \) sweep out the interval \((0, 1)\) as \( \lambda \) and \( \mu \) take all possible values for the disturbance modes.

We may finally set out the frequency equation as follows after withdrawing common factors

\[
(f + n V_o) \left[ \left( \frac{\mu \pi}{R_o} \right)^2 + n^2 + \left( \lambda \pi \right)^2 \left( \frac{b + a}{2h} \right)^2 \right] [1 - (f + n V_o)^2]^{-1}
\]

from the columns \( n = m \eta, m(\eta + 1) \):

\[
\begin{vmatrix}
(F - 1)^2 - x & -ye^{i\phi} & 0 \\
-y e^{-i\phi} & F^2 - x & -ye^{i\phi} \\
0 & -ye^{-i\phi} & (F + 1)^2 - x
\end{vmatrix} = 0,
\]

(19)

where

\[
F = \frac{f}{m V_o} + \eta,
\]

(20)

\[
x = \left( \frac{S_1}{S} \right) (1 - \beta) + \left( \frac{S_2}{S} \right) \beta,
\]

(21)

\[
y = \theta_m \left( \frac{S_1}{S} \right) (1 - \beta),
\]

(22)

\[
S_1 = \frac{S \theta_0}{m^2 V_o^2},
\]

(23)

\[
S_2 = \frac{S}{m V_o},
\]

(24)

3. Stability criteria for the wave régime

Inspection of Eq. (19) shows that the frequency equation may be written in the form

\[
F^6 - (2 + 3x)F^4 + (1 + 3x^2 - 2x^2)F^2 - (1 - x)[x(1 - x) + 2y^2] = 0,
\]

(25)

The necessary and sufficient condition for stability of the basic motion is that the roots of Eq. (25) must be real. These conditions are equivalent to the set of inequalities

\[
\begin{align*}
J(y, x) & \geq 0, \\
1 + 3x^2 & \geq 2y^2, \\
(1 - x)[x(1 - x) + 2y^2] & \geq 0,
\end{align*}
\]

(26)

where \( J(y, x) \equiv (2y^2)^3 + (12x - 17.75)(2y^2)^2 + (48x^2 - 52x + 2)(2y^2) + 4x(4x - 1)^2 \)

may be regarded as a cubic in \((2y^2)\) with coefficients dependent upon \( x \).

Our procedure runs as follows. First, condition (26) will be re-cast according as to whether \( x \geq 1 \) or \( x < 1 \). Then the \( x - y \) stability relations will be interpreted in terms of \( S - \beta \) criteria. Finally the stability curves \( S(m) \) will be tested for all \( \beta \).
Figure 2. Theoretical stability diagram of $2y^2$ against $x$. Shaded areas represent unstable values of the parameters.

(a) Stability curves in $x - y$ parameters

Inequalities in Eq. (26) reduce to

$$2y^2 \leq \begin{cases} x^2 - x, & x \geq 1, \\ 2y_0^2, & x < 1, \end{cases}$$

(27)

where $y_0$ is the largest non-negative zero of $f(y, x)$ such that

$$2y_0^2 \leq 1 + 3x^2.$$ 

(28)

The $x - y$ stability diagram is shown as Fig. 2 in which shaded areas represent unstable regions. For a given perturbation value $\beta$, $x$ is a decreasing function of $S$. As $S$ increases from a low value, $x$ decreases from a high value and the flow first becomes unstable when

$$2y^2 = x^2 - x.$$ 

(29)

We consider this to be the upper criterion for stability.

(b) The upper stability curve in $S - \beta$ parameters

By substituting for $x$ and $y$ from Eq. (21) and Eq. (22) into Eq. (29), the stability criterion may be set down as follows. The flow is stable with respect to perturbation value $\beta$ if $S \leq S_3(\beta)$ where

$$\frac{S_1}{S_3} \left( S_1 \left( 1 - 2 \frac{T_m}{T_{m2}} \right) - 1 \right) (1 - \beta)^2 + \frac{S_1}{S_3} \left( S_1^2 - 1 \right) + \frac{S_2}{S_3} \left( S_2 - 1 \right) \beta (1 - \beta) + \frac{S_2}{S_3} \left( \frac{S_2^2}{S_3^2} - 1 \right) \beta^2 = 0.$$ 

(30)

It may be shown that the necessary and sufficient condition for stability with respect to all perturbation modes $\beta \epsilon [0, 1]$ may be written

$$S_3 \leq \text{minimum} \ (S_1^* , S_2)$$

where the equation for $S_1^*$ is

$$S_1^* = S_1 [1 - 2\Theta^2_m(S_1^*)].$$

(31)

It may be seen in the Appendix that $S_1^*$ has the form $c_1/m^2$ and $S_2$ the form $c_2/m$ where $c_1$ and $c_2$ are constants. Putting together the lower criterion $S_0$ and the upper criterion $S_3$, we have finally
\[
(mR_0 + \frac{\pi^2}{mR_0})K \frac{2\omega(b - a)^2}{(m^2 + \omega^2)} \equiv S_0 \leq S \leq S_3 \equiv \text{minimum} \left( \frac{c_1}{m^2}, \frac{c_2}{m} \right)
\]  \hspace{1cm} (32)

(c) **Comparison of experimental with theoretical results**

Schematic theoretical stability curves are shown as Fig. 3.

![Theoretical stability diagram](image)

**Figure 3.** Theoretical stability diagram of S against wave number m. The upper critical curve corresponds to the curve \(2y^3 = x^2 - x\) in Fig. 2.

As has been suggested by other workers, the lower theoretical curve \(S = S_0(m)\) underestimates the corresponding experimental value at each \(m\). Guided by some lower transition results due to Brindley (1960) whose work is principally a viscous extension of the paper of Davies (1956), a modified form of \(S_0(m)\) has been entered in Fig. 3. Essentially the values of \(S_0(m)\) given by Eq. (32) have been increased by a factor of 100.

The curve \(S = S_0(m)\) decreases with increasing \(m\) indicating that lower wave numbers remain stable at higher values of \(S\), a feature in accord with the experimental transitions of Fig. 1. We may imagine that the wave transitions operate in the following way. Starting with a high \(S\), low \(m\) wave flow the heating contrast (and hence \(S\)) is lowered until the curve \(S_0\) is reached at which the particular wave number can no longer exist. In I Davies regards the critical curve \(S_0\) as a heating instability curve at which the wave pattern changes due to the amount of heat available to maintain a new form. We infer from Fig. 3 that as \(S\) is reduced still further, a higher wave number flow develops, which when \(S_0\) is again reached, gives way to a still higher wave number as \(S\) is lowered. This process continues until the maximum wave number \(m^*\) is reached for which, according to Davies, \(m^*R_0 \approx \pi\). (This estimate agrees quite well with the earlier experimental value due to Hide.) If at this point we continuously raise the heating contrast, \(S\) correspondingly increases and wave number \(m^*\) remains stable according to Fig. 3 until \(S_0(m^*)\) is reached at which the flow is dynamically unstable to small disturbances. Wave numbers lower than \(m^*\) are stable and as \(S\) is increased still further we infer from the diagram that a lower steady wave number state emerges. This process is continued as before until wave number one becomes unstable, at which point we infer that spiral flow is resumed.

It must be noted that the transition curve \(S\) cannot be properly applied to high wave number flow because of our approximation \(m/m^*\) small. Qualitative agreement is certainly apparent between Figs. 1 and 3. Quantitative agreement is much more tentative and depends upon values of \(c_1\) and \(c_2\), estimated in the Appendix. Using the results therein

\[
S_1^* = \frac{1.46}{m^2}, \quad S_2 = \frac{9.29}{m},
\]  \hspace{1cm} (33)

and it follows that \(S_3 = S_1^*\) in Fig. 3.
Differential heating in a fluid annulus

The upper theoretical curve $S_3$ gives fair agreement with the upper transition curves but it must be pointed out that the quantity $S_3(m)$ is sensitive to the value of $\varepsilon$ defined in the Appendix so the agreement may be fortuitous.

(d) Conclusion

Theoretical transitions have been found which qualitatively accord with the experimental transitions of Fultz. The position of the upper theoretical stability curve $S = S_3(m)$ depends upon the value of $\varepsilon$, which in turn depends on knowledge of viscous effects. An inviscid analysis cannot completely predict the stability characteristics of the flow and a non-linear viscous treatment of the basic steady flow is called for.

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Appendix

(a) Mean quantities of the steady régime

The solutions in $I$ for the dimensional velocity and temperature fields are

\[
T' = (\Delta T)_e + \frac{K(\Delta T)_e}{2\omega m(b - a)^2 S} \left\{ \frac{L}{A} + \phi \cos (m\theta - \psi) \right\},
\]

\[
rV' = \frac{K}{m} \frac{\partial \ell}{\partial R} + \frac{d \phi}{dR} \sin (m\theta - \psi) + \frac{B}{A} \frac{d \phi}{dR} \cos (m\theta - \psi)
\]

\[ (34) \]
where \((\Delta T)_v\) is the vertical temperature difference and the functions \(l(R, \xi), L(R, \xi), \phi(R), B(\xi)\) and \(A(\xi)\) remain to be discussed. We have taken here the simplest case \(\psi\) constant examined in \(I\).

Since the transport of westerly momentum is proportional to \(d\phi/dR\), this case corresponds to zero momentum transport. Davies points out that the detailed features of the stability diagram are not influenced by momentum transfer (in particular the curve \(S_1(m)\)) and we conclude that the case \(\psi\) constant, though not the most practical, contains the essential stability features of the model.

Define the mean of a function \(x(R, \xi)\) by \([x]\) where

\[
[x] = \frac{1}{R_o} \int_{-R_o}^{R_o} x(R, \xi) dRd\xi.
\]

On comparing Eq. (34) with the corresponding expressions in the set (7) and using Eq. (3) we obtain

\[
\Theta_o = S \left( \frac{b-a}{h} \right)^2 \frac{(\Delta T)_v}{(\Delta T)_h} + \frac{K}{2 \omega m h^2} \left[ \frac{\partial L}{\partial \xi} \right] \]

(36)

\[
V_o = \frac{2K}{m \omega (b+a)^2} \left[ \frac{\partial l}{\partial R} \right] \]

(37)

\[
\Theta_o \Theta_m = \frac{K}{4m \omega h^2} \left[ \phi \right] \left( \frac{1}{A(1)} - \frac{1}{A(0)} \right) \]

(38)

\[
V_m = \frac{K}{m \omega (b+a)^2} \left[ \frac{d\phi}{dR} \right] \left[ \frac{B}{A} \right] \]

\[
V_{-m} = \frac{K}{m \omega (b+a)^2} \left[ \frac{d\phi}{dR} \right] \]

(39)

Eq. (36) may be dealt with in approximate form since in \(I\) the function \(L\) is assumed to vary slowly with height. The term \(\partial L/\partial \xi\) will therefore be neglected in comparison with the term proportional to \((\Delta T)_v/(\Delta T)_h\).

For \(V_o\) we first establish a relation expressing the condition that the temperature difference across the cylinders is \((\Delta T)_h\) at some level. Substituting the integral of \((I\ 1.51)\) with respect to \(R\) into Eq. (34) leads to

\[
\int_{-R_o}^{R_o} (j - \frac{1}{4} \phi^2) dR = \frac{2 \omega m (b-a)^2}{K} SA(\xi^*),
\]

(40)

where \(\xi^*\) is the level at which the horizontal temperature difference is \((\Delta T)_h\). The result of Eq. (40) may be substituted into \((I\ 1.53)\) which gives an expression for \(\partial l/\partial R\). On substituting this expression into Eq. (37) we obtain

\[
V_o = \frac{4 \epsilon S (b-a)^2}{R_o (b+a)},
\]

(41)

where

\[
\epsilon = \left[ \frac{B}{A} \right] A(\xi^*).
\]

(42)

The function \(1/A(\xi)\) is proportional to the heat flow at the cylindrical boundaries. The simplest type for \(A(\xi)\) which permits a temperature phase change in the vertical and provides an atmospheric analogue in which the heat supplied at the Equator diminishes with height would appear to be given by

\[
\frac{1}{A(\xi)} = (1 - \xi)e^{-\xi}.
\]

(43)

If \((\Delta T)_h\) is defined to be the maximum horizontal temperature difference, then this occurs
at \( \zeta^* = 0 \), at which \( 1/A(0) = 1 \). Functions \( A(\xi) \) and \( B(\xi) \) are connected by the relation (1 1.54) namely

\[
\frac{d}{d\xi} \left( \frac{B}{A} \right) = \frac{1}{A}.
\]  
(44)

On performing the integrations of Eq. (44) using Eq. (43) we have

\[
e = 1 - 2e^{-1} + \delta,
\]  
(45)

where \( \delta \) is a constant of integration. In I \( \delta \) was left arbitrarily constant. For the case \( A(\xi) \equiv 1 \), an estimate for the corresponding \( \delta \) made by Rogers (1962) gave the value approximately equal to 0.1. In the absence of further evidence, it is proposed to adopt this value here. Consequently we take

\[
e = 0.17.
\]  
(46)

To evaluate \( \Theta_e \Theta_m \), we require the solution for \( \phi(R) \) supplied by (I 4.20) involving the elliptic function

\[
\phi = \frac{4Kk}{R_o} \text{sn}\left(-\frac{2KR}{R_o}, k\right),
\]  
(47)

where \( K(k) \) is the complete elliptic integral

\[
K = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{-1/2} d\theta.
\]

Performing the integration of Eq. (47) and substituting the result into Eq. (38) yields

\[
\Theta_e \Theta_m = -\frac{K}{2\omega m h^2 R_o} \ln \left( \frac{1 + k}{1 - k} \right),
\]  
(48)

since

\[
\frac{1}{A(1)} - \frac{1}{A(0)} = -1,
\]

Finally, since \( \phi(0) = \phi(-R_o) = 0 \), it is clear from Eq. (39) that \( V_m \) and \( V_{-m} \) are zero.

(b) Insertion of basic data into primary parameters

We use data from the experiments of Fultz (1959) and set

\[
\begin{align*}
& a = 25 \text{ mm}, \quad b = 49.5 \text{ mm}, \quad h = 130 \text{ mm}, \\
& \omega = 1.5 \text{ rad s}^{-1}, \quad K = 1.4 \times 10^{-4} \text{ mm}^2 \text{ s}^{-1}.
\end{align*}
\]  
(49)

We note that the slight difference between the angular velocities of the cylinders and the wave pattern has been neglected. The ratio of \((\Delta T)_e \) to \((\Delta T)_H \) is not given by the solution in \( I \).

For axially symmetric steady flow, Hide (1967) showed that

\[
(\Delta T)_e / (\Delta T)_H \approx 2/3,
\]  
(50)

and we shall use this value for the wave régime flow also. Two quantities \( S_1 \) and \( S_2 \) defined by Eqs. (23) and (24) become

\[
S_1 = \frac{(\Delta T)_e}{(\Delta T)_H} \frac{R_o(b + a)^4}{(b - a)^2 h^2 \omega^2} \cdot \frac{1}{m^2} \approx 2.04 \frac{m^2}{m^2},
\]  
(51)

\[
S_2 = R_o \left( \frac{b + a}{b - a} \right)^2 \cdot \frac{1}{m} \approx 9.29 \frac{m}{m}.
\]  
(52)

(c) Dependence of \( \Theta_m \) on \( S \)

Connecting together Eq. (48) with modified Eq. (36) leads to
\[ \Theta_m = -\frac{K}{2\omega(b-a)^2S\rho_0} \frac{(\Delta T)_H}{(\Delta T)_v} \ln \frac{1 + k}{1 - k} \]  

(53)

where from (4.30)

\[ S = \frac{K}{2\omega(b-a)^2\rho_0^2} (m^2R_0^2 + \sigma^2) \]  

(54)

and where \( \sigma^2 \) is an explicit function of \( k \) such that as \( k \to 1 \),

\[ \sigma^2 \sim 4 \ln \left( \frac{16}{1 - k^2} \right) \]  

(55)

In the range \( 0 \leq k < 1 \), \( -\Theta_m^2 \) is an increasing function of \( k \) and hence

\[ \Theta_m \to -(\Delta T)_H/4(\Delta T)_v = -3/8 \]  

(56)

When \( S = 0(S_0) \) then the data of Eq. (49) when substituted into Eq. (54) gives an order of magnitude relation for \( \sigma^2 \) in the form

\[ \frac{1}{m^2} \simeq \frac{10^{-4}}{m} \frac{(m^2R_0^2 + \sigma^2)}{m^2} \]

from which

\[ \sigma^2 = O(10^4/m) \]  

(57)

Hence the form Eq. (55) relating \( \sigma^2 \) to \( k \) may be submitted into Eq. (53) to yield finally

\[ \Theta_m = \frac{3}{8} \{ 1 - O(m^310^{-4}) \} \]  

(58)

(d) Approximation for \( S_1^* \)

The quantity \( 2\Theta_m^2 \) increases with \( S \) from a value zero at \( S = S_0 \) and tends to \( 9/32 \) as \( S \to \infty \). It follows from Eq. (31) that \( S_1^*/S_1 = O(1) \) for which values

\[ \frac{2\Theta_m^2}{9/32} = 1 - O(m^310^{-4}) \]  

(59)

Since our study is restricted to the case \( m < m^*/2 \) and since the highest values of \( m^* \) observed have been of the order of 15 in a narrow annulus it is clear that \( m^3/10^4 < 1 \). We may therefore neglect the second term on the right-hand side of Eq. (59) and set

\[ S_1^* \simeq \frac{23}{32} S_1 = \frac{1.46}{m^2} \]  

(60)