The geostrophic drag coefficient and the 'effective' roughness length

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SUMMARY

An 'effective' roughness length is defined for use over heterogeneous terrain as the roughness length which homogeneous terrain would have to give the correct surface stress over a given area. A method is suggested to compute geostrophic drag coefficient, wind-contour angle and surface heat flux, given this roughness length, latitude, geostrophic wind speed and insolation or ground-air temperature differences.

1. INTRODUCTION

In order to incorporate surface friction into large-scale mathematical models of the atmosphere, geostrophic drag coefficients and angles between surface wind and free-flow contours must be estimated. We will here define the geostrophic drag coefficient $c_D$ by $u_w/V_o$, where $u_w$ is the surface friction velocity and $V_o$ the geostrophic wind.

Under barotropic conditions, not too close to the Equator, the wind-contour angle and the geostrophic drag coefficient are known to be functions of the surface Rossby number, $Ro = V_o/\nu$, and a suitable measure of buoyancy, $\mu$, defined by

$$\mu = \frac{k u_w}{f L}.$$  \hspace{1cm} (1)

Here, $f$ is the Coriolis parameter, $z_o$ the roughness length, $k$ von Kármán's constant and $L$ the Monin-Obukhov length. Thus, for example, the Rossby number and the geostrophic drag coefficient are related by (Blackadar and Panofsky 1969)

$$\ln Ro = A(\mu) - \ln c_D + \left(\frac{k^2}{c_D} - B(\mu)\right)^{1/2}. \hspace{1cm} (2)$$

The functions $A(\mu)$ and $B(\mu)$ have been determined empirically, most recently by Clarke (1970a). These empirical estimates have a great deal of scatter. For practical purposes, they can be fitted by

$$A(\mu) = 4.5 \quad \text{for } \mu \leq -50$$

$$A(\mu) = 0.00144 \mu^3 - 0.144 \mu + 0.9 \quad \text{for } \mu > -50$$

and

$$B(\mu) = 1.0 \quad \text{for } \mu \leq -75$$

$$B(\mu) = 0.00052 \mu^4 + 0.093 \mu + 4.5 \quad \text{for } \mu > -75.$$  \hspace{1cm} (3)

A difficulty with applying the above expressions in practice is that the geostrophic wind, the required surface stress and $\mu$ represent space averages, over areas bounded by the grid points for the computing scheme used in the model. On the other hand, quantities like $z_o$ and $L$ have usually been determined from masts or towers, and are thus local parameters. The purpose of this note is to suggest space-average parameters which can be used to estimate the momentum loss of the air to the ground.

2. DEFINITION OF EFFECTIVE ROUGHNESS LENGTH

We will first consider situations of strong winds, where $\mu$ approaches zero. Then the drag coefficient and the wind-contour angle are functions only of $Ro$. The only parameter needed in these functions which is not usually a space average is $z_o$. Since we wish to determine the downward transport of momentum as function of $z_o$, we will now define the 'effective' roughness length as that roughness length which homogeneous terrain would have in order to produce the correct space-average downward flux of momentum near the ground, with a given wind near the ground.

In order to obtain the effective roughness length, then, we could measure the average stress...
near the ground over the areas in question, and then apply the theory of wind profiles in neutral air over homogeneous terrain to derive the roughness length. Although such roughness lengths will have been computed under neutral conditions, they will be assumed to be invariant with changing $\mu$.

As pointed out by Charnock (personal communication), it would have been more consistent, for applications to the large-scale problem, to define the effective roughness length in terms of geostrophic wind rather than the actual wind near the surface. However, geostrophic winds were not available and the roughness lengths derived in this paper should be essentially the same.

3. Computation of effective roughness length

Measurements are needed to yield average stresses over areas typically used in large-scale modelling, that is, with linear dimensions of order of several hundred kilometers. Such observations are lacking. However, analogous statistics can be derived from the plane-based measurements of project LO-LOCAT over 30 km line segments. If the terrain has some degree of large-scale homogeneity, then 30 km linear averages should not be too different from averages over areas of larger dimensions.

In project LO-LOCAT, the U.S. Air Force engaged the Boeing Company (Wichita) to fly aircraft at 75 m and 225 m three times a day from four bases for the purpose of obtaining turbulence statistics. The four locations were selected to cover a variety of topographic conditions, such as plains, desert, low mountains, high mountains, swamp and sea. From each location, the flights consisted of eight legs. Turbulence statistics were determined separately for 330 seconds of flight time (30 km) along each leg.

The four airplanes participating in this programme were equipped to measure the three orthogonal wind components with a minimum sampling interval of 10 m, and mean wind by Doppler-radar. In addition, infra-red surface temperature was measured along each leg. At the beginning and end of each leg, temperatures were measured at heights of 30 m and 300 m. For details of the instrumentation, see the report by the Boeing Company, Gunter et al. (1969).

The project reports provided computations of standard deviations of vertical velocities, but not, originally, momentum fluxes. However, there is considerable evidence that, in neutral air,

$$\sigma_w = a \cdot u_{sl}.$$  \hspace{1cm} (5)

Here, $u_{sl}$ is the local friction velocity. The agreement between different measurements of the constant $a$ in Eq. (5) in the surface layer is good; the best estimate for $a$ is about 1.3. The behaviour of the constant at greater heights is less certain. Yokoyama (1970) found negligible variation of the constant up to 400 m height. He gives the value as 1.15. However, this number is based on a regression equation of $\sigma_w$ on $u_{sl}$, leading to a relatively small value. Inspection of Yokoyama's observations suggests that a value of 1.3 fits at least as well. Further, stresses have now been determined from 6 runs of project LO-LOCAT at 225 m, yielding an average constant of 1.37. Therefore, Eq. (5) with $a = 1.3$ is believed to be consistent with all existing turbulence data in the atmospheric boundary layer, in neutral air.

The ratio of $\sigma_w$ to $u_w$ appears to be essentially independent of Richardson number $Ri$ for $-4 < Ri < 0.25$. For more unstable air, the ratio is larger.

From the large number of standard deviations of vertical velocity computed at 75 and 225 m under various meteorological conditions 'effective' roughness lengths can then be computed, given Eq. (5) and the shape of the wind profile, provided that near-neutral conditions can be assumed. In practice, the standard deviations published by LO-LOCAT personnel were first corrected for low-frequency and high-frequency filtering produced by the measurement and reduction procedures involved.

4. The wind profile up to 100 m

Under neutral conditions, the familiar logarithmic wind profile is valid in the surface layer, that is, up to 10 to 30 m. The logarithmic profile implies constant stress with height and linear increase of mixing length or 'eddy scale' with height. Actually, the stress slowly decreases with height, and the mixing length increases non-linearly, less rapidly than in the surface layer. Blackadar and Panofsky (1969) have constructed a wind profile theory, expected to be valid to at least 100 m, which is based on the assumption that the wind direction does not change with height.

The variation of stress with height can be obtained from the $x$-component of the equation of motion along the wind direction near the ground:
SHORTER CONTRIBUTIONS

\[ f(v - v_g) = -\frac{\partial}{\partial z} u_{\sigma} \]  \hspace{1cm} (6)

or, if we neglect the turning of wind in the lowest 100 m, from

\[ \frac{\partial}{\partial z} u_{\sigma} = f v_g . \]  \hspace{1cm} (7)

Here \( v_g \) is the component of the geostrophic wind at right angles to the surface wind direction. According to the theory of the planetary boundary layer (Blackadar and Tennekes 1968), \( v_g \) is given by the equation

\[ v_g = -u_0 B(\sigma)/k. \]  \hspace{1cm} (8)

Integration of Eq. (7), and substitution of Eq. (8) yields

\[ u_{\sigma} = u_0 \sqrt{1 - f z B / k u_0} . \]  \hspace{1cm} (9)

Expanding Eq. (9) in a Taylor series in \( f z \) leads to

\[ u_{\sigma} = u_0 - f z B / 2 k + (f z B)^2 / 8 k^2 u_0. \]  \hspace{1cm} (10)

In practice the quadratic term is always small. For example, under neutral conditions, with \( u_0 \) equal to 1 m s\(^{-1}\) and \( B \) about 5, in middle latitudes, the term is of order 0.2 cm s\(^{-1}\) at \( z = 100 \) m. This is completely negligible. We will therefore drop the quadratic term and write

\[ u_{\sigma} = u_0 - f z B / 2 k \]  \hspace{1cm} (11)

where \( B \) is given by Eq. (4). Thus, for example, in neutral air:

\[ u_{\sigma} = u_0 - 6 f z. \]  \hspace{1cm} (12)

The greatest uncertainty in this equation is not due to the omission of the quadratic term, but due to the scatter in the measurements of \( B \).

The differential equation for the wind profile can be written

\[ \frac{\partial u}{\partial z} = \frac{u_{\sigma}}{\lambda} \]  \hspace{1cm} (13)

Here, \( \lambda \) is a quantity which has dimensions of length and represents 'eddy size' or 'mixing length'. As shown from spectra of vertical motion (Busch and Panofsky 1968) the horizontal scale of vertical motion is proportional to height near the ground and, at higher levels, tends to a constant. Blackadar and his co-workers found that the same distribution is valid for the vertical distribution of \( \lambda \); in order to fit wind profiles through the planetary boundary layer, they have suggested that under neutral conditions

\[ \lambda = k z \left( 1 + \frac{k z f}{c u_0} \right)^{-1} \]  \hspace{1cm} (14)

The non-dimensional constant, \( c \), has the numerical value of 0.0063, determined so as to fit observed wind-contour angles and geostrophic drag coefficients.

After substitution of Eq. (12) and (14) into (13) and integrating, we find an expression for the wind profile (omitting a small term as \( f z^2 \)):

\[ u = \frac{u_0}{k} \ln \frac{z}{z_0} + f z \left( \frac{1}{c} - \frac{B}{2 k^2} \right) . \]  \hspace{1cm} (15)

Note that the effect of the variation of stress with height only contributes about 10 per cent to the second term on the right of Eq. (15). Hence the uncertainty in \( B \) is much less important than any uncertainty in \( c \).

With the previously given numerical estimates of \( B \) and \( c \), Eq. (15) becomes:

\[ u = \frac{u_0}{k} \ln \frac{z}{z_0} + 144 f z. \]  \hspace{1cm} (16)

Eq. (16) should be generally useful in the analysis of strong winds over homogeneous terrain, up to 100 m. For example, if \( u - 144 f z \) is plotted as function of \( \ln z \), the slope will be \( u_0/k \) and the intercept \( z_0 \). This equation was derived in a different manner by Blackadar (see Blackadar et al. 1969).
Some more recent unpublished work on tower wind profiles suggests that the deviation from the logarithmic profile is actually less than 144 $\text{fs}$. This discrepancy would imply that the hypothetical distribution of $\lambda$ as given by Eq. (14) is not correct. However, this matter has not been settled.

Eq. (16) is valid only for purely mechanical turbulence. The observations of $\sigma_{w}$, on the other hand, covered unstable and neutral conditions. In order to make use of Eq. (16), the assumption was made that for wind speeds of and above 15 m s$^{-1}$ the contribution of buoyancy to the turbulence could be neglected. Therefore, the observed ratio $\sigma_{w}/u$ was plotted as function of wind speed, a smooth line was drawn through the points, and a value of $\sigma_{w}/u$ was read from this curve at a wind speed of 15 m s$^{-1}$.

With this wind speed, and a height of 75 m, Eq. (16) can be written after substitution of Eq. (5)

$$\ln \frac{z}{z_{o}} = \frac{0.48}{\sigma_{w}/u + 0.004}.$$  \hspace{1cm} (17)

From this equation, effective roughness lengths $z_{o}$ were derived. These, and the $\sigma_{w}/u$ ratios are given in Table 1.

<table>
<thead>
<tr>
<th>Table 1. Effective roughness lengths</th>
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<tbody>
<tr>
<td>$\sigma_{w}/u$</td>
</tr>
<tr>
<td>Plains</td>
</tr>
<tr>
<td>Low mountains</td>
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<tr>
<td>High mountains</td>
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In the case of high mountains, the limit of $\sigma_{w}/u$ was poorly determined from the relatively few observations, and hence the roughness length is extremely uncertain.

The roughness lengths in Table 1 must be regarded just as preliminary estimates. There are many reasons why Eq. (17) may not be adequate. First, actual wind profiles may be more nearly logarithmic than Eq. (16) suggests. Subject to the extreme assumption that the logarithmic profile is valid up to 75 m height, the roughness lengths would have been 0.32, 0.56 and 0.87 m, respectively, in the three categories. The true values are probably intermediate.

Equally serious, perhaps, is the assumption that near-neutral conditions can be assumed when the wind at 75 m is 15 m s$^{-1}$ or more; all observations were made at day time, and the convective contribution to turbulence at 75 m may on occasion have been significant. Still, Eq. (5) is valid up to negative Richardson numbers as large as 0.4.

The hypothesis that heat convection can be neglected with strong winds is strengthened by dispersion experiments. Pasquill (1962) suggests that, even with wind speeds (at 10m) as low as 6 m s$^{-1}$, convection becomes important only under conditions of extremely large insolation. With much higher wind speeds as used here, the effect of mechanical turbulence must be even more dominant.

Eventually, effective roughness lengths should be obtained from directly measured area-average Reynolds stresses in a sufficient number of regions so that good estimates of effective roughness lengths can be made over all land surfaces.

Given the effective roughness length and the geostrophic wind, the surface stress in neutral air can now be found from Eq. (2), with values of $A$ and $B$ given in Eq. (3) and (4). Similarly, the cross-contour angle $\alpha$ follows from

$$\sin |\alpha| = \frac{B}{k} c_{D}.$$  \hspace{1cm} (18)

5. APPLICATION TO THE LARGE-SCALE PREDICTION PROBLEM IN UNSTABLE AIR

In order to estimate geostrophic drag coefficients and wind-contour angles under general conditions, planetary Richardson numbers $\mu$ must be estimated, again, as area averages. The selection of the 'external' parameters, from which $\mu$ can be determined, is not unique and none of
the obvious parameters are completely satisfactory. In the following, two different possibilities will be given.

From the definition of $\mu$, see Eq. (1), it follows, that

$$\mu = \frac{H}{f V_\theta^2 c_\theta} \left( \frac{\theta}{\theta} \right)$$

where $g$ is gravity, $\rho$ density, $\theta$ potential temperature, and $c_\theta$ specific heat at constant pressure. For constant $\mu$, $H/f V_\theta^2$ is only a rather slowly varying function of $V_\theta$ and $I$ only. Assuming that over land the turbulent flux of sensible heat is dominantly controlled by the insolation $I$, it is suggested here that it is perhaps sufficient to regard $\mu$ as a function of $V_\theta$ and $I$ only. Thus Fig. 1 shows how $\mu$ is related to these two variables. This figure is based mostly on observations at O'Neill, Nebraska (see Lettau and Davidson 1957) and at Cape Kennedy (supplied by NASA). Obviously, the figure cannot be used at night. During windy nights, $\mu$ is effectively zero and on light-wind nights, perhaps frictional effects can be neglected altogether.

![Figure 1. Isotherms of $\mu$ as a function of insolation $I$ and geostrophic wind velocity $V_\theta$.](image)

Given $\mu$ from Fig. 1, $A(\mu)$ and $B(\mu)$ can be found from Eqs. (3) and (4). Further, with the effective values of $\alpha$, the geostrophic drag coefficient $c_\theta = u_\theta/V_\theta$ is determined. The wind-contour angle $\alpha$ then follows from Eq. 18. Finally, the surface flux of sensible heat can be found from the definition of $\mu$, which can be written

$$H = \frac{\mu f u_\theta^2 c_\theta \rho \theta}{g h^3}$$

On some occasions, the surface temperatures may be available from radiation measurements (from satellites or airplanes). In numerical prediction models they are sometimes determined from the radiation balance of the Earth's surface and are therefore also available for future time steps. Although these results are not extremely accurate and show errors up to a few degrees, may be used to find $\Delta \theta$, the difference of the temperature $\theta$, at the top of the boundary layer (known in numerical large-scale prediction models) and $\theta_0$, the temperature at the Earth's surface. With $\Delta \theta$ as an 'external' parameter, $\mu$ is then

$$\mu = \frac{H}{f V_\theta^2 c_\theta} \left( \frac{\theta}{\theta} \right)$$

where $T_\theta$ is the scaling temperature $-H/c_\theta \rho u_\theta k$. $T_\theta/\Delta \theta$ is again a function of Rossby number $R_o$ and $\mu$, which was given by Monin and Zilitinkevich (1967) in the form

$$\frac{T_\theta}{\Delta \theta} = \frac{1}{\ln R_o c_\theta - C(\mu)}$$

Here, $C(\mu)$ is another function of $\mu$ which also has been given by Clarke (1970a). A curve fitted to
his empirical estimates, can be expressed algebraically by

$$C(\mu) = \begin{cases} 7.5 & \text{for } \mu \leq -100 \\ -0.0065 \mu^2 - 0.13 \mu + 1.0 & \text{for } \mu > -100. \end{cases} \quad (23)$$

Eqs. (2), (21), and (23) form a set of three equations with five variables. The two dependent variables $c_D$ and $T_e/\Delta \theta$ can be eliminated to give $\mu$ as a function of $Ro$ and $S$ defined by $\frac{g \Delta \theta}{\theta f V_\theta}$, formed from 'external' parameters only. As a result, and in view of Eqs. (2) and (18), $c_D$ and the wind-contour angle are functions only of $S$ and $Ro$. Clarke (1970b) has produced nomograms for these functions. (These nomograms are reproduced here as Figs. 2 and 3 with Clarke's permission. They

![Figure 2: A nomogram showing the quantity $V_\theta/u_\star$ as a function of $S$ and $\log_{10}Ro$ $S = (g \Delta \theta/\theta + \epsilon g \Delta q)/(f V_\theta)$. Here, this $\epsilon$ is 0.62 and $\Delta q$ the difference between moisture at the ground and moisture at the free atmosphere.](image)

![Figure 3. A nomogram showing the angle $\alpha_\theta$ (degrees) between the wind at the top of the boundary layer, and the surface stress in terms of $S$ and $\log_{10}Ro$.](image)
also contain corrections for the effect of water vapour on buoyancy.) Eq. (22) can then be used in the form

$$H = -\rho c_p \frac{u^*_k}{\ln \text{Ro} \ c_p - C(\mu)} \Delta \theta$$  \hspace{1cm} (24)

to evaluate the flux of sensible heat near the ground. Use of Eq. (24) will, of course, result in more accurate values for the heat flux $H$ than estimates from Eq. (20) will be, because an error in $\mu$ of a factor of two will change $H$ in unstable situations only by 10 to 20 per cent and in stable conditions at most 50 per cent. Heat fluxes during stable conditions are, however, almost negligibly small so that the increasing error produced by the uncertainty of $\mu$ is not serious. Alternatively, $H$ can be found from Clarke’s nomogram of $(-\rho c_p V_g \Delta \theta H)^{1/2}$ as function of $\text{Ro}$ and $S$. (This nomogram is reproduced here as Fig. 4.)

Figure 4. A nomogram showing the quantity $(-\rho c_p V_g \Delta \theta H)^{1/2}$ as a function of $S$ and $\log_{10} \text{Ro}$.

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