The surface drag contribution to the depth of Atlantic, and European, depressions

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SUMMARY

Estimates are made of the forced $\omega_F$-components arising from surface frictional inflow within the circulation of an 'average' Atlantic and an 'average' European depression. When allowance is made for the high curvature flow by the inclusion of a centripetal acceleration term in the equation of motion, it is found that the $\omega_F$-contribution at the top of the boundary layer, is much the same in each case. The quasi-geostrophic $\omega$-equation is solved to obtain the three-dimensional $\omega_F$-field accompanying the average depression. This gives a value of $\omega_F$ at the centre of the depression, and using this in the geostrophic vorticity equation it is found that surface friction makes a positive contribution of about 4 mb day$^{-1}$ to the central pressure tendency in each case.

1. INTRODUCTION

It is established from numerical weather prediction and general circulation studies that surface frictional drag is a significant dynamical process. However, quantitative estimates of its effect upon the deepening and filling of depressions are not widely given. Although such estimates might best be obtained from extensive calculations with good numerical models involving different methods of parameterization it is of interest to carry out a study by more elementary methods. The concept adopted is the familiar one of a boundary layer of limited thickness in a steady state such that the surface drag is balanced by the Coriolis acceleration of the associated ageostrophic motion integrated through the layer. The divergence of the ageostrophic component then provides a value of $\omega = dp/dt$ at the lower boundary of the dynamical model, taken as 1,000 mb. The surface pressure tendency is obtained, by approximation, from the 950 mb height tendency computed dynamically from an ' $\omega$-equation'. In this respect, it is, perhaps, worth remarking that the pressure tendency at the surface cannot be derived as the hydrostatic effect of the frictional convergence as this is accompanied by an almost balancing divergence above the boundary layer - the surface pressure tendency contribution is the result of the small imbalance between the two.

2. THE SELECTED AVERAGE DEPRESSIONS

In order to investigate the effect of differences between the average values of the surface drag coefficients for the north-east Atlantic Ocean and Europe, calculations are applied to two 'average' depressions near their state of maximum depth. These 'average' depressions were obtained by taking the mean radii and mean maximum depths of two large samples of depressions which were almost wholly over the north-east Atlantic Ocean or Europe, respectively, during 1965, 1966, 1967. A discussion of these statistics, obtained from working charts stored at the Meteorological Office, Bracknell, is given elsewhere (Dodd 1971).

For the present discussion they simply provide an 'average' depression which is reasonably sufficient for the purpose. The mean radius and mean maximum depth of the Atlantic depressions are 718 km and 21-3 mb respectively, the similar values for the European depressions being 758 km and 14-1 mb.

3. THE CALCULATION OF $\omega$

The basic equation is that of Sawyer (1959)

$$\omega_F = \frac{g \rho}{f} \left\{ \frac{\partial}{\partial y} [kV_x (u_x \cos \alpha - v_y \sin \alpha)] - \frac{\partial}{\partial x} [kV_y (u_x \sin \alpha + v_y \cos \alpha)] \right\} .$$

(1)
g being the gravitational acceleration, \( \rho \) the density, \( f \) the Coriolis parameter, \( k \) the geostrophic drag coefficient, \( x, y \) horizontal co-ordinates, \( \alpha \) the angle between the geostrophic and surface winds, and \( V_s \) being defined by \( V_s = \sqrt{u_s^2 + v_s^2} \). For the circulations of deep depressions the geostrophic approximation used in the derivation of Eq. (1) is a poor one, and the equation is, therefore, modified to apply to gradient wind conditions, including a centrifugal acceleration term. Thus, writing the equation of motion, with no tangential accelerations, as

\[
\frac{\mathbf{V}}{r_s} \mathbf{k} \times \mathbf{V} = \frac{1}{\rho} \frac{\partial \tau}{\partial z} - f \mathbf{k} \times \mathbf{V} + f \mathbf{k} \times V_t .
\]  

(2)

\( \mathbf{V} \) being the horizontal wind velocity, \( r_s \) the radius of the particle trajectory, \( \mathbf{k} \) the unit vertical vector, \( \tau \) the frictional stress, and \( z \) the vertical co-ordinate, and approximating \( V/\tau_s \) by \( V_{gr}/\tau_s \), \( V_{gr} \) being the gradient wind, defined by \( V_{gr} = (rf/2)(-1 + \sqrt{1 + 4r^2/\tau f}) \), and \( \tau_s \) the isobaric radius, assumed to be the trajectory radius. Eq. (2) becomes

\[
f \mathbf{k} \times \mathbf{V}^* - f \mathbf{k} \times V_\theta = \frac{1}{\rho} \frac{\partial \tau}{\partial z} .
\]  

(3)

where \( \mathbf{V}^* = \mathbf{V} (1 + V_{gr}/\tau_s f) \).

Integrating through the boundary layer and taking the divergence of both sides of Eq. (3)

\[
\int_0^H \nabla \cdot \rho \mathbf{V}^* \, dz = \int_0^H \nabla \cdot (\mathbf{k} \times \tau_0) .
\]  

(4)

the subscripts 0, \( H \) referring to the base and top of the boundary layers, respectively.

Assuming \( \tau_0 = k \rho V_{gr} \), and that the surface wind makes an angle \( \alpha \) with the isobars, Eq. (4) gives, for a circular depression:

\[
\omega_{FHF} = \frac{-k \rho V_{gr} \cos \alpha}{f(1 + V_{gr}/\tau f)} \left( \frac{V_{gr}}{\tau_s} + 2 \frac{\partial V_{gr}}{\partial \tau} \right) .
\]  

(5)

The main difficulty in using Eq. (3) is the choice of an appropriate value for the geostrophic drag coefficient, \( k \). This is a controversial matter which continues to receive much attention. In the event, the values given by Cressman (1960) were chosen. These are estimated average values, according to geographical position, and an average of the Cressman values, at the centre of each depression, was adopted for the average depression of each type, i.e. Atlantic and European. In accordance with Sawyer (1959) \( \alpha \) was made 30° over Europe, and 60° over the Atlantic.

The results of the calculations using both Eq. (1) and Eq. (5) are given in Table 1. These

| TABLE 1. VALUES OF \( \omega_{FHF} \) AT THE HALF-RADIUS OF AN AVERAGE ATLANTIC, AND AN AVERAGE EUROPEAN, DEPRESSION |
|-----------------|-----------------|-----------------|
| Equation used   | North-east Atlantic | Europe          |
|                 | \( \omega_{FHF} \) (mb s\(^{-1} \times 10^{-3}) | \( \omega_{FHF} \) (mb s\(^{-1} \times 10^{-3}) |
| Eq. (1)         | -1.3            | -1.0            |
| Eq. (5)         | -0.7            | -0.7            |

show the importance, in regions of large streamline curvature, of the inclusion of the centrifugal acceleration term and the gradient flow approximation in the basic equations. However, the most striking result is that, when allowance is made for the curvature of the flow, the \( \omega_{FHF} \)-values are not significantly different for the average Atlantic or European depression. This result is consistent with the view, Kung (1968), that the effect of a large drag coefficient on the stress value is nearly cancelled by the smaller average wind speed which accompanies it.

4. ANALYTICAL SOLUTION OF THE \( \omega \)-EQUATION \( \omega_F(r,p) \)

If it is assumed that the isobaric configurations for an average depression are circular, the \( \omega \)-equation, with an external surface drag forcing function, but no internal forcing functions, can be solved analytically, subject to certain assumptions.
The forced component of the ω-equation may be expressed

$$\nabla^2 (\omega_F) - \frac{f^2}{g} \frac{\partial^2 \omega_F}{\partial p^2} = 0$$

(6)

σ being the static stability, defined by $\sigma = (1/gp) \partial \theta / \partial p$,

θ being the potential temperature.

Rewriting the equation in cylindrical co-ordinates, r, θ, p,

$$\gamma^2 \frac{\partial^2 \omega_F}{\partial p^2} = -\left( \frac{\partial^2 \omega_F}{\partial r^2} + \frac{1}{r} \frac{\partial \omega_F}{\partial r} + \frac{1}{\tau^2} \frac{\partial^2 \omega_F}{\partial \theta^2} \right)$$

(7)

γ being defined by $\gamma^2 = -f^2/\sigma g$, a, taken as constant. For a circular depression, radius a, the boundary conditions are assumed to be

$$\omega_F (r, \theta, 0) = 0$$

(8)

$$\omega_F (a, \theta, p) = 0$$

(9)

$$\omega_F (r, \theta, p) = \omega_F (r)$$

(10)

Assuming circular symmetry $\omega_F$ may be expressed

$$\omega_F = R(r) P(p)$$

Substitution in Eq. (7) gives:

$$\frac{\gamma^2}{p} \frac{d^2 P}{dp^2} + \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{\tau R} \frac{dR}{dr} = -Q^2$$

(11)

Q being a constant.

Eq. (11) gives

$$\frac{d^2 P}{dp^2} = \left( \frac{Q}{\gamma} \right)^2 P = \delta^2 P$$

(12)

δ being defined by $\delta = Q/\gamma$, and

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + Q^2 R = 0$$

(13)

Eq. (13) has solutions in Bessel functions. However, a simple case, which is not unrealistic, is obtained by assuming the boundary condition expressed by Eq. (10). The complete solution is

$$\omega_F (r, p) = S J_0 (Qr) (e^{\delta p} - e^{-\delta p})$$

(14)

S being a constant.

The Bessel function formulation is only a possible model, but it is convenient, and probably realistic. To illustrate this point $J_0(X)$ is plotted against X in Fig. 1 and shows a kind of variation.

![Graph](image)

Figure 1. Graph of the first order Bessel function $J_0(X)$ against X.
with \( X \) that would be expected for \( \omega_F \), i.e. rising from zero on the circumference of the depression to a maximum value at its centre. Eq. (14) has been evaluated for 100 mb levels between 1,000 mb and 600 mb, the results obtained, for an average Atlantic and European, depression, being presented in Table 2. A value for \( S \) was obtained by substituting \( \omega_F \) (350 km, 1,000 mb) = \(-0.7 \times 10^{-3}\) mb s\(^{-1}\), Table 2, 350 km being the depression half radius. The value used for

| TABLE 2. Variation of \( \omega_F \) with height and different static stabilities |
|-----------------|-----------------|-----------------|
| pressure (mb)   | with            | with            |
|                 | \( \sigma = -1.5 \times 10^{-7} \) c.g.s. | \( \sigma = -1.5 \times 10^{-8} \) c.g.s. |
| 1,000           | -0.70           | -0.70           |
| 900             | -0.69           | -0.61           |
| 800             | -0.35           | -0.55           |
| 700             | -0.24           | -0.45           |
| 600             | -0.17           | -0.37           |

Calculations were made at the half radius 350 km of an average depression. \( \sigma = -1.5 \times 10^{-7} \) c.g.s. corresponds to the static stability parameter used by the Meteorological Office in their 3-level model.

\[ \delta = 3.4 \times 10^{-8} \]

\[ Q = 3.4 \times 10^{-8} \]

the static stability, \( \sigma \), was that corresponding to the value of \( \Gamma \) used by the British Meteorological Office (Bull 1966), where \( \Gamma \) is defined by \( \Gamma = g \sigma / R \), \( R \) being the gas constant; \( p \) was taken as 800 mb, i.e. the mid-pressure of the layer.

It is seen from Table 2 that \(| \omega_F |\) decreases rapidly with height, being only half its 1,000 mb value at 800 mb. Tables 2 and 3 illustrate the effect on \( \omega_F \) of varying \( \sigma \) and \( r \). It is seen that a reduction in the magnitude of \( \sigma \) by a factor 10 significantly reduces the damping of \(| \omega_F |\) with height; and that \(| \omega_F |\) decreases more rapidly with height for small depressions than for large ones.

| TABLE 3. Effect of depression size upon \( \omega_F \) |
|-----------------|-----------------|-----------------|-----------------|
| pressure (mb)   | \( \frac{1}{2} \) radius of depression | \( \frac{1}{2} \) radius of depression | \( \frac{1}{2} \) radius of depression |
|                 | 175 km | 350 km | 700 km |
| 1,000           | -0.70 | -0.70 | -0.70 |
| 900             | -0.27 | -0.49 | -0.58 |
| 800             | -0.13 | -0.35 | -0.48 |
| 700             | -0.07 | -0.24 | -0.39 |
| 600             | -0.03 | -0.17 | -0.31 |

Calculations used

\[ \sigma = -1.5 \times 10^{-7} \] c.g.s.

5. Surface pressure change produced by surface friction

For a depression near its state of maximum depth the advection of vorticity near the surface is small and the vorticity equation can be approximated by

\[
\nabla^2 \frac{\partial \omega}{\partial t} = \frac{f^2 \omega}{g \partial p}.
\]
Since Eq. (15) is linear in $\phi$ it is possible to evaluate the height tendency resulting from the effect of surface drag alone. Assuming a sinusoidal variation in $z$, i.e. $z = M \cos 2 \pi x/L \cos 2 \pi y/L$, $M$ and $L$ being constants, Eq. (15) gives

$$\frac{\partial z}{\partial t} = -\frac{f^2 L^2 \partial \phi}{8 \pi^2 g \partial p}.$$  

Taking $L = 4a$, Eq. (16) gives

$$\frac{\partial z}{\partial t} = -\frac{2a^2 f^2 \partial \phi}{\pi^2 g \partial p}.$$

or

$$\frac{\partial p}{\partial t} = -\frac{2a^2 f^2 p (\partial \phi_{1000} - \partial \phi_{500})}{\pi^2 100}.$$  

$p_0$ being the pressure at the centre of the depression, the subscript numbers, 1,000 and 900 signifying the 1,000 mb and 900 mb levels. Using the data derived from Eq. (5), Table 1 and Eq. (14) a positive pressure tendency of about 4 mb day$^{-1}$ is obtained. This value may be compared with actual average rates of deepening, and filling, calculated for the 1965, 1966, 1967 sample of depressions. The average European depression is found to deepen by 9-5 mb day$^{-1}$ in the 24 hr period immediately prior to maximum depth, and fill by about 5-5 mb day$^{-1}$ in the 24 hr period immediately following; the similar values for an average Atlantic depression are 11 mb day$^{-1}$, and 7 mb day$^{-1}$ respectively.

6. Concluding remarks

The computations described show that surface frictional drag in deep depressions provides a significant contribution to the surface pressure tendency which is of similar magnitude to the total rates of pressure change. This is the rate at which the depressions would fill if other dynamical processes were suddenly removed. The fact that the actual rate of filling, following maximum depth, is greater than that contributed by the surface drag may suggest that the drag coefficients chosen are too small, or, as is generally believed, that there are other dissipative processes in the free atmosphere. It is found that the smaller drag coefficient is apparently fully compensated by the stronger winds that accompany Atlantic, as compared with European, depressions.

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References


