Ekman boundary layer interactions in a numerical model of the general circulation

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SUMMARY

A representation of an 'Ekman' type of boundary layer (assumed to extend from 1,000 mb to 900 mb) is inserted into a two-level, quasi-geostrophic, general circulation numerical model. An approximate time-dependent analytic solution of this boundary layer is matched to the 'free atmosphere' regime of the model through the vertical velocities produced at the grid points. By using this analytic solution the essential physical characteristic of the observed continuous increase with height of scale length of turbulence in the boundary layer is represented. By comparing time integrations of two models, only one of which contains an Ekman layer, it is shown that large-scale flow characteristics, expressed in terms of growth rate of zonal and eddy kinetic energy, are sensitively dependent on the presence of such a layer in a numerical model.

1. Introduction

Delsol, Miyakoda and Clarke (1971) have recently studied in a numerical model the effects, on the scale of the general circulation, of parameterizations of various processes related to boundary layer physics. They evaluate the comparative effects of taking different roughness parameters over land and sea; they include Monin-Obukhov treatments of turbulent transfer processes in constant flux surface layers, and they use a Richardson number dependent parameterization for Ekman layer processes. However, they use the discret level method of numerical models. This method leads of course to a good simulation of processes whose characteristic length scale is considerably larger than the grid distance; e.g., baroclinic waves of length 4,000 km are adequately resolved in a general circulation model using a 300 km 'horizontal' grid spacing, but one of the main remaining unsolved problems consists of representing sub-grid scale systems. In the Delsol-Miyakoda-Clarke model the vertical mixing length is assumed to be 30 m and yet the first sampling level is at 75 m, the next is at 640 m; consequently the computing model used by these authors is not capable of resolving the characteristic vertical component of eddy size in the constant stress and so-called 'Ekman' layers of the model. In fact it is generally considered (see e.g. Lumley and Panofsky 1964) that the characteristic eddy size of the mixing system in the planetary boundary layer is a linear function of height above the surface; recently Taylor, Warner and Bacon (1971) have suggested that it varies with the square root of the height. A basic physical characteristic of the boundary layer then is that the scale length of eddy size increases continuously with altitude, and consequently it would seem that a valid mathematical description can only be achieved through an analytic solution. From a large scale point of view the important question of course is to determine with what degree of refinement do we need to parameterize the constant stress and 'Ekman' boundary layers in order to describe adequately the rate of growth of large-scale flow characteristics. It is not possible to answer this particular question by using a two-level model, but by comparing the results of two model systems, of the type used by Everson and Davies (1970), one of which contains an analytic Ekman representation and the other does not, it can be demonstrated that large-scale flow characteristics of the model are sensitively dependent on the presence of such a layer.

2. A time dependent analytic solution of the 'Ekman' type boundary layer

The planetary boundary layer exists in the atmosphere in various forms, varying strongly with changes in the large scale flow pattern; its definition and main known characteristics have been discussed recently by Charnock and Ellison (1967) and by Sheppard (1970). Although an extremely important interaction takes place between the surface and the upper reaches of the troposphere through the agency of deep convection, the boundary layer commonly consists of
(a) a constant stress zone, and (b) an ‘Ekman’ type layer. The two-level models used recently by Everson and Davies are limited by a constant static stability condition and are only capable of examining the Ekman component of the boundary layer. However, this is an important element in atmospheric dynamics and the model is capable of providing an initial approximate estimate of the extent of dependence of changes in large-scale flow characteristics in a model on Ekman boundary layer parameterization, by simply noting the differences produced in previous results due to the introduction of a boundary layer representation.

The mathematical-computing procedure used in carrying out this exercise consists basically of matching an analytic boundary layer solution to a numerical model. It is of general interest, as it is clearly capable of extension to high vertical resolution systems.

The boundary layer assumptions on which the numerical experiment is based are as follows.

(a) The non-linear inertia terms are small compared to the free atmosphere values and are therefore neglected in the boundary layer but not of course in the ‘free’ atmosphere numerical model.

(b) Constant static stability is assumed, and therefore Monin-Obukhov variations (of the type discussed by Delsol, Miyakoda and Clarke) are not included at this stage. (c) The boundary layer is taken to consist entirely of a constant thickness ‘Ekman’ layer from 1,000 mb to 900 mb and covering the whole domain. (d) A constant eddy viscosity is assumed. The basic philosophy of course is to establish an initial working model on which we can construct higher resolution systems. Even in the absence of deep convection effects, the results in fact suggest that the large scale free atmosphere – boundary layer interaction is sensitively dependent on a boundary layer representation.

The basic equations used in the boundary layer are, using the usual convention,

\[
\frac{\partial u}{\partial t} - fu = -\frac{1}{\rho} \frac{\partial p}{\partial x} + K \frac{\partial^2 u}{\partial z^2},
\]

(1)

\[
\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} + K \frac{\partial^2 v}{\partial z^2},
\]

(2)

and

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,
\]

(3)

where \(K\) denotes a constant eddy viscosity, representing turbulence in the Ekman layer (despite the difficulties inherent in using this concept, it does not seem possible at the present time to formulate a predictive scheme in any other way), and \(w\) is taken to be small compared to \(u\) and \(v\).

The terms \(K \frac{\partial^2 u}{\partial z^2}\) and \(K \frac{\partial^2 v}{\partial z^2}\) in Eqs. (1) and (2) are taken to represent vertical fluxes in the Ekman layer of horizontal momentum, averaged spatially over the horizontal grid spacing and over the period of a time step. From atmospheric diffusion theory (Pasquill 1962) we would expect \(K\) to be a function of \(z\), consistent with a strong dependence of all surface turbulent processes on scale height. However, for simplicity of analysis in setting up an initial model, \(K\), due to surface turbulence, has been kept constant up to the top of the boundary layer, \(z = h\), and zero at \(z \geq h\).

In order to obtain an analytic solution we now express \(u, v, w\) in the forms

\[
u = u_0 e^{i\sigma t}, \quad v = v_0 e^{i\sigma t}, \quad p = p_0 e^{i\sigma t},
\]

(4)

where \(u_0, v_0, p_0\) are functions of \(x, y, z\) and \(\sigma\) will vary from one time step to another in the numerical model but kept constant over the period of each time step: these steps turn out in the computational scheme of the numerical free atmosphere part of the model to be very much smaller than the period of \(\sigma\). Substitution of Eq. (4) into Eqs. (1) and (2) and dropping the suffixes leads to

\[
\frac{i}{\sigma} u - fu = -\frac{1}{\rho} \frac{\partial p}{\partial x} + K \frac{\partial^2 u}{\partial z^2},
\]

(5)

\[
\frac{i}{\sigma} v + fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} + K \frac{\partial^2 v}{\partial z^2},
\]

(6)

the terms \(i\sigma u\) and \(i\sigma v\) being additional to the classical Ekman formulation. Using \((u_\sigma, v_\sigma)\) to denote component velocities at the height \(z = h\) (corresponding to 900 mb), where we take the Ekman \(K\) to be zero, they can be regarded as geostrophic. Expressing \(u_\sigma\) and \(v_\sigma\) in terms of the appropriate pressure gradients, Eqs. (5) and (6) can then be written in the form

\[
\frac{d^2}{dz^2} (u + iv - u_\sigma - iv_\sigma) = \frac{i(\sigma + f)}{K} (u + iv - u_\sigma - iv_\sigma),
\]

(7)
and choosing for convenience the solution which does not increase indefinitely with \( z \), we obtain
\[
u + iv - u_v - iv_v = B e^{-\alpha t + \theta_t},
\]
where
\[
a = \left( \frac{\alpha + f}{2K} \right)^{\frac{1}{2}}.
\]

Dealing firstly with the lower boundary condition, at \( z = 0 \), the angle between the surface wind and the geostrophic wind is an important parameter; we denote the angle between the surface wind direction and the \( x \)-axis by \( \alpha \) and we write \( \beta \) for the angle between the (geostrophic) wind at 900 mb and the \( x \)-axis. The relationship between \( \alpha \) and \( \beta \) is a complex one and can be determined as follows. Let \( G \) denote the magnitude of the geostrophic wind, and write \( B = Ce^{\nu} \), \( u + iv = V \); then Eq. (8) becomes
\[
V - Ge^{\beta} = Ce^{\nu} - (1 + i\alpha)G.
\]
If \( D \) is the magnitude of the surface wind, then
\[
V = De^{\alpha}, \quad \text{at} \quad z = 0,
\]
and
\[
De^{\alpha} = Ce^{\nu} + Ge^{\beta}.
\]
Equating real and imaginary parts of the two sides of Eq. (11), we then obtain a relationship between \( C \) and \( \gamma \),
\[
G \sin (\beta - \alpha) = C \sin (\alpha - \gamma).
\]
The angles \( \alpha \) and \( \gamma \) are connected by using the assumption that \( V \) is parallel to \( dV/dz \) at \( z = 0 \), i.e. that the surface wind is directed along the surface Reynolds stress. This gives
\[
\gamma = \alpha - 5\pi/4.
\]
The numerical values of \( G \) and \( \beta \) are known at each grid point after each time step from the "free" atmosphere numerical model, so that if \( \alpha \) can be found then the complex constant \( B \) is known from Eqs. (12) and (13). To evaluate \( \alpha \) use is made of the relationship between surface stress and surface velocity. Surface stress \( (\tau_\nu)_{z=0} \) is given by \( Kp(dV/dz)_{z=0} \), which on using Eq. (9) reduces to
\[
(\tau_\nu)_{z=0} = 2Kp a G \sin (\alpha - \beta).
\]
An equation for \( \alpha \) is then arrived at by expressing \( \tau \) in an alternative form. The formulation of the 'Reynolds' stress \( \tau_\nu \) at the lowest level, 4, of the model (1,000 mb) is chosen to simplify the mathematical-computational problem and is based on the assumption (following Phillips 1956) that \( \tau_\nu = -k\nu \), \( \tau_v = -k\nu \), where \( k \) is a constant to be determined; the magnitude of the resultant stress is then given by
\[
|\tau_\nu| = k\nu c^4,
\]
where \( c^4 = u^4 + v^4 \). Analyses of observations, however, suggest that at the actual surface \( (z = 0) \)
\[
|\tau|_{z=0} = \kappa \nu c^2,
\]
where \( \kappa \) is an empirical constant varying widely over land and sea surfaces and of average climatological value 0.003. Following Phillips again, a numerical value of \( k \), suitable for numerical experiments, is then obtained in terms of this value of \( \kappa \) by taking a characteristic value of \( c = 0.7z_v \), where \( z_v \) is taken to be a typical average geostrophic velocity value (8 m s\(^{-1}\)). We then have
\[
k = \kappa (0.49)c_v, \quad \text{and}
\]
\[
|\tau|_{z=0} = \kappa (0.49)c_v \nu c.
\]
The surface wind speed \( c \) is then expressed in terms of \( G \) from Eq. (11), using Eq. (13), in the form
\[
c = G[1 - 2 \sin(\alpha - \beta) \cos(\alpha - \beta)]^{\frac{1}{2}}.
\]
Substituting Eqs. (18) and (14) into Eq. (17) gives finally an equation for \( \alpha \),
\[
2Kd \sin(\alpha - \beta) = \kappa (0.49)c_v [1 - 2 \sin(\alpha - \beta) \cos(\alpha - \beta)]^{\frac{1}{2}}.
\]
5 \times 10^3 \text{ cm}^2 \text{s}^{-1} at 2 \text{ m} over smooth sea and long grass respectively for a surface wind speed of 5 \text{ m s}^{-1}; Priestley, (1959), estimates values over land of 1.5 \times 10^5 \text{ cm}^2 \text{s}^{-1} and 7 \times 10^4 \text{ cm}^2 \text{s}^{-1} at 500 \text{ m} and at 900 \text{ m} respectively). The numerical value of $K$ chosen for the numerical experiment turns out to be particularly convenient as Eq. (19) simply reduces to
\[2 \sin (2(\alpha - \beta)) - \cos (2(\alpha - \beta)) = 1,\] giving solutions $2(\alpha - \beta) = \pi$ and $2(\alpha - \beta) = \sin^{-1}4/5$, i.e. $\alpha - \beta = 26^\circ 34^\prime$, which is reasonably near real average atmospheric values.

Expressions for horizontal components of velocity in the boundary layer then follow from Eq. (9):
\[u = G \cos \beta + \sqrt{2} Ge^{-ax} \sin (\alpha - \beta) \cos (\gamma - az),\] and
\[v = G \sin \beta + \sqrt{2} Ge^{-ax} \sin (\alpha - \beta) \sin (\gamma - az),\] where every term on the right-hand sides of Eqs. (21) and (22) is known, since $\beta$ is obtained from the 'geostrophic' model wind values at each grid point from the numerical solution.

The link between the boundary layer solution and the numerical model is now expressed in terms of the vertical velocity produced at the 'top' of the boundary layer; this is given by
\[w = -\int_0^h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz,\] where $h$ is the height of the 'top' of the boundary layer, given in the model by the height at which the resultant of $u$ and $v$ first becomes parallel to the geostrophic vector, i.e. where
\[\gamma - az = \beta \pm 2n\pi,\] the first positive solution occurring when $n = 1$, and leading to
\[h = \frac{1}{a} \left( \frac{3 \pi}{4} + \alpha - \beta \right),\] which is numerically very near to $\pi/a$. With our choice of $K$ and the characteristic model values for $\sigma$ and $f$, this leads to a numerical value of $h$ very close to 1 km (i.e. roughly at 900 mb). Then assuming that variations in $G$, the magnitude of the geostrophic wind speed, are greater than those of $\alpha$ and $\beta$, Eq. (23) leads to
\[w = -\int_0^\sigma \left[ \frac{\partial G}{\partial x} \left( \cos \beta + \sqrt{2} \sin (\alpha - \beta) e^{-ax} \cos (\gamma - az) \right) \right. \] \[\left. + \frac{\partial G}{\partial y} \left( \sin \beta - \sqrt{2} \sin (\alpha - \beta) e^{-ax} \sin (\gamma - az) \right) \right] dz.\] From integration of Eq. (26) and substitution of appropriate numerical values we finally obtain:
\[w = -\frac{1}{a} \left[ \frac{\partial G}{\partial x} \left( 3.14 \cos \beta + 0.32 \sin \gamma + 0.32 \cos \gamma \right) \right. \] \[\left. + \frac{\partial G}{\partial y} \left( 3.14 \sin \beta + 0.32 \sin \gamma - 0.32 \cos \gamma \right) \right].\] From this expression, numerical values for $\omega = dp/dt$ can be calculated at the top of the boundary layer and used to match the boundary layer solution to the overlying numerical model.

3. Procedure for matching the boundary layer solution to the numerical model values

In order to reduce the computational complexities in our initial experiment the 'free' atmosphere model was taken to extend from 900 mb upwards, this being consistent with the boundary layer analysis of Section 2. From a number of trial runs the stability of the computing technique was found to depend sensitively on retaining a vertical symmetry in the pressure levels. Consequently the upper level of the model was taken at 100 mb and $\omega$ is set to zero there. The thermodynamic equation was applied, as before, at 500 mb while the dynamical equations were applied at 300 mb (characterizing the 100 mb to 500 mb level) and at 700 mb (characterizing the 500 mb to 900 mb level). The analytic 'Ekman' solution applied to the 900 mb to 1,000 mb layer.
The finite difference equations are modified from the previous experiments to allow for a boundary layer inclusion by writing

\[
\left( \frac{\partial \omega}{\partial p} \right)_{300 \text{ mb}} = \frac{\omega_2}{\Delta p} \quad \text{as before, (28)}
\]

and

\[
\left( \frac{\partial \omega}{\partial p} \right)_{700 \text{ mb}} = \frac{\omega_E - \omega_2}{\Delta p} \quad \text{ (29)}
\]
a new term, where \( \omega_E \) is the value of \( \omega \) at the 'top' of the boundary layer as given by Eq. (27); \( \Delta p \) is now reduced from 500 mb (the value used by Everson and Davies 1970) to 400 mb.

Two energy transformations of great interest, and additional to those appearing in our previous model (Everson and Davies 1970), can be deduced from the system of equations. One describes a transfer from zonally averaged kinetic energy in the free atmosphere into the boundary layer and is given in the usual convention (see e.g. Phillips 1956) by

\[
\{ K \cdot BL \} = \frac{f_0}{\Delta p} \int \omega_E \psi_1 \, dx,
\]

where \( f_0 \) is the constant Coriolis parameter of a \( \beta \)-plane model, \( \psi_1 \) denotes stream function at level 3 (700 mb), \( dx \) denotes an element of area in the horizontal; the averages, indicated by bars, are taken over lines of latitude, and the integration is taken over the whole spatial domain. The other transformation describes a similar transfer from the eddy kinetic energy in the free atmosphere into the boundary layer,

\[
\{ K' \cdot BL \} = \frac{f_0}{\Delta p} \int \omega_E' \psi_1' \, dx.
\]

These two formulae describe the large-scale model interaction between the boundary layer and the 'free' atmosphere part of the model.

4. Results of the Numerical Experiments

Two long period numerical experiments of 26 days duration were carried out. One experiment included an 'Ekman' boundary layer of the type discussed in Section 2 (the 'Ekman' model), and another (as in previous experiments by Everson and Davies 1970) assumed a linear profile of \( \omega \) from zero at 1,000 mb to a value \( \omega_2 \) at 500 mb in order to deduce an appropriate value of \( \omega_E \) at 900 mb in the free atmosphere system. A comparison of the results enables us to evaluate directly the effects on large-scale flow of introducing a boundary layer.

An interesting computational instability immediately occurred in the 'Ekman' model through the introduction of a matching \( \omega_E \). This can be explained by looking at the nature of the system of finite difference equations. We see that there are two sets, one describing mean values (averaged over latitudes) and one giving the deviations from these means. In the model used previously by Everson and Davies (1970) \( \tilde{\omega} \) and \( \omega \) were never evaluated explicitly through the finite difference equations but were always substituted into the thermodynamic equation. However, in our 'Ekman' model \( \omega_E \) occurs explicitly in both mean and deviation form. We notice in dealing with the stream function and vorticity quantities that the deviations \( \psi' \), \( \eta' \) are small compared with mean values \( \tilde{\psi} \), \( \tilde{\eta} \), but the \( \omega \) values, on the other hand, are found to be extremely small compared to \( \omega' \) since the upward and downward vertical velocities tend to cancel one another out over a latitude 'circle'. Consequently the addition of an 'Ekman' \( \omega_E \) has added considerable sensitivity to the stability of the system of equations. This is to be expected since the length scale of \( \omega \) is of the order of 1 km (the boundary layer thickness), whereas the length scale of the \( \psi \)'s is 300 km (the grid size), and a calculated single grid point \( \omega \) value may not be numerically consistent with the general model structure. In fact the computation turned out to be highly unstable, but when numerical values of \( \omega \) and \( \omega' \) were used after averaging over the nine points surrounding a particular grid point, the computation was stable and the resulting flow regime behaved realistically. The model equations were then integrated for 26 days with this modification included in the computing scheme using the 'Ekman' model and the 'linear \( \omega \) profile' model for comparison purposes.

The results obtained are shown in Figs. 1, 2, 3, 4. The dominant feature of the 'linear \( \omega \) profile' model is that the \( \tilde{K} \) value increases at a highly unrealistic rate, almost doubling in about 15 days.
Figure 1. Computed global eddy kinetic energy (units as in Phillips 1956) for (a) the 'Ekman' model, indicated --- ---, and (b) for the 'linear w profile' model, indicated --- ---.

Figure 2. Computed global zonal kinetic energy (units as in Phillips 1956) for (a) the 'Ekman' model, indicated --- ---, and (b) for the 'linear w profile' model indicated --- ---.

Figure 3. Computed transfer of zonal kinetic energy of the 'free' atmosphere into the 'Ekman' layer (units as in Phillips 1956).
This result is due to the fact that this model does not transfer significant energy downwards into the boundary layer space (1,000 mb — 900 mb) to be dissipated at the surface, and consequently $K$ builds up at first in an unrealistic manner, although the remaining energy conversions behave in a similar fashion to previous work; the three cell meridional structure and the easterly-westery surface pattern manifest themselves by day 18 and an error check (Phillips 1956) showed relatively small values throughout the period of integration.

An examination of Fig. 2 shows that the rate of increase of $K$ is far more controlled and is nearer the probable value in the real atmosphere, as suggested by high resolution model results. This behaviour is due to an average energy transfer downwards from the 'free' model atmosphere into the 'Ekman' layer, leading to dissipation at the surface and so avoiding an unrealistic initial build up of $\bar{K}$; the variation with time of this downward transfer is shown in Fig. 3, noting that it is effectively zero in the linear model. This average downward transfer of westerly momentum is of course a dominant average feature of the planetary boundary layer of the real atmosphere in mid-latitudes. We note also that the ratio of $K'$ to $\bar{K}$ is greatly increased and its value therefore moves nearer to the observed atmospheric value by introducing an 'Ekman' layer into the model.

We note finally that, although our numerical model with an 'embedded Ekman layer' involves many approximations and is concerned with only one aspect of planetary boundary layer characteristics, it does demonstrate the rather sensitive dependence of large scale flow characteristics on the presence of a boundary layer parameterization in a numerical model. Our numerical experiments also lead to two other general points of interest. (a) The procedure of obtaining a time dependent analytic solution of an 'Ekman' boundary layer matched to a numerical 'free atmosphere' model has been shown to be workable in a two layer model; this suggests that it is capable of extension to higher resolution models. It is planned to incorporate the system in a four-level model with variable static-stability and with associated varying characteristics such as boundary layer thickness, using the Monin-Obukhov theory. (b) If in a higher resolution model it can be shown that the large scale dynamics are sensibly dependent on boundary layer parameterization, then it will be difficult to describe this physical characteristic properly by using a finite difference discrete level system, and it may well be important to allow for the continuous variation with height of the scale length of boundary layer turbulence by the use of an analytic form of solution.

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REFERENCES


