The critical condition for the maintenance of turbulence in stratified flows

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SUMMARY

Theoretical models for predicting the critical Richardson number for the maintenance of turbulence in a stratified atmosphere are re-examined in the light of recent observations. It is proposed that the critical condition may first arise from the failure of the maintenance of steady shear in the presence of buoyancy forces. An expression for the critical flux Richardson number is derived from the consideration of the equations of Reynolds stress and turbulent energy. This gives a range of 0.15 – 0.25 for $R_{cr}$, in agreement with the results of direct observations.

1. INTRODUCTION

Various perturbation analyses (e.g. Taylor 1931; Miles 1961) have shown that a laminar layer of a density stratified fluid is stable with respect to small perturbations if Richardson number exceeds a value of 1/4, in all parts of the flow. The flow may become unstable if $R_f < 1/4$ at any level (Collver 1970). A critical Richardson number in the above sense does not necessarily imply that in an already existing turbulent shear flow, the turbulence would disappear as $R_f \to 1/4$. Energy may be available for turbulence even when $R_f > 1/4$ (Businger 1969; Badgley 1969).

The problem of reverse transition from turbulent to laminar flow is, at present, not amenable to rigorous mathematical treatment. Several simple models (Ellison 1957; Townsend 1958) have been used to predict critical conditions. These predictions have, however, been based on somewhat speculative assumptions which are not borne out by actual observations.

2. THEORETICAL BACKGROUND

From a simple balance of turbulent energy, Richardson (1920) suggested that $R_{cr}$ = 1, for just no turbulence. He ignored, however, the viscous dissipation of energy, which has subsequently been observed to be important even under strongly stratified conditions.

Ellison (1957) proposed a theoretical model for the surface layer of the atmosphere, based on the equations of turbulent energy, mean square temperature fluctuations and turbulent heat flux. Assuming steady state and neglecting advection and diffusion terms, these equations become

\[ -u'w' \frac{\partial U}{\partial z} + \frac{\theta'}{T} w' \theta' = \epsilon \]  
\[ -\bar{w'} \theta' \frac{\partial T}{\partial z} = \epsilon_\theta \]  
\[ -\bar{w'}^2 \frac{\partial T}{\partial z} + \frac{\theta'}{T} \bar{w'}^2 + \frac{1}{\rho} p' \frac{\partial \theta}{\partial z} = \epsilon_{\text{vort}}. \]

Here, $U$ is the mean velocity at the height $z$; $u'$, $v'$ and $w'$ are velocity fluctuations in $x$, $y$ and $z$ directions; $T$ and $\theta'$ are the mean and fluctuating temperature; $\rho$ is the mean density; $p'$ the fluctuating pressure; $g$ the acceleration due to gravity; and $\epsilon$, $\epsilon_\theta$ and $\epsilon_{\text{vort}}$ denote the molecular dissipation terms in Eqs. (1), (2) and (3).
Ellison (1957), apparently, neglected the pressure strain term in Eq. (3) and formally expressed each dissipation term as the ratio of the quantity concerned to its decay time, i.e.,

\[ \epsilon = \frac{\bar{q}^2}{2T_2}; \quad \epsilon_\theta = \frac{\bar{\theta}^2}{2T_1}; \quad \epsilon_{u\theta} = -\frac{\bar{w} \bar{\theta}'}{T_3} \]  

in which, \( \bar{q}^2 = \bar{u}^2 + \bar{v}^2 + \bar{w}^2 \), and the time scales \( T_1, T_2 \) and \( T_3 \) are so defined that in the absence of producing terms the mixing action of turbulence would begin to destroy \( \bar{q}^2/2, \bar{q}^2/2 \) and \( \bar{w} \bar{\theta}' \), respectively, at the rates equal to \( 1/T_1, \), etc. After substituting from Eq. (4) into Eqs. (1) to (3), Ellison obtained an expression for the ratio of the exchange coefficients of heat and momentum:

\[ \alpha = \frac{K_H}{K_M} = \frac{1}{2} \frac{T_3}{T_2} \frac{\bar{q}^2}{(\bar{u} \bar{w})^2} \left( 1 - R_f \frac{T_1}{T_2} \frac{\bar{q}^2}{\bar{w}^2} - R_f \right) \frac{1}{(1 - R_f)^2} \]  

where, \( R_f = \frac{g}{T} \frac{\bar{w} \bar{\theta}'}{(\bar{u} \bar{w})^2} \frac{\partial U}{\partial z} \), is the flux Richardson number. Further, speculate that the ratios \( T_1/T_2, T_3/T_2 \), and \( \bar{q}^2/\bar{w}^2 \) may be roughly constant, independent of stability, he inferred from Eq. (5) that \( \alpha \) decreases with the increase in stability, approaching zero under critical conditions and also that

\[ R_{f_{cr}} = \left( 1 + \frac{T_1}{T_2} \frac{\bar{q}^2}{\bar{w}^2} \right)^{-1} \]  

Empirical values of \( T_1/T_2 = 1 \), and \( \bar{q}^2/\bar{w}^2 = 5 \cdot 5 \), then, lead to \( R_{f_{cr}} \sim 0 \cdot 15 \). Recognizing that under strongly stable conditions, the ratio \( \bar{q}^2/\bar{w}^2 \) may be greater than its value of 5·5 in near-neutral conditions, Ellison predicted even a smaller value for \( R_{f_{cr}} \).

Townsend (1958) has proposed a similar model for a non-developing flow far from the restraining boundaries. In the absence of radiative heat transfer his model is essentially based on Eqs. (1) and (2), in which the dissipation terms are expressed as

\[ \epsilon = \frac{(\bar{w} \bar{\theta}')^2}{L_e}; \quad \epsilon_\theta = \frac{\bar{\theta}^2}{(\bar{w} \bar{\theta}')^2} \frac{3L_\theta}{L_e}. \]  

Then, one can obtain from Eqs. (1) and (2),

\[ R_f (1 - R_f) = 3N R_l \]  

in which, \( R_f = \frac{g}{T} \frac{\partial T}{\partial z} \left( \left( \frac{\partial U}{\partial z} \right)^2 \right) \), is the gradient form of Richardson number, and

\[ N = \frac{L_\theta}{L_e} \left( \frac{k_{\theta w}}{k_{\theta w}} \right)^2 \]  

\[ k_{\theta w} = -\frac{\bar{u} \bar{w}'}{\bar{w}^2}, \quad k_{\theta w} = \frac{\bar{w} \bar{\theta}'}{(\bar{w} \bar{\theta}')^{1/2} (\bar{w}^2)^{1/2}} \]  

Assuming \( N \) to be a constant, independent of stability, and considering Eq. (8) as a quadratic equation in \( R_f \), Townsend obtained the following conditions for the maintenance of steady turbulence (rather, the conditions for obtaining real values of \( R_f \):

\[ R_f \leq \frac{1}{12N} \]  

\[ R_f \leq \frac{1}{2} \]  

Further, assuming that \( N = 1 \), Eq. (9a) gives a value of \( R_{f_{cr}} = 1/12 \).

Even though the two models discussed here are presumably valid for somewhat different flow regimes, there is an obvious contradiction in their implied behaviour of \( \alpha \) with increasing stability. Ellison's theory implies that \( \alpha \) decreases monotonically with increase in stability, approaching to zero as \( R_f \to R_{f_{cr}} \). On the other hand, Townsend's model implies that \( \alpha \) actually increases from a value of 3 in near-neutral conditions,
which can be seen from Eq. (8), to 6 under critical conditions. This serious difference needs to be resolved in the light of recent experiments.

3. Experimental evidence

Almost all the recent observations in the surface layer of the stable atmosphere (Gurvich 1965; Webb 1970; Oke 1970; Businger, Wyngaard, Izumi and Bradley 1971) indicate $\alpha$ to remain constant, close to unity. Therefore, Ellison's interpretation of the critical condition viz. $\alpha = 0$ when $R_f = R_{for}$, is certainly not correct. The laboratory studies (Ellison and Turner 1960; Webster 1964; Arya and Plate 1969) do show $\alpha$ to be a decreasing function of stability. But, there is no evidence that it would approach to zero instead of some other constant value as indicated in Fig. 1.

![Stability Group](image)

Figure 1. $K_h/K_M$ as a function of stability in the wind tunnel.

The laboratory data used in this and other Figures are the result of experiments performed by the author in a wind tunnel. The tunnel has a test section 25 m long and 1.8 $\times$ 1.8 m$^2$ in cross-section. Over the last half of the length, the floor consists of an aluminium plate which can be heated or cooled. The air in the recirculating tunnel can also be heated or cooled for obtaining different stability conditions in the boundary layer developing on the floor of the test section. For the experiments reported here, a temperature difference of about $-45^\circ$C was maintained between the floor and the ambient air. Average ambient velocities for runs classified into stability groups I, II and III, were 9.2, 6.1 and 3.0 m s$^{-1}$, respectively. Turbulent intensities and fluxes in the boundary layer were measured using an elaborate hot-wire technique described elsewhere (Arya and Plate 1969; Arya 1968).

The difference between the laboratory and the atmospheric trends of $\alpha$ may be attributed to the fact that in the former stronger stability conditions have become necessarily associated with very low Reynolds numbers. Model conditions are better realized in the atmosphere if the winds are steady and the underlying surface is fairly homogeneous.

For directly checking the assumptions of Ellison's model, the ratios $T_3/T_2$ and $T_4/T_2$ were determined from wind tunnel measurements. Figs. 2 and 3 show that there is strong dependence of these quantities on $R_f$. Because of the observed independence of $\alpha$ on $R_f$ in the stable atmosphere, $T_3/T_2$ is expected to increase with stability even more rapidly than indicated by the wind tunnel results. Therefore, Eq. (5) cannot be considered to express
\[ \alpha \] as an explicit function of \( R_f \), and the conclusions drawn by Ellison about \( R_{terr} \) were in fact based on wrong assumptions.

A serious error may have been introduced in Ellison's model by his neglect of pressure interaction term in Eq. (3). If local isotropy can be assumed to exist, which is a fair assumption for large Reynolds number flows such as in the atmosphere, one would expect that in the budgets of shear stress and heat flux, dissipation terms will be negligible, and pressure interaction terms will essentially balance the production and the buoyancy terms.

Coming back to Townsend's theory, one may object to its implication that in near-neutral conditions, \( \alpha = \alpha_o = 3 \), and that it increases to value of 6 as the critical condition is approached. This is a consequence of the assumption that \( N = L_o/L_e \left( k_{\theta e}/k_\theta \right)^2 \), is close to unity and is independent of stability. If the model is applicable at all to the surface layer, it follows from the definition of \( L_e, L_o \), etc., that \( N = \alpha_o/3 \), which for \( \alpha_o = 1.35 \) (Businger et al. 1971) would give \( R_{terr} = 0.19 \). Beals (1970) arrived at the same value of \( R_{terr} \) by a different approach using direct measurements of turbulent intensities and fluxes. The conditions in Eqs. (9a) and (9b) are, however, based on the assumption that \( N \) is independent of stability, which is not confirmed by Beals' measurements. One can also write Eq. (8) as

\[ 1 - R_f = \frac{3}{\alpha} N, \]  

(10)
in which case, $R_f$ is always real and no conditions are given for the maintenance of steady turbulence. Recognizing further the observed fact that in stable conditions $a$ is more or less independent of stability, Eq. (10) indicates that $N$ cannot possibly be a constant. The validity of Townsend’s model to the surface layer is, therefore, questionable.

Several investigators have attempted to determine $R_{fcr}$ or $R_{fer}$ from direct observations in the surface layer (Lyons, Panofsky and Wollaston 1964; Webb 1970; Oke 1970; Businger et al. 1971). The criterion of what may be called the ‘critical condition’ has also varied greatly. For example, Lyons et al. (1964) have examined wind and temperature records when nocturnal inversion begins to be destroyed by the onset of turbulence. They did not find a unique $R_{fcr}$ for the onset of turbulence, but observed that it must lie between 0.2 and 0.5. Webb (1970) and Businger et al. (1971), have given a value of $R_{fer} \approx 0.2$, which is based on their observation that wind and temperature profiles are of log-linear form even under fairly strong stability conditions and therefore, $K_M = ku_a z(1 - \beta R_f)$, approaches zero as $R_f$ approaches a value of $\beta^{-1}$ ($\beta$ is the empirical constant in the log-linear formula). Webb has also observed that before this limit is quite reached profiles deviate from the log-linear form, so that $K_M$ though quite small, will actually not vanish at $R_f = \beta^{-1} \sim 0.2$ (see also Oke 1970). Oke (1970) has recently reviewed much of the existing experimental evidence from the strongly stratified (stable) surface layer of the atmosphere. He concludes that there is a threshold value of $R_{fcr} \sim 0.1$, beyond which turbulent transport of horizontal momentum decreases rapidly, and a maximum value, $R_{f \max} \sim 0.3$, at which turbulent motion ceases entirely.

4. Proposed model

The critical condition in a stably stratified flow is generally thought to arise from the circumstance that a steady state turbulence structure can no longer be maintained because of the presence of buoyancy forces. It may be caused by buoyancy drawing more energy from turbulence than is available to the latter, or by the failure of the mechanism by which mean flow energy is being converted steadily into turbulence. This latter possibility seems more likely in view of the observed fact that even in extreme stability conditions, buoyancy draws only a fraction of the energy that is available, the rest being dissipated by viscosity. On the other hand, buoyancy may inhibit the production of Reynolds stress to the extent that steady state conversion of mean flow energy into turbulence would no longer be possible.

It has been suggested in the literature (Lyons et al. 1964; Oke 1970) that turbulence may not be maintained because molecular dissipation of energy into heat may exceed the balance of production and buoyancy terms. From physical considerations, however, it is not clear how this can happen. Dissipation is a passive process and can occur only to the extent that the available energy gets transferred down the spectrum from large (energy containing) eddies to small (dissipative) eddies. There is nothing in the system that can increase the final dissipation beyond what is available by way of spectral transfer, say, through the inertial subrange. With the increase in the work done against buoyancy forces, less and less energy will be available for dissipation, until the latter would become zero when $R_f = 1$ (if turbulence could be assumed to exist at all up to this point). Thus, the consideration of the energy equation alone does not give any new criterion for the maintenance of turbulence other than that suggested originally by Richardson (1920). Similarly, no criterion is expected from the balance of mean square temperature fluctuations as shown earlier in our discussion on Townsend’s (1958) model.

In the stably stratified atmospheric surface layer in which the advection and diffusion of turbulence can be neglected in the local balance of energy, the maintenance of turbulence is intimately related to the maintenance of steady shear because the latter is the only source of turbulent energy. Therefore, it appears more appropriate to consider the budget of turbulent shear stress:

$$\frac{\partial (-w' \cdot u' \cdot w')}{\partial t} = w'^2 \frac{\partial U}{\partial z} - \frac{1}{\rho} \frac{\partial}{\partial x} \left( \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial x} \right) - \frac{g}{\theta'} u' \theta' + \epsilon_{uuw}. \quad (11)$$
The dissipation term in the above equation is always quite small ($\epsilon_{w_e} = 0$, for locally isotropic turbulence), and can be neglected. The ratio of the buoyancy term to production term is a stability parameter which is related to the flux Richardson number as

$$ S = \frac{g}{T} \frac{w' \theta'}{w'^2} = - \frac{w' \theta'}{w'^2} k_w R_f. \quad \ldots \quad (12) $$

Observations in the atmosphere (Zubkovski and Tsvang 1966; Weseley, Thurtell and Tanner 1970), as well as in the laboratory (Arya 1968) indicate that the ratio $-w' \theta'/w'^2$ is about 3-5 in near neutral conditions and increases further with increase in stability. The change in $k_w$, which is about 0.6 in neutral conditions, is comparatively small. Therefore in stable conditions the parameter $S$ is several times $R_f$. Recognizing that the pressure transfer term in Eq. (11) is also quite large (this term essentially balances the production term in neutral conditions), and has the same sign as the buoyancy term, the critical condition is likely to arise first from the failure of maintaining an equilibrium shear stress even when $R_f$ is much less than unity. An indirect support to our hypothesis is provided by observations in the atmosphere (Oke 1970), which indicate non-dimensional momentum exchange coefficient decreasing very rapidly for $R_f > 0.1$.

For determining the critical value of $R_f$ from Eq. (11), one needs to know the pressure-strain term. This has never been measured directly. It is well known, however, that pressure fluctuations act in a way so as to resist any tendency of the flow to become anisotropic. Thus, pressure strain terms will act in opposition to other terms e.g., shear and stratification, which give an anisotropic structure to the flow. The role of these terms becomes more evident when one considers the budgets of energy in different components. Here, we have terms like $1/\rho p' \frac{\partial u_i}{\partial x_i}$, etc., which transfer energy from the largest to smaller components. After assuming that the rate of this transfer is proportional to the degree of anisotropy, and the turbulence is locally isotropic ($\epsilon_u = \epsilon_v = \epsilon_w = \epsilon/3$), Rotta (1951) expressed the pressure strain terms as

$$ \frac{1}{\rho} p' \frac{\partial u_i}{\partial x_i} = - C \left( \frac{u_i^2 - \frac{1}{3} q^2}{q^2} \right) \epsilon, \quad \ldots \quad (13) $$

in which, $u_i (i = 1, 2, 3) = u, v, w$ and $x_i = x, y, z$.

Eq. (13) has been supported by measurements in grid turbulence (see Rotta 1962), which indicate a value of $C \sim 6$. For locally isotropic turbulence in a uniform mean flow, Eq. (13) leads to

$$ \frac{1}{\rho} p' \left( \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial x} \right) = - 2C \ \frac{u' w'}{q^2} \ \epsilon. \quad \ldots \quad (14) $$

When the Reynolds number of the flow is sufficiently large as in the atmosphere, it is reasonable to assume local isotropy, so that in Eq. (11), $\epsilon_{w_e} \sim 0$. In neutral conditions, then Eq. (11) gives

$$ \frac{1}{\rho} p' \left( \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial x} \right) = \frac{w'^2}{2} \frac{\partial U}{\partial z}. \quad \ldots \quad (15) $$

Comparing Eqs. (14) and (15) and recognizing that under neutral conditions, $\epsilon = -u' w' \frac{\partial U}{\partial z}$, we obtain

$$ C = \frac{1}{2} \left[ \frac{u'^2 q^2}{(u' w')^2} \right] \text{neutral}. \quad \ldots \quad (16) $$

We will extend Rotta's hypothesis to the case of stratified flow and assume that $C$ is a constant, independent of stability. A similar hypotheses has been used by Monin (1965).

After substituting from Eqs. (12) and (14) into Eq. (11), and making use of the energy balance equation, we obtain the condition for the maintenance of steady state turbulence as
where,

\[ f = \frac{w^2 q^2}{(u' w')^2}, \quad \quad \quad (18a) \]

\[ g = \frac{w' \theta'}{w \theta} \cdot \frac{u' w'}{w^2} \quad \quad \quad (18b) \]

According to the Monin-Obukhov similarity theory the quantities \( f \) and \( g \) must be universal functions of the stability ratio \( \xi = z/L \), where \( L \) is the Obukhov length. The study by Monin (1965) shows that \( f \) and \( g \) will approach to some constant values, say \( f_\infty \) and \( g_\infty \), as \( \xi \to \infty \), or as the critical condition is approached. In that case, we have from Eq. (17)

\[ R_{fer} = \frac{1 - (f_0 / f_\infty)}{g_\infty - (f_0 / f_\infty)}. \quad \quad \quad (19) \]

Where, \( f_0 \) is the value of \( f \) under neutral conditions and Eq. (16) has been used for \( C \). The value of \( R_{fer} \) can be determined from Eq. (19), by substituting the measured values of \( f_0 / f_\infty \) and \( g_\infty \). In view of the large scatter in the reported measurements of turbulence under stable conditions, we express \( f \) and \( g \) in terms of the anisotropy and correlation coefficients as

\[ f = (1 + A_v^2 + A_{v'w'})/k_{uw^2} \quad \quad \quad (20) \]

\[ g = \frac{k_{w\theta}}{k_{vw}} \cdot \frac{k_{u\theta}}{k_{uw}} \cdot A_{uw^2} \quad \quad \quad (21) \]

Where,

\[ A_v = (\bar{v'}^2/\bar{u'}^2)^{1/2}; \quad A_w = (\bar{w'}^2/\bar{u'}^2)^{1/2}. \quad \quad \quad (22a) \]

\[ k_{uw} = |u' w'|/(\bar{u'}^2 \bar{w'}^2)^{1/2}; \quad k_{u\theta} = |u' \theta'|/(\bar{u'}^2 \bar{\theta'}^2)^{1/2}; \quad k_{w\theta} = |w' \theta'|/(\bar{w'}^2 \bar{\theta'}^2)^{1/2}. \quad \quad \quad (22b) \]

From Eq. (20), one has

\[ f_0 / f = \left( \frac{k_{uw}}{k_{uw\theta}} \right)^2 \left[ 1 + A_{v'w'}^2 + A_{v'w''}^2 \right] \left[ 1 + A_v^2 + A_{w}^2 \right]. \quad \quad \quad (23) \]

Where, subscript \( o \) indicates the value under neutral conditions.

The asymptotic value of the ratio in the brackets can be determined from the expressions derived by Monin (1965) for anisotropy coefficients, i.e.,

\[ \frac{1}{A_v^2} = 1 + \left( \frac{1}{A_v^2 - 1} \right) \frac{\phi(\xi)}{\phi(\xi) - \xi} \quad \quad \quad (24) \]

\[ \left( \frac{A_{uw}}{A_v} \right)^2 = \left( \frac{A_{v}}{A_v} \right)^2 - \left( \frac{1}{A_v^2 - 1} \right) \frac{\xi}{\phi(\xi) - \xi} \quad \quad \quad (25) \]

Here, \( \phi = (h^2/\nu) \partial U/\partial z \), is the dimensionless wind shear, which for strongly stable conditions (\( \xi \to \infty \)) is given as (Webb 1970; Businger et al. 1971):

\[ \phi = \beta \xi; \quad \beta \sim 5 \quad \quad \quad (26) \]

Taking empirical values of \( A_{vo} \approx 0.80 \), and \( A_{wo} \approx 0.60 \), we determine from Eqs. (24) to (26) that \( A_{v'w'} \approx 0.77 \), \( A_{v'w''} \approx 0.50 \), and \( (1 + A_v^2 + A_{wo}^2)/(1 + A_v^2 + A_{wo}^2) \approx 1.08 \). The latter ratio will not differ substantially even if one takes somewhat different values of \( A_{vo} \) and \( A_{wo} \).

Until recently, there has been a great lack of good quality measurements in the atmosphere under strongly stable conditions. Figs. 4 and 5 show the results of recent measurements from Kansas tower, which are used here by courtesy of Dr. D. A. Haugen of AFCRL, Cambridge (see also Haugen, Kaimal and Bradley 1971; Businger et al. 1971).
Wind tunnel data (Arya 1968) are also represented for comparison. In spite of the large scatter, we note that Fig. 4 is in general agreement with the theoretical results of Eqs. (24) and (25). Fig. 5 indicates that \( k_{uw} \) decreases from a value of about 0.35 in neutral conditions to about 0.25 in near critical conditions. The latter figure may be as low as 0.20, as some of the data points would indicate. With this uncertainty, the ratio \( f_0/f_\infty \) is empirically estimated to lie between 0.30 and 0.60.

Next, we consider the quantity \( g = (k_{uw}/A_w^2)(k_{uw}/k_{w0}) \). For strong stability conditions, the data of Figs. 4 and 5 indicate that the ratio \( k_{uw}/A_w^2 \) approaches a constant value of about 0.75. The plot of \( k_{uw}/A_w^2 \) vs \( R_f \), not represented here showed much less scatter than that of Figs. 4 and 5, so that the above estimate is considered good to within \( \pm 10 \) per cent. Wind tunnel measurements by Webster (1964) and the author (see Fig. 6) suggest that \( k_{uw}/k_{w0} \) may become as large as 4. The results of Weseley et al. (1970) indicate a value of about 3, although there is considerable scatter in their data points. The best one can say from the available data is that the quantity \( g_\infty \) in Eq. (19) lies somewhere between 2 and 3.

With the above estimated range of values for \( f_0/f_\infty \) and \( g_\infty \), Eq. (19) gives an estimated range of 0.15 - 0.40 for \( R_{trer} \), which, according to our definition, represents the maximum for which a steady state (non-decaying) turbulence structure can be maintained at any
height in the surface layer. One can further reduce the upper limit to 0.25, since above this value of the Richardson number the flow has been shown to be inherently stable with respect to small perturbations (fluctuations indeed are very small when \( R_f > 0.2 \), e.g., see Oke 1970).

The critical condition will, probably, first occur near the top of the surface layer, or, even at a higher level in the stable boundary layer, and may gradually proceed downwards after turbulence dies down in the upper layers. It is also quite possible, as pointed out by Lyons et al. (1964) and Oke (1970), that there is not a unique value of \( R_{fer} \), or \( R_{fer} \) in the surface layer and that this will increase with increasing height. But, the reason given by them for this to happen is not very convincing as it is based on the wrong premise that the critical condition arises because of dissipation exceeding the balance of production and buoyancy terms. One can reason out from Eq. (19) that \( R_{fer} \) would be height dependent, if \( g_{so} \) and/or \( f_{so} f_{so} \) are height dependent. After examining the available data we could not detect any significant variation with height of the latter. Wind tunnel observations, however, show a marked decrease in \( g \) with increasing height. This is because the horizontal heat flux \( u' \theta' \) drops rapidly away from the surface, while \( u' \theta' \) remains fairly constant in the surface layer. If the same is true for the atmosphere, one would expect \( g_{so} \) to decrease with height and, consequently, \( R_{fer} \) to increase.

5. Conclusions

Theoretical models proposed by Ellison (1957) and Townsend (1958) are contradictory in their implied behaviour of \( \alpha \) (the ratio of the exchange coefficients of heat and momentum) with stability. In the former, the prediction of \( R_{fer} \) has been based on the assumption that \( \alpha_{cr} = 0 \), and the ratios of time scales of decay of the fluctuating quantities are independent of stability, which are not borne out by actual measurements. The assumptions in Townsend’s model imply that \( \alpha_{cr} = 6 \), which is also unrealistic.

It is proposed that the critical condition will first arise from the failure of maintaining an equilibrium (steady state) shear stress. The equations of Reynolds stress and turbulent energy yield an expression for \( R_{fer} \) in terms of correlation and anisotropy coefficients. Using empirical information on these, a range of 0.15 – 0.25 for \( R_{fer} \) is predicted. This agrees with the values of the critical Richardson number estimated by other investigators from direct observations in the surface layer of the atmosphere. It is suggested that \( R_{fer} \) may not be unique, but might increase with height from the surface.

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