Non-Boussinesq effects and further development in a model of upper tropospheric frontogenesis

By B. J. HOSKINS

Advanced Study Program, National Center for Atmospheric Research

(Manuscript received 1 November 1971; in revised form 8 February 1972)

Summary

In previous papers, upper tropospheric frontogenesis has been exhibited in a model in which a horizontal deformation field acts on a temperature distribution in a Boussinesq two fluid system. The basic approximation made is that the flow along the incipient front is geostrophic. This approximation allows the reduction of the problem to that of solving a non-linear second order partial differential equation. Here this equation is solved by a slightly different technique. Non-Boussinesq effects are included and are shown to produce a strengthening of the upper-tropospheric front and its associated jet stream. Further deformation is allowed and the development of a more realistic front described.

List of symbols not defined in the text

- $p$: pressure
- $p_0$: standard pressure, taken as 1,000 mb
- $\rho$: density
- $\rho_0$: standard density at $p = p_0$
- $\theta$: potential temperature
- $\theta_0$: standard potential temperature at $p = p_0$
- $\phi$: geopotential
- $k$: unit vertical vector
- $u$: ‘horizontal’ velocity ($u, v, 0$)
- $w$: $Dz/Dt$
- $v$: ‘total’ velocity = $u + wk$
- $g$: acceleration due to gravity
- $f$: Coriolis parameter
- $R$: gas constant
- $H_s$: scale height = $R \theta_0/g = p_0/(\rho g)$
- $\gamma$: ratio of specific heats
- $\kappa$: $(\gamma - 1)/\gamma$

1. Introduction

In previous papers (Hoskins 1971; Hoskins and Bretherton 1972, hereafter referred to as I and II respectively), the formation of fronts in the atmosphere has been studied in some detail. The fundamental approximation made was the justifiable one of geostrophic balance across the incipient front. It was remarked that quasi-geostrophic theory which assumes also a geostrophic balance along the front is certainly not valid in a description of frontogenesis. A general model based on the classic horizontal deformation field, in particular, produced solutions exhibiting the formation of realistic surface and upper tropospheric fronts. In this model a velocity field $u = -ax$, $v = ay$ acts on a potential temperature field independent of $y$ and with $x$ gradients initially so small that the associated thermal winds are negligible. As shown in II, Kelvin’s circulation theorem suggests the use of a new independent variable in the $x$ direction whose introduction produces a great conceptual and analytic simplification. To find the solution at any particular time given an

*The National Center for Atmospheric Research is sponsored by the National Science Foundation.
†Present Address: Department of Geophysics, University of Reading, England.
initial state, one need not determine the solution for all intermediate times but need only solve a second order elliptic equation. If the deformation field acts on a model stratosphere-troposphere atmosphere then the solutions given in I exhibit the formation of a tongue of stratospheric air pushing down into the troposphere, a jet stream and other features of upper tropospheric fronts observed by e.g., Reed (1955) and Reed and Danielsen (1959).

In I the solutions were obtained making the Boussinesq approximation. Changes in density following a fluid particle are ignored in this traditional simplification. Scale analysis suggests that this is a reasonable approximation in the early stages of upper tropospheric frontogenesis because the height scale of the motion is much smaller than the scale height of the atmosphere. However, in the final stages of the extreme process described by Reed (1955) in which stratospheric air descends to 700 mb, clearly this condition would not be satisfied. It is thus of interest to determine the errors produced by the approximation and to exhibit solutions in which it is not made.

To this end, in this paper we re-examine the model of upper tropospheric frontogenesis with the initial conditions of Experiment 4 of I. A slightly different method of solution is used giving results very similar to the previous ones. Non-Boussinesq effects are then introduced and are found to produce a strengthening of the upper tropospheric front and jet stream and a weakening of the surface front. This is consistent with a study of the vorticity equation.

Finally, the deformation field is allowed to act for a further few hours and the development of a more intense upper air front is described.

2. The problem and method of its solution

In this Section we first present a summary of the problem and its analytical investigation as presented in II. The notation used is fairly standard but is given in the list of symbols.

We use the primitive equations with hydrostatic balance in the vertical and introduce as vertical co-ordinate:

\[ z = [1 - (p/p_0)^{\gamma}] H_0 \kappa. \]

\( z \) is not, in general, equal to the physical height. However, since increments in \( z \) and height \( h \) are connected by

\[ \theta \, dz = \theta_0 \, dh, \]

for an adiabatic atmosphere of potential temperature \( \theta_0, z = h \). This co-ordinate has some advantages in analytical work. In particular the 'thermal wind' relations give linear proportionality between the shear of the geostrophic wind and the potential temperature gradient, independent of density or pressure. We define a 'pseudo-density'

\[ \tau(z) = \rho_0 \left( \frac{p}{p_0} \right)^{1/\nu} = \rho_0 \left( 1 - \frac{\kappa z}{H_0} \right)^{1/(\nu - 1)}, \]

so that

\[ \rho = (\theta_0/\theta) \tau. \]

Then

\[ \rho \, dh = \tau \, dz, \]

and so mass conservation becomes

\[ \nabla \cdot (\tau \mathbf{v}) = 0. \]

We consider fluid between two surfaces \( z = \text{const.} \) (\( p = \text{const.} \)): \( z = 0, H \) say, and look for a solution

\[ u = -ax + u', \]

\[ v = a\gamma + v', \]

\[ w = w', \]

\[ \phi = f \pi xy - (\alpha^2 + \dot{\alpha}) y^2/2 - (\alpha^2 - \dot{\alpha}) x^2/2 + \phi', \]

\[ \theta = \theta'. \]
where all primed quantities are independent of y and z is a slowly varying function of t only. We assume that \( u' \), \( v' \), \( w' \) are negligible everywhere initially and for all time at infinite \( x \). The equation of hydrostatic balance is

\[
g \frac{\partial \theta'}{\partial \theta} = \frac{\partial \Phi'}{\partial x},
\]

(1)

and the ' geostrophic cross-front balance'

\[
f v' = \frac{\partial \Phi'}{\partial x}.
\]

(2)

Combining Eqs. (1) and (2) we have the thermal wind relation

\[
\frac{f \partial v'}{\partial x} = \frac{g}{\theta_0} \frac{\partial \theta'}{\partial x}.
\]

(3)

For consistency, the initial potential temperature \( \theta_t (x_t/L_t, z) \), say, must have \( L_t \) large enough that the associated thermal winds are negligible.

In the \( x \) direction we use the ' geostrophic ' co-ordinate (see II)

\[
X = x + v' f.
\]

For notational convenience we use \( Z = z \), \( T = t \) to denote the vertical co-ordinate and time respectively so that \( \partial / \partial Z \) is a derivative at constant \( X \). It can easily be shown that the thermal wind relation (Eq. (3)) is conserved: there exists a function \( \Phi' (= \phi' + \frac{1}{2} v'^2) \) such that

\[
g \frac{\partial \theta'}{\partial \theta} = \frac{\partial \Phi'}{\partial Z}, \quad f v' = \frac{\partial \Phi'}{\partial X}.
\]

(4)

In the initial state of negligible \( v' \) (time \( t_i \)), we have \( X = x_t \). The \( X \) velocity of a fluid particle may be found from the \( y \) equation of motion which can be rewritten

\[
\frac{DX}{Dt} = -v X.
\]

Thus in the \( X,Z \) plane a fluid column moves in the \( X \) direction with the basic deformation velocity and at time \( T \) its position is

\[
X = x_t \exp \left( -\int_{t_i}^T \frac{dX}{X} \right) = x_t \exp [-\beta(T)], \text{ say.}
\]

(5)

The vertical component of absolute vorticity is

\[
\zeta = f + \frac{\partial v}{\partial x} = \frac{f}{1 - \frac{f}{\theta_0} \frac{\partial \theta'}{\partial X}}.
\]

(6)

In adiabatic, frictionless motion Ertel's potential vorticity \( q \) is conserved. In the \( X,Z \) co-ordinate system

\[
q = \frac{1}{\theta} \frac{\partial \theta'}{\partial Z}.
\]

(7)

From Eqs. (4), (6), and (7), we have

\[
\frac{1}{f^2} \frac{\partial^2 \Phi'}{\partial X^2} + \frac{f \theta_0}{g q} \frac{\partial^2 \Phi'}{\partial Z^2} = 1.
\]

(8)

Since \( q, g \theta'/\theta = \partial \Phi'/\partial Z \) and \( X \) are known for all time on each fluid particle from their initial values, the problem is well posed, the boundary conditions for Eq. (8) being

\[
\frac{\partial \Phi'}{\partial Z}(X, \text{bdry, } T) = \frac{g}{\theta_0} \theta_t(X/L, \text{bdry}),
\]

where \( L \) is the geostrophic length scale \( L_0 \exp [-\beta(T)] \).
The only non-linearity remaining in the mathematical problem is that \( q \) is known not as a function of \( X \) and \( Z \), but as a function of \( X \) and \( \frac{\partial \Phi'}{\partial Z} \).

As described in II, the solutions shown in I were obtained by taking the \( Z \) derivative of Eq. (8) and solving for \( \theta' \). However, we now consider the problem for \( \Phi' \). Instead of \( q \) we consider the conserved quantity analogous to the Brunt-Väisälä frequency in the initial state:

\[
N^2 = \frac{g}{\theta_0} \frac{d \theta}{d Z} = \frac{g}{f \theta_0} \tau(z_I) q.
\]

We also set \( \mu = \tau(z_I)/\tau(Z) \). Since \( z_I \) is a known function of \( \theta \) and \( X \), \( \mu \) is a known function of \( X \), \( Z \) and \( \frac{\partial \Phi'}{\partial Z} \). Eq. (8) may now be written

\[
\frac{1}{f^2} \frac{\partial^2 \Phi'}{\partial X^2} + \frac{\mu}{N_1^2} \frac{\partial^2 \Phi'}{\partial Z^2} = 1.
\]

As before, we model the stratosphere-troposphere system by two fluids of initially uniform Brunt-Väisälä frequencies, that of the upper fluid \( (N_2) \) being much larger than that of the lower fluid \( (N_1) \). The internal surface of discontinuity models the tropopause. If the initial potential temperatures on \( Z = 0 \), the tropopause, and \( Z = H \) are \( \theta_t(X/Z_0), \theta_x(X/Z_0) \), and \( \theta(\infty) \), respectively, and the initial height of the tropopause \( h_t(X/Z_0) \), then

\[
\theta_t = \theta_1 + N_1^2 \theta_0(g) h_t,
\]

\[
\theta_x = \theta_t + N_2^2 \theta_0(g)(H - h_t) = \theta_1 + (\theta_0/g) [N_2^2 H - (N_2^2 - N_1^2) h_t].
\]

In order to have no unrealistic front formation at the 'lid', \( h_t \) is chosen such that \( \theta_x \) is constant.

On \( z = 0 \) and \( H \) the boundary conditions at time \( t \) are

\[
\frac{\partial \Phi'_1}{\partial Z} = \frac{g}{\theta_0} \theta_t \left( \frac{X}{L} \right) \quad \text{and} \quad \frac{\partial \Phi'_2}{\partial Z} = \frac{g}{\theta_0} \theta_x, \quad \text{respectively.}
\]

On the unknown internal boundary we have that \( \theta \) is given and \( \nu \) is continuous. Therefore

\[
\frac{\partial \Phi'_1}{\partial Z} = \frac{g}{\theta_0} \theta'_t \left( \frac{X}{L} \right) = \frac{\partial \Phi'_2}{\partial Z} \quad \text{and} \quad \Phi'_1 = \Phi'_2.
\]

The resulting mathematical problem for the solution at time \( T \) is shown in Fig. 1. The side boundary conditions would ideally be

\[
\frac{\partial \Phi'}{\partial Z} (\pm \infty, Z, T) = \frac{g}{\theta_0} \theta'_t (\pm \infty, Z).
\]

---

Figure 1. The mathematical problem.
An arbitrary function of \( X \) still occurs in the solution. However, this may be removed by noting the identity (see Appendix)

\[
\int_0^H r v' \, dZ = 0. \tag{10}
\]

This states that the additional \( y \) momentum in a fluid column, initially vertical, is identically zero for all time.

The problem is solved numerically much as in II. The boundary condition at infinity is replaced by one at finite \( X \):

\[
\frac{\partial \Phi'}{\partial Z}(\pm X_0, Z, T) = \frac{g}{\theta_0} \theta' \left( \pm \frac{X_0}{L}, Z \right).
\]

The position of the tropopause is fixed and we relax towards the solution of the problem as posed above except that at the tropopause we impose only \( \frac{\partial \Phi'_1}{\partial Z} = \frac{\partial \Phi'_2}{\partial Z} \) and \( \Phi'_1 = \Phi'_2 \). After a certain number of steps we move the tropopause such that the common value of \( \frac{\partial \Phi'}{\partial Z} \) there is likely to be nearer the required value. The method is then repeated.

The Boussinesq approximation is to set \( \mu \equiv 1 \) in Eq.(9) and neglect the change in volume of a fluid element. This is expected to be a good approximation if the height scale of the motion is much smaller than the scale height of the atmosphere (\( \sim 8 \) km). With the Boussinesq approximation, Eq. (10) is no longer exact. A zero for \( \nu' \) may be found by requiring that the total extra northward momentum of the fluid is zero.

The initial temperature distribution and the values of constants assumed in this paper are as in Experiment 4 described in I. There is a surface potential temperature contrast \( \sim 49^\circ K \) and a weak minimum in potential temperature on the cold side of the transition zone. The compensating initial tropopause is at 293 mb on the cold side, descends to 304 mb at the minimum in potential temperature and rises to 220 mb on the warm side. The constants used are:

\[
N_1 = 10^{-2} s^{-1}, N_2 = 3 \times 10^{-2} s^{-1}, f = 10^{-4} s^{-1},
\]

\[
\theta_0/g = 30 \text{ m}^{-1} \text{s}^2 \text{K}, R = 287 \text{ m}^2 \text{s}^{-2} (\text{K})^{-1},
\]

\[
H = 1.52 H_s + \text{lid at 135 mb}.
\]

3. COMPARISON WITH A PREVIOUS SOLUTION OF THE BOUSSIÈRESQ PROBLEM

The advantage of the present method is the simplicity of the boundary conditions at the tropopause. The previous method demanded a matching across this unknown surface which involved the slope of the surface. Clearly the present method must be more stable and convergence made easier. However, since quantities such as vorticity now involve taking second derivatives and their values are not now matched across the tropopause, their values may be expected to be slightly rougher than before. To present a comparison we apply both methods to Experiment 4 to I.

In Fig. 2 is shown the shape the tropopause would have had if there had been no induced ageostrophic motions. Below is exhibited the solution (B6) obtained by the method described in this paper. The main features of interest are:

(i) the tongue of stratospheric air with large gradients in velocity and temperature across it;

(ii) the jet stream;

(iii) the surface front.

These features are extremely similar to those described in detail in I. For a comparison between a solution based on the previous method (16) and the present solution (B6), we refer to Tables 1 and 2. They give the extrema of physical quantities at or near the surface and tropopause respectively. It should be noted that the extrema of \( \partial \theta/\partial x, \partial \theta/\partial z, \text{Ri in} \)
both Tables and ζ in Table 1 represent interior grid point values and hence underestimate (overestimate) the values at the surface in question. Table 1 shows that the surface front in I6 and B6 has developed in an identical manner. The same is true of the upper-air front except for slight differences in the height of the base of the tongue and the maximum vorticities. After 1,000 iterations the maximum movement of the tropopause in non-dimensional units is $8 \times 10^{-3}$ and $8 \times 10^{-6}$ in I6 and B6 respectively. This shows the increased convergence and indicates that the height in the present experiment (B6) is more reliable. Given that the rest of the solution is almost identical, increased descent of the tropopause implies increased stretching of vortex lines above and decreased stretching below. Consequently, we expect larger vorticities above and small vorticities below the tropopause. However, some of the difference in vorticities shown may be due to the inherent smoothing in I6 and some roughness in the vorticities in B6.
TABLE 2. VALUES OF PHYSICAL QUANTITIES AT OR NEAR THE TROPOPAUSE. (2 AND 1 REFER TO ABOVE AND BELOW THE TROPOPAUSE RESPECTIVELY)

<table>
<thead>
<tr>
<th></th>
<th>I6</th>
<th>B6</th>
<th>NB6</th>
<th>NB5·5</th>
<th>NB5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base of tongue</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maximum descent of tropopause mb</td>
<td>407</td>
<td>411</td>
<td>425</td>
<td>438</td>
<td>460</td>
</tr>
<tr>
<td>$v_{\text{max}}$ [ms$^{-1}$]</td>
<td>111</td>
<td>115</td>
<td>128</td>
<td>139</td>
<td>159</td>
</tr>
<tr>
<td>$L_2$ max</td>
<td>46</td>
<td>46</td>
<td>50</td>
<td>51</td>
<td>52</td>
</tr>
<tr>
<td>$L_1$ max</td>
<td>3.6</td>
<td>4.4</td>
<td>4.8</td>
<td>5.2</td>
<td>8.2</td>
</tr>
<tr>
<td>$(\frac{\partial \theta}{\partial z})_{\text{max}}$ (10$^4$K km$^{-1}$)</td>
<td>2.5</td>
<td>2.3</td>
<td>2.0</td>
<td>2.1</td>
<td>2.3</td>
</tr>
<tr>
<td>$(\frac{\partial \theta}{\partial z})_{\text{min}}$ (10$^4$K km$^{-1}$)</td>
<td>7</td>
<td>7</td>
<td>10</td>
<td>11</td>
<td>16</td>
</tr>
<tr>
<td>$(\frac{\partial \theta}{\partial x})_{\text{max}}$ (10$^4$K km$^{-1}$)</td>
<td>-4</td>
<td>-4</td>
<td>-5</td>
<td>-5</td>
<td>-6</td>
</tr>
<tr>
<td>$(\frac{\partial \theta}{\partial x})_{\text{min}}$ (10$^4$K km$^{-1}$)</td>
<td>8</td>
<td>8</td>
<td>9</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>$R_{\xi, \text{min}}$</td>
<td>0.79</td>
<td>0.96</td>
<td>0.52</td>
<td>0.41</td>
<td>0.26</td>
</tr>
<tr>
<td>$R_{\xi, \text{min}}$</td>
<td>-0.84</td>
<td>-0.84</td>
<td>-0.96</td>
<td>-0.82</td>
<td>-0.60</td>
</tr>
</tbody>
</table>

Figure 3. The non-Boussinesq solution (NB6). Notation as in Fig. 2.

4. NON-BOUSSINESQ EFFECTS

In Fig. 3 is shown the solution (NB6) when the Boussinesq approximation $\mu = 1$ is not made. Comparison with Fig. 2 indicates that, as expected, non-Boussinesq effects are not dramatic. However, closer inspection shows that the tongue of stratospheric air has descended a further 14 mb and is somewhat narrower. The jet stream has strengthened, while the jet associated with the surface front has weakened. The surface front has also weakened. A detailed comparison between B6 and NB6 is given in Tables 1 and 2. Clearly non-Boussinesq effects increase the realism of our model of the upper tropospheric front and its jet stream.

Defining the vorticity to be

$$\zeta = f k + \nabla \times u,$$

the vorticity equation may be written

$$\frac{D\zeta}{Dt} = (\zeta \cdot \nabla) v - k \frac{g}{\theta_0} \nabla \theta - \lambda \zeta. \quad \ldots \quad \ldots \quad (11)$$
where
\[ \lambda = \nabla \cdot \mathbf{v} = -\frac{1}{r} \frac{d\gamma}{dz} \frac{w}{1 - \kappa z / H_s} \gamma H_s. \]

However, using the Boussinesq equations, we would obtain Eq. (11) with \( \lambda = 0 \). Thus the non-Boussinesq term indicates an increase in vorticity in descending air and a decrease in ascending air. Clearly this is consistent with our findings of increased development of the upper tropospheric front which is formed in descending air and decreased development of the surface front whose formation is associated with ascending air.

5. Further Development

We now present two solutions of the same model at later times. These two solutions correspond to geostrophic length scales 5.5/6 and 5/6 of the value prescribed in the above solution NB6 and will be denoted by NB5.5 and NB5 respectively. In other words, under the action of the basic deformation field alone, a fluid element of length 6l in the x direction at the time of NB6 would have shrunk to 5.5l at the time of NB 5.5 and 5l at the time of NB5. Giving \( \kappa \) the small value of \( 10^{-1} \text{s}^{-1} \), the time between NB6 and NB5.5 is 2.4 hours and between NB5.5 and NB5 is 2.6 hours.

Details of the surface and upper tropospheric fronts in NB5.5 and NB5 are given in Tables 1 and 2. The solution NB5 is shown in Fig. 4 together with the shape the tropopause would have had in the absence of ageostrophic motions. The surface front develops as expected and we now confine our remarks to the formation of the upper tropospheric front. A comparison of Figs. 3 and 4 shows that the tongue of stratospheric air has pushed deeper into the troposphere and become narrower. The gradients in velocity and temperature in the base of the tongue have become larger. The velocity of the jet has, however, increased only slightly. This may be understood from the thermal wind relation (Eq. (3)).

---

Figure 4. Above: The geostrophic contraction of the tropopause. Below: The non-Boussinesq solution at this time (NB5). Notation as in Fig. 2, plus particle paths from NB6. The basic deformation motion is shown below the lower surface.
implies that $\psi$ is a vertical integral of the potential temperature gradient. Over most of the fluid this latter quantity changes little.

The geostrophic co-ordinate acts as a boundary layer co-ordinate for fronts in that lines $X = \text{const.}$ crowd together in regions of large vorticity. Despite this, the tongue becomes narrow even in the $XZ$ plane. Because of the grid size used, second derivatives at the tropopause become rather rough in the NB5 solution, as is shown by the maximum vorticity for the upper layer given in Table 2.

As in I, the minimum in potential temperature on the tropopause occurs near the base of the tongue. As was shown in II, this indicates no tendency to form discontinuities in upper air fronts, unlike surface fronts (see I and II).

The Richardson numbers shown in Table 2 indicate an increasing likelihood of overturning in the base of and underneath the tongue. Endlich and McLean (1957) have noted that this is a preferred region for clear air turbulence.

Also shown in Fig. 4 are some fluid particle paths from their positions in NB6. The paths are much as described in I and agree well with observations of humidity and cirrus clouds by Vuorela (1953), Shafer and Hubert (1955) and others. Taking $\alpha = 10^{-8} \text{s}^{-1}$, the maximum descent shown underneath the tropopause is at an average rate of 9 mb hr$^{-1}$ for the five-hour period. If the deformation were such as to weaken the front ($\alpha < 0$) then motions would be in the reverse direction.

6. Conclusion

We have exhibited the development of a model upper tropospheric front due to non-Boussinesq effects and further squeezing. This development is summed up in Table 2. The final solution given (NB5) seems to model quite realistically the phenomena observed by Reed (1955), Reed and Danielsen (1959) and others.

References


Vuorela, L. 1953 'On the air flow connected with the invasion of upper tropical air over north-western Europe,' Geophysica, 4, pp. 105-130.

Appendix

In II it is shown that by introducing a stream function $\psi$ such that

$$ru' = \partial\psi/\partial z, \quad rw' = -\partial\psi/\partial x,$$

and using the geostrophic co-ordinate $X$, the $y$ equation of motion may be written

$$r(\vartheta + \alpha) u' + f \partial\psi/\partial Z = 0.$$

Here

$$\vartheta = \partial/\partial T - aX \partial/\partial X$$

is the derivative moving with a fluid column. Since $r$ is a function of $z$ only,
\[(\mathcal{D} + \kappa) \int_0^H ru' \, dZ = -f [\psi]_0^H. \]

But \(w' = 0\) on \(Z = 0\), \(H\) implies that \(\psi\) is constant on each boundary. The boundary condition \(u' = 0\) at \(x = \pm \infty\) implies that this constant is the same for both boundaries i.e. \([\psi]_0^H = 0\). Since the integral is initially zero, we must have, for all time,

\[\int_0^H ru' \, dZ = 0.\]

The additional \(y\) momentum in a fluid column is identically zero for all time.