Some experiments in dynamic initialization for a simple primitive equation model

By C. TEMPERTON
Meteorological Office, Bracknell

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SUMMARY

A number of experiments in dynamic initialization are carried out within the framework of a relatively simple primitive equation model. Some previously suggested variants of initialization schemes based on the Euler-backward timestep are tested and found wanting. A new scheme is proposed, based on averaging the solutions of the primitive equations at time-levels separated by a small interval. This scheme is tested, with encouraging results.

Theoretical considerations and experimental evidence indicate that both mass-to-wind and wind-to-mass adjustment should be catered for in a dynamic initialization scheme. This facility can easily be incorporated into the scheme proposed here, and tests confirm the advantages of doing so.

1. Introduction

During the past decade, the use of primitive equation models for simulating and forecasting atmospheric motions has become well-established. A problem arose early in the development of such models concerning the specification of initial conditions for numerical integrations: if the mass and wind fields are not properly balanced, gravity waves are excited. These may attain unrealistic amplitudes, with damaging effects on the meteorological modes which the integration is supposed to be simulating. The task of specifying suitably balanced initial fields has become known as the 'initialization problem'.

An early approach, carried over from the days of filtered models, was to assume that the required balance could be expressed as a diagnostic relationship between the mass and wind fields. The simplest such relationship is the geostrophic approximation; more sophisticated diagnostic relationships lead to a hierarchy of balance equations for the rotational part of the wind (Houghton and Washington 1969; Baumhefner 1970), and to various forms of the $\omega$-equation for the divergent part (Houghton, Baumhefner and Washington 1971). For simple models, a detailed analysis was given by Phillips (1960).

This approach suffers from a number of drawbacks: the resulting diagnostic equations are often difficult and time-consuming to solve; to achieve consistency with the model's finite-difference equations of motion involves further complications; and incorporation of secondary physical effects (e.g. convection, heating, friction and diffusion) into the diagnostic approach is virtually impossible. In particular, it is clear that the required balance is partly a function of the model itself.

These considerations led to a possible alternative approach, that of determining a balanced initial state by making use of the modelling equations themselves – the concept of dynamic initialization. The two papers which pioneered this approach were by Miyakoda and Moyer (1968), and Nitta and Hovermale (1969). Both were concerned primarily with obtaining a wind field in balance with a given mass field.

In Miyakoda and Moyer's initialization scheme, the first and second time derivatives of the divergence are constrained to be zero, as in the derivation of the $\omega$-equation; but instead of explicitly solving a diagnostic equation, the primitive equations are used in an iterative fashion, to alter the wind field slowly until the constraints on the divergence are satisfied. The principal computational drawback to this scheme is that it involves the solution of Poisson equations, repeated many times. Also, the restrictions placed on the divergence are reasonable for synoptic-scale motions outside the Tropics, but of doubtful validity in other situations.
In the scheme proposed by Nitta and Hovermale, on the other hand, no constraints are placed on the divergence; instead, the modelling equations are used to integrate backwards and forwards around the initial time, using a timestep scheme designed to damp the high-frequency gravity-wave modes while retaining the lower-frequency balanced meteorological modes. Besides lifting the restrictions imposed by the Miyakoda-Moyer scheme, the Nitta-Hovermale scheme is more adaptable in that provision can be made, if desired, for the mass field to adjust to the wind field, rather than the other way round. The main disadvantage of the scheme is the somewhat slow rate of convergence towards the required balanced solution.

In this paper, we shall follow Nitta and Hovermale in seeking to achieve balance by damping the unwanted gravity waves, rather than by imposing some assumed diagnostic relationship. In particular, we shall examine some suggested variants of the Nitta-Hovermale scheme. In addition, a new dynamic initialization scheme will be proposed and tested, in the hope of achieving a faster rate of convergence than with the previous schemes.

Finally, it may be appropriate at this point to mention briefly the closely related problem of four-dimensional data assimilation now exciting considerable interest (Williamson and Kasahara 1971; Mesinger 1972) as non-synchronous observations on a global scale become more general. Any progress made with the initialization problem at a single time-level is likely to be helpful in attacking the full four-dimensional problem.

2. The model

(a) Choice of model

In selecting a model as a test-bed for initialization experiments, it seemed appropriate to begin with the simplest model capable of simulating atmospheric motions with some degree of realism. In particular it should possess those characteristics relevant to the experiments (the existence of both Rossby and gravity-wave modes in the solutions), and should describe fully two-dimensional flow in the presence of nonlinear interactions. A natural choice was the so-called free-surface model describing the motions of an inviscid incompressible fluid with vanishing stress at the upper boundary. The geometry of the model was chosen to be that of a beta-plane, the fluid being confined in a channel between rigid free-slip walls to the north and south, with cyclic boundary conditions imposed in the zonal direction.

The continuous equations for this model, expressed in momentum form, are:

\[
\frac{\partial (\phi u)}{\partial t} + \frac{\partial}{\partial x} (\phi u^2) + \frac{\partial}{\partial y} (\phi uv) - f \phi v + \phi \frac{\partial \phi}{\partial x} = 0
\]  \quad (1)

\[
\frac{\partial (\phi v)}{\partial t} + \frac{\partial}{\partial x} (\phi uv) + \frac{\partial}{\partial y} (\phi v^2) + f \phi u + \phi \frac{\partial \phi}{\partial y} = 0
\]  \quad (2)

\[
\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x} (\phi u) + \frac{\partial}{\partial y} (\phi v) = 0
\]  \quad (3)

where \( \phi \) is the geopotential height of the free surface, \( u \) and \( v \) are the zonal and meridional components of the flow, \( f \) is the Coriolis parameter, and \( x, y, t \) have their usual meanings. The equations are solved numerically over a grid of 32 \( \times \) 32 points.

(b) Finite-differencing in space

The finite-difference scheme used conserves momentum in the advective terms, and total mass over the domain, and also conserves total energy apart from time-truncation effects. It is the Cartesian analogue of that described by Grinner and Shaw (1967) and is equivalent to Grammelvedt's (1969) Scheme B.
Making the following definitions of finite-difference operators:

\[
\delta_x F = (F(x + \frac{1}{2} \Delta x) - F(x - \frac{1}{2} \Delta x))/\Delta x
\]

\[
F^x = \frac{1}{2}(F(x + \frac{1}{2} \Delta x) + F(x - \frac{1}{2} \Delta x))
\]

where \( \Delta x \) is the grid increment in the x-direction, and with \( \delta_y F, F^y \) similarly defined, the equations are written:

\[
\frac{\partial (\phi u)}{\partial t} + \delta_x (\phi u^x u^x) + \delta_y (\phi u^y u^y) - f\phi u + \phi \delta_x \phi^x = 0
\]  \hspace{1cm} (4)

\[
\frac{\partial (\phi v)}{\partial t} + \delta_x (\phi u^x v^x) + \delta_y (\phi v^y v^y) + f\phi u + \phi \delta_y \phi^y = 0
\]  \hspace{1cm} (5)

\[
\frac{\partial \phi}{\partial t} + \delta_x \phi u^x + \delta_y \phi v^y = 0
\]  \hspace{1cm} (6)

where \( \phi, u, v \) are all defined at each grid point, without staggering.

The rigid walls forming the northern and southern boundaries of the channel lie halfway between two grid lines. In addition to Eqs. (4) to (6), we require boundary conditions at the walls. For the continuous case, the free-slip rigid wall condition implies \( v = 0 \) at the boundary; in the finite-difference scheme we represent this by \( v^y = 0 \). Conservation of total energy over the domain of integration then requires that we set \( \delta_y \phi = 0 \) at the wall; no boundary condition is called for on \( u \). Cyclic continuity provides the boundary conditions in the x-direction.

(c) \textit{Finite-differencing in time}

During the course of this study, reference will be made to several time-stepping procedures. For the sake of convenience, they are summarized here together with their relevant properties. In general we will consider finite-difference approximations to the differential equation:

\[
\frac{\partial X}{\partial t} = F(X)
\]

where \( F \) is a function of an arbitrary variable \( X \). In the following definitions, superscripts refer to time levels:

(i) Forward timestep:

\[
X^{n+1} = X^n + \Delta t F(X^n)
\]

(ii) Centred (leapfrog) timestep:

\[
X^{n+1} = X^{n-1} + 2\Delta t F(X^n)
\]

(iii) Euler-backward timestep:

\[
X^n = X^n + \Delta t F(X^n)
\]

\[
X^{n+1} = X^n + \Delta t F(X^*)
\]

(iv) Modified Euler-backward timestep:

\[
X^n = X^n + \frac{1}{2} \Delta t F(X^n)
\]

\[
X^{n+1} = X^n + \Delta t F(X^{**})
\]

The forward timestep is unstable for all values of \( \Delta t \), and is used only to start an integration. For the free-surface model, define:
\[ \omega = \left| \frac{u}{\Delta x} + \frac{v}{\Delta y} \right| + c\left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right)^{\frac{1}{2}} \]

where \( c \) is the characteristic gravity wave speed, equal to the square root of the mean geopotential height. Then for the centred and Euler-backward timesteps, the linear stability criterion is \( \omega \Delta t \leq 1 \), while for the modified Euler-backward scheme, the criterion is \( \omega \Delta t \leq \sqrt{2} \).

Provided the stability criterion is satisfied, the centred timestep neither damps nor amplifies the solution; both forms of the Euler-backward scheme selectively damp the higher-frequency modes. For fuller details of these timestepping procedures, see for example Kurihara (1965).

(d) Separation of rotational and divergent components

From time to time we shall refer to the rotational and divergent components of the flow field. The finite-difference form of the well-known Helmholtz decomposition theorem states the following: let \((u,v)\) be a wind field specified at a set of grid points. Define the finite-difference vorticity \( \xi \) and divergence \( \nabla \) of the field in the natural way:

\[ \xi = -\delta_y u^{x} + \delta_x v^{y} \]
\[ \nabla = \delta_x u^{x} + \delta_y v^{y}. \]

Then we can separate the field \((u,v)\) into two components, say \((u_1, v_1)\) and \((u_2, v_2)\), satisfying the following:

\[ u = u_1 + u_2; \quad v = v_1 + v_2 \]
\[ -\delta_y u_1^{y} + \delta_x v_1^{x} = \xi; \quad \delta_x u_1^{x} + \delta_y v_1^{y} = 0 \]
\[ -\delta_y u_2^{y} + \delta_x v_2^{x} = 0; \quad \delta_x u_2^{x} + \delta_y v_2^{y} = \nabla. \]

We refer to \((u_1, v_1)\) as the rotational part of the wind field, and to \((u_2, v_2)\) as its divergent part. The decomposition is achieved by defining, at the same set of grid points, a stream function \( \psi \) and a velocity potential \( \chi \), and setting:

\[ u_1 = -\delta_y \psi^{y}; \quad v_1 = \delta_x \psi^{x} \]
\[ u_2 = \delta_x \chi^{x}; \quad v_2 = \delta_y \chi^{y}. \]

We obtain a Poisson equation for \( \psi \):

\[ \delta_x (\delta_x \psi^{x}) + \delta_y (\delta_y \psi^{y}) = \xi \]

where the finite-difference operator on the left-hand side has the form of the usual five-point approximation to the Laplacian, but corresponds to a grid with space increments \(2\Delta x, 2\Delta y\). This form gives rise to problems with the boundary conditions, but with the geometry of the grid used in the present model these can be overcome. The details will not be discussed here. Finally, note that the decomposition theorem can similarly be applied to the momentum field \((\phi u, \phi v)\).

(e) Linearized balance equation

At two stages during this study, we use a linearized balance equation to derive a wind field approximately in balance with a given geopotential field. The equation is formed by dropping the advective terms from Eqs. (4) and (5), rewriting the pressure gradient terms, taking the finite-difference curl of the equations, and assuming that:

\[ D = \delta_x \phi u^{x} + \delta_y \phi v^{y} = 0 \quad \text{and} \quad \frac{\partial D}{\partial t} = 0. \]
The resulting equation for the momentum stream function $\psi$ is:

$$f\nabla^2\psi + \beta\partial\psi^y = \frac{1}{2}\nabla^2(\psi^2)$$

(7)

where $\nabla^2 = \delta_{xx} + \delta_{yy}$ is the usual five-point approximation to the Laplacian operator, and $\beta = \partial f/\partial y$. Since the north and south walls of the channel must be streamlines of the flow, we set $\psi^y = \text{constant}$ at each wall. Having solved Eq. (7) over the domain, e.g. by successive over-relaxation, we set:

$$\phi_u = -\delta_y\psi^y; \quad \phi_u = \delta_x\psi^x.$$  

(8)

Note that the assumption $D = 0$ has yielded a purely rotational momentum field. The use of Eqs. (7) and (8) ensures that:

$$\frac{\partial\psi}{\partial t} = 0 \quad \text{and} \quad \frac{\partial^2\psi}{\partial t^2} \approx 0$$

are satisfied initially.

3. Design of experiments

The aim of an initialization procedure is to provide suitably balanced mass and wind fields from which to start a forward integration. In testing such a procedure, we encounter the following difficulty: suppose for example that an arbitrary mass field has been specified. We do not know in advance what the corresponding balanced wind field should be, and so we cannot measure the success of the initialization procedure by comparing the 'initialized' and 'balanced' wind fields.

To circumvent this difficulty we follow Nitta and Hovermale (1969) and Mesinger (1972), and let the model generate its own balanced set of fields, which can then be used as control data for testing various initialization schemes.

The control integration was started with an analytically prescribed geopotential field similar to that used by Grammelveldt (1969) in his test of finite-difference schemes. Corresponding initial winds were computed from the linearized balance equation.

The following values were assigned to various parameters of the model:

$$\Delta x = 2 \times 10^3 \text{ m}; \quad \Delta y = 2 \times 10^5 \text{ m}; \quad \Delta t = 600 \text{ s};$$

$$f_0 = 2 \times 10^{-4} \text{ s}^{-1}; \quad \beta = 1.5 \times 10^{-11} \text{ m s}^{-1}; \quad \Phi = 2 \times 10^4 \text{ m}^2\text{s}^{-2}$$

where $f_0$ is the value of the Coriolis parameter in the middle of the channel, $\beta = \partial f/\partial y$, and $\Phi$ is the mean geopotential height of the free surface.

The model was then integrated forwards for two days, using an Euler-backward timestep throughout. By this time, the selective damping properties of the Euler-backward scheme appeared to have virtually removed any gravity waves implicit in the initial fields, leaving only the slowly evolving 'meteorological' modes consisting of mutually balanced geopotential and wind fields. The initial (Day 0) and final (Day 2) geopotential fields from this control integration are shown in Fig. 1.

The stage was thus set for testing a number of initialization schemes. Given the geopotential field 'observed' at Day 2 of the control integration, the aim of each scheme was to find a corresponding balanced wind field; as the initialization proceeded, the initialized wind field was compared with that observed in the control integration. Besides a prescribed geopotential field, an initialization scheme requires a first-guess wind field. This was provided by the linearized balance equation as formulated in Section 2(e). Mesinger (1972) has warned against the use, in testing initialization schemes, of a first-guess wind field which is already close to the observed field; however, it seems reasonable that in any practical application the first guess should be provided by some simple form of balance equation, or at least by the use of the geostrophic relationship.

So far, we have merely followed the experimental procedure of previous investigators. It seems important, though, to bear in mind our characterization of mutually balanced
mass and wind fields – that they provide initial conditions for an integration from which gravity-wave modes are effectively absent. A simple measure such as the r.m.s. magnitude of the vector difference between the initialized and observed wind fields is not adequate in assessing how close an initialization scheme has come to providing the required balance, even when we consider that our observed fields are themselves balanced.

With this consideration in mind, another test was applied: the initialized wind components, together with the observed geopotential field from Day 2 of the control integration, were used as initial conditions for a further integration of 24 hours. In this integration, the centred timestep was used throughout (after an initial forward timestep); because of the lack of damping inherent in the centred timestep scheme, any imbalance between the mass and wind fields can only be removed through the mechanism of gravity waves, which are allowed full play. During this continued integration, the parameter which was monitored to indicate the presence or otherwise of gravity waves was the r.m.s. mass divergence, equivalent by Eq. (3) to the r.m.s. rate of change of geopotential height.
DYNAMIC INITIALIZATION

Figure 2. r.m.s. mass divergence during continued integrations with different initial wind fields: as observed during control integration (-----); and the rotational part only of the observed wind field (-----).

As an important first application of this test, Fig. 2 shows the r.m.s. mass divergence during a continued integration using the observed fields themselves as initial conditions. This test establishes the 'synoptic' level of this parameter, and the smoothness of the graph confirms that the observed data provided by the control integration are indeed well-balanced.

Also shown in Fig. 2 is the evolution of the r.m.s. mass divergence during a second continued integration, using as initial conditions the observed geopotential field together with the rotational part only of the observed momentum field. Clearly, this integration suffers from a considerable amount of noise caused by the presence of gravity waves. This demonstrates Phillips' (1960) point that the divergent component of the flow field forms an essential part of the balance. It also illustrates the point made above concerning the inadequacy of the r.m.s. error as a measure of success of an initialization scheme; the r.m.s. difference between the initial wind fields for the two integrations shown in Fig. 2 is only 0.074 ms⁻¹.

4. EXPERIMENTS USING EULER-BACKWARD SCHEMES

The basis of the initialization scheme proposed by Nitta and Hovermale (1969) was to integrate around a central time-level, using the Euler-backward timestep to damp the high-frequency modes, and periodically restoring the mass field in order to force the wind field to adjust to it. In detail, one cycle of the process consists of integrating forwards for \( N \) timesteps, backwards for \( 2N \) timesteps, then forwards again for \( N \) timesteps; in Nitta and Hovermale's own experiment \( N = 1 \) was used. The mass field is restored at some stage during the cycle – normally at each passage through the time origin.

In the experiments reported below, the modified form of the Euler-backward timestep was used, as this was found to give a considerably higher rate of convergence. Also, in view of the less stringent stability criterion associated with this scheme, the timestep was increased to 900s.

Nitta and Hovermale admitted that the convergence of their scheme was discouragingly slow. Mesinger (1972) has suggested two possible improvements: (i) to increase the iteration amplitude \( N\Delta t \), and (ii) to restore the mass field only partially, but to do so more frequently, i.e. not only when passing through the time origin.

Four experiments were carried out to test these ideas: Fig. 3 plots, for each experiment, the decrease in the r.m.s. wind 'error' (i.e. the r.m.s. magnitude of the vector difference between the computed and observed fields). Since the rate of convergence should be measured in terms of computing time rather than in terms of the number of initialization cycles, the unit along the x-axis is the basic timestep. On this reckoning, each modified Euler-backward timestep is equivalent to three basic timesteps, because the derivatives have to be evaluated three times.
Figure 3. r.m.s. wind error during initialization experiments using various Euler-backward schemes.

Experiment 1A uses the basic scheme; the iteration amplitude is one timestep, and the mass field is restored at each passage through the time origin. Experiment 1B differs in that the mass field is only partially restored; at each passage through the central time, the current value of the geopotential is replaced by a weighted mean:

$$\phi = \frac{2}{3}\phi_{\text{observed}} + \frac{1}{3}\phi_{\text{current}}$$  \hspace{1cm} (9)

Fig 3 shows that this procedure has actually slowed down the rate of convergence slightly as compared with the basic scheme 1A.

In Experiment 1C, the iteration amplitude was increased to $4\Delta t (=1 \text{ hr})$, and the mass field was completely restored at each passage through the time origin. Again, this procedure is seen to be detrimental to the rate of convergence.

Finally, in Experiment 1D, the iteration amplitude remained at $4\Delta t$, but the geopotential field was partially restored, using Eq. (9), at every other timestep during the cycle. This corresponds approximately to Mesinger's recommendations, and Fig. 3 shows that their use in conjunction has indeed improved the rate of convergence.

We now take the wind fields computed by the two ‘best’ schemes (the basic scheme 1A with $N = 1$ and the 'improved' scheme 1D with $N = 4$ and partial restoration of the height field every other timestep) after equal amounts of computation, and apply to each our second test, the 24-hour continued integration with centred timesteps. Fig. 4 shows the rather striking results of this comparison; although the r.m.s. error in the wind field computed by scheme 1D was slightly smaller than that computed by scheme 1A, its performance in the subsequent forward integration is markedly worse. Gravity wave activity is almost as great as in the integration started with a purely rotational flow field (Fig. 3), while the wind field computed by scheme 1A yields an integration in which gravity wave activity has been reduced to acceptable levels.

In order to investigate this result further, the rate of convergence during the initialization process was determined separately for the rotational and divergent components of the wind field. The results are plotted in Fig. 5, which reveals that although scheme 1D improved the convergence of the rotational wind field, the important divergent component was handled better by the basic scheme 1A. The reason for this is not difficult to see: by partially restoring the height field (which applies to the central time-level) at off-centre time-levels, we are artificially reducing the rate of change of geopotential, and thus reducing the magnitude of the divergence.
Figure 4. r.m.s. mass divergence during continued integrations using wind fields derived from initialization experiments 1A (-----) and 1D (-----).

Figure 5. r.m.s. rotational and divergent wind errors during initialization Experiments 1A and 1D.
We are forced to conclude that we have been unable to improve on the basic Nitta-Hovermale scheme. Moreover, the amount of computing time used during the initialization scheme 1A is equivalent to that required for a forward integration of over three days using centred timesteps. Clearly a faster scheme is called for.

5. EXPERIMENTS USING AVERAGING SCHEMES

The experiments reported in Section 4 made use of the selective damping properties of the modified Euler-backward timestep to remove the unwanted high-frequency gravity-wave modes. Another possible way of doing this arises naturally by considering the solution of Eqs. (1) to (3) at two time-levels separated by a suitable small time interval. If the solutions at the two time-levels are meaned, the low-frequency modes will be relatively unaffected, while the high-frequency modes will be damped. Consideration of typical values of the frequencies of meteorological and gravity-wave modes of the solution of Eqs. (1) to (3) offers the hope that a scheme based on this idea might be able to perform the required separation.

Accordingly, we can define a family of new initialization schemes. In the basic scheme, one iteration is carried out as follows: beginning at a central time-level \( t_0 \) we integrate forwards to \( t_0 + N\Delta t \), using a forward timestep followed by \( (N - 1) \) centred timesteps. We also integrate backwards from \( t_0 \) to \( t_0 - N\Delta t \) using a similar procedure. To obtain the new estimate of the fields at time-level \( t_0 \), we restore the control geopotential field, and replace the wind fields by

\[
 u = \frac{1}{2}(u(t_0 + N\Delta t) + u(t_0 - N\Delta t)) \\
 v = \frac{1}{2}(v(t_0 + N\Delta t) + v(t_0 - N\Delta t)).
\]

Two possible variants of this process will be considered here. Firstly, the simple forward and backward timesteps used to start each cycle have an undesirable amplifying effect, and might profitably be replaced by Euler-backward steps in each time-direction. Secondly, the iteration amplitude \( N\Delta t \) could be changed from one iteration to the next, in the hope of selectively damping waves of different frequencies.

Fig. 6 shows the results of four such experiments, together with the results of Experiment 1A for comparison. Again, the unit along the \( x \)-axis is the basic timestep, proportional

![Figure 6. r.m.s. wind error during initialization experiments using basic Euler-backward scheme (1A) and various averaging schemes (2A-2D).](image-url)
to total computing time. In all cases, each cycle was started with simple forward and backward timesteps; substitution of Euler-backward timesteps proved to be not worth the extra computation involved.

Experiments 2A, 2B and 2C differ only in the iteration amplitude, which is set respectively to $3\Delta t$, $6\Delta t$ and $9\Delta t$. In Experiment 2D, the iteration amplitude was set at $12\Delta t$ for the first iteration, and successively decreased by $\Delta t$ for each iteration; the process was stopped after the tenth iteration, for which the amplitude was $3\Delta t$. Fig. 6 shows that this procedure was not beneficial; the optimum rate of convergence was achieved in Experiment 2B, with a fixed iteration amplitude of $6\Delta t$, equal to 1 hr. Note that Experiment 2C, with an iteration amplitude of $9\Delta t$, actually exhibits divergence from the required solution; this illustrates the danger of extending too far the interval between the time-levels at which the variables are meant.

The principal result of this set of experiments is that we have achieved a considerably faster rate of convergence using 'averaging' schemes than with the Euler-backward schemes represented by the results of Experiment 1A, also shown on Fig. 6.

Finally, the effect of the initial wind error was examined by running an experiment (2E) using the same scheme as Experiment 2B, but in this case the first-guess wind field was taken from Day 5 of the control integration. This gave rise to an initial r.m.s. wind error some ten times greater than that yielded by the linearized balance equation. The results of the initialization run are plotted in Fig. 7; there is no evidence to suggest that our use of the linearized balance equation, with its rather small error, has led us to be over-optimistic in our assessment of the initialization scheme.

It remains to apply our second test of the initialized data. The wind fields computed after 25 cycles of Experiment 2B were used, in conjunction with the observed geopotential, as initial conditions for a further 24-hour integration using centred timesteps. As before, the r.m.s. mass divergence was monitored during this integration, and plotted on Fig. 8, which also repeats the results of the corresponding integration using observed winds as initial data. Note that the scale is more open than was used earlier for corresponding graphs.

The results (Fig. 8) show that we have still failed to eliminate entirely the unwanted gravity-wave modes. In the following Sections we shall investigate from a theoretical point of view why this should be so, and indicate a possible remedy. The results of Experiment 3, also plotted in Fig. 8, will be discussed in Section 8.

![Figure 7](image_url)  
Figure 7. r.m.s. wind error during initialization Experiment 2E—using the same scheme as Experiment 2B but with a large initial error.
6. Linearized Adjustment Theory

Washington (1964) has investigated the mechanism which tends to bring the mass and wind fields into a state of mutual balance. The mechanism, sometimes called the adjustment process, is of fundamental importance in understanding what takes place during dynamic initialization. Washington's analysis was essentially for a one-dimensional system; the following analysis represents an extension to a fully two-dimensional situation.

We consider the following linearized form of Eqs. (1) to (3):

\[
\frac{\partial u}{\partial t} = fv - \frac{\partial \phi}{\partial x} 
\]

\[
\frac{\partial v}{\partial t} = -fu - \frac{\partial \phi}{\partial y} 
\]

\[
\frac{\partial \phi}{\partial t} = -\Phi \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) 
\]

where all the variables have been previously defined; the Coriolis parameter \( f \) is now assumed to be constant.

Suppose that the fields can be represented as sums of Fourier components, e.g.:

\[
u(x, y, t) = \sum_{kl} u_{kl}(t) \exp (ikx + ily).
\]

By virtue of the linearity of the system (10) to (12), we can consider a particular pair of wave numbers \((k, l)\), and for convenience drop the subscripts from the Fourier coefficients. The transformed system thus becomes:

\[
\frac{\partial u}{\partial t} = fv - ik\phi 
\]

\[
\frac{\partial v}{\partial t} = -fu - il\phi 
\]

\[
\frac{\partial \phi}{\partial t} = -\Phi(iku + ilv) 
\]

where \( u, v, \phi \) are now to be understood as Fourier coefficients. Next we apply the spectral version of the Helmholtz decomposition theorem, and set:

\[
u = -il\psi + ik\chi; \quad \psi = ik\phi + il\chi
\]

where \( \psi \) is the stream function and \( \chi \) is the velocity potential. Substitution of Eq. (16) into Eqs. (13) to (15) yields the system:
\[ \frac{\partial \phi}{\partial t} = -fx \]  

(17)

\[ \frac{\partial X}{\partial t} = f\phi - \phi \]  

(18)

\[ \frac{\partial \phi}{\partial t} = \Phi(k^2 + l^2) X \]  

(19)

Strictly speaking, this system is not capable of describing the adjustment process, since it has a general solution which oscillates indefinitely without change of amplitude. However, the solution can be made to converge towards a balanced stationary state by including a damping term in the equation for the velocity potential; in spectral form, Eq. (18) is replaced by:

\[ \frac{\partial \phi}{\partial t} = f\phi - \phi - \nu(k^2 + l^2) X \]  

(18a)

where \( \nu \) is a diffusion coefficient.

Consider the stationary state of the modified system consisting of Eqs. (17), (18a) and (19), when

\[ \frac{\partial \phi}{\partial t} = \frac{\partial X}{\partial t} = \frac{\partial \phi}{\partial t} = 0. \]

Clearly, we must have:

\[ \chi_s = 0; \quad \phi_s = f\psi_s \]  

(20)

where the subscript \( s \) denotes the stationary value. Eq. (20) shows that, in the stationary state, the geopotential and wind fields are in geostrophic balance. An invariant quantity analogous to potential vorticity can be formed from Eqs. (17) and (19). Define

\[ \Omega = -\Phi(k^2 + l^2) \phi - f\phi \]

and we see that \( \partial \Omega/\partial t = 0. \)

If the system (17) to (19) is solved as an initial-value problem with the initial fields not in geostrophic balance, we can now relate the initial and stationary values of \( \Omega \) by \( \Omega_s = \Omega_i \), i.e.

\[ \Phi(k^2 + l^2) \psi_s + f\phi_s = \Phi(k^2 + l^2) \psi_i + f\phi_i \]  

(21)

where the subscript \( i \) indicates the initial value. Substituting Eq. (20) into Eq. (21) gives:

\[ \{\Phi(k^2 + l^2) + f^2\} \psi_s = \Phi(k^2 + l^2) \psi_i + f\phi_i. \]  

(22)

Finally, we construct a hypothetical stream function \( \psi_i = f^{-1} \phi_i \) in geostrophic balance with the initial geopotential field. Eq. (22) becomes:

\[ \psi_s = \frac{\Phi(k^2 + l^2) \psi_i + f^2 \psi_i}{\Phi(k^2 + l^2) + f^2} \]

i.e.

\[ \psi_s = \alpha \psi_i + (1 - \alpha) \psi_i \]  

(23)

where

\[ \alpha = \Phi(k^2 + l^2)/\{\Phi(k^2 + l^2) + f^2\} \]  

(24)

Eq. (23) demonstrates the relationship between the stationary and initial values of the variables. If \( \Phi(k^2 + l^2) \gg f^2 \), then \( \psi_s \approx \psi_i \) and \( \phi_s \approx f\psi_i \), i.e. the mass field adjusts to the initial wind field. If \( \Phi(k^2 + l^2) \ll f^2 \), then \( \psi_s \approx \psi_i \) and \( \phi_s \approx \phi_i \), i.e. the wind field adjusts to the initial mass field.

The physical significance of Eq. (23) is made clearer by transforming back from wave number space into \((x, y)\)-space. Let the wavelengths in the \( x \)- and \( y \)-directions of the Fourier component \( \psi_{k_l} \) be \( \lambda_x \) and \( \lambda_y \) respectively. Then

\[ k = 2\pi/\lambda_x; \quad l = 2\pi/\lambda_y \]  

(25)
Following Washington, we define a critical wavelength \( \lambda_c \) by:

\[
\lambda_c = \frac{2\pi}{f} \sqrt{\Phi} .
\]  

(26)

We also define the effective scale \( \lambda_{xy} \) of the Fourier component \( \psi_{kl} \) by:

\[
\lambda_{xy} = \left( \frac{1}{\lambda_x^2} + \frac{1}{\lambda_y^2} \right)^{-\frac{1}{2}} .
\]  

(27)

Substituting Eqs. (25) to (27) into Eq. (24), we obtain:

\[
\alpha = \lambda_c^2 (\lambda_x^2 + \lambda_{xy}^2) .
\]  

(28)

Thus the relationship between the stationary stream function \( \psi_s \) and the initial values of the variables is seen to depend upon the ratio of the two-dimensional effective scale to the critical wavelength; if \( \lambda_{xy} \ll \lambda_c \), then \( \psi_s \approx \psi_t \), while if \( \lambda_{xy} \gg \lambda_c \), \( \psi_s \approx f^{-1}\psi_t \). Eqs. (23) and (28) form the two-dimensional analogue of Eq. (9) of Washington (1964) (in which there is an error; the minus signs should be replaced by plus signs). The importance of the extension to two-dimensions is that for the effective scale \( \lambda_{xy} \) to be greater than the critical wavelength \( \lambda_c \), we must have both \( \lambda_x > \lambda_c \) and \( \lambda_y > \lambda_c \). In other words, however great the wavelength in one direction, unless the wavelength in the other direction also exceeds the critical wavelength, the component will behave as a small-scale one.

The discussion in this Section has concentrated on the continuous case; in a gridpoint model, the argument has to be slightly modified. Ökland (1970) has pointed out that the use of spatial finite differences instead of continuous derivatives will result in the smallest-scale components behaving like large-scale components in that the wind field will tend to adjust to the mass field. If the spatial derivatives in Eqs. (10) to (12) are replaced by centred differences, the whole argument still carries through if we define the effective scale \( \lambda_{xy} \) by:

\[
\lambda_{xy} = 2\pi \left[ \frac{1}{(\Delta x)^2} \sin^2 \frac{2\pi \Delta x}{\lambda_x} + \frac{1}{(\Delta y)^2} \sin^2 \frac{2\pi \Delta y}{\lambda_y} \right]^{-\frac{1}{2}} .
\]

7. Discussion

The implication of the analysis of Section 6 for the dynamic initialization process is that forcing the wind field to adjust to the mass field is 'unnatural' for the small-scale components involved. This does not mean that adjustment will not take place at these smaller scales; rather, that the distribution of energy within the system, with respect to the critical wavelength, will affect the overall success of the forced adjustment. This point can be clarified by a more detailed analysis of the results of one of our initialization experiments.

The error fields of \( u \) and \( v \) obtained during Experiment 2B were subjected to a two-dimensional Fourier analysis. By Parseval's identity for finite two-dimensional Fourier series, the sum of the squares of the Fourier coefficients of a field defined at grid points is equal to the sum of the squares of the grid-point values themselves. Using this theorem, we can determine the contributions from various scales to the total mean square error. The results are shown in Fig. 9, where the mean square error is split up into contributions from large-scale \( (\lambda_{xy} > \lambda_c) \), medium-scale \( (\lambda_c > \lambda_{xy} > \lambda_c/2) \) and small-scale \( (\lambda_c/2 > \lambda_{xy}) \) components. It will be seen that the adjustment takes place at much the same rate at all scales. For the values of \( f \) (the Coriolis parameter in mid-channel) and \( \Phi \) used in the model, the critical wavelength is approximately \( 4.44 \times 10^6 \text{m} \). In fact these values were chosen to provide a reasonable proportion of the 'energy' \( (u^2 + v^2) \) at both large and small scales. Fourier analysis of the \( u \) and \( v \) fields at Day 2 of the control integration showed that the proportion of energy on large scales \( (\lambda_{xy} > \lambda_c) \) was 59 per cent. The proportion of energy on the aliassed smallest scales discussed at the end of Section 6 turned out to be negligible.
Figure 9. Contributions to mean square wind error from various scales during initialization Experiment 2B. (L) large-scale ($\lambda_{xy} > \lambda_c$); (M) medium-scale ($\lambda_c > \lambda_{xy} > \frac{1}{2}\lambda_c$); (S) small-scale ($\frac{1}{2}\lambda_c > \lambda_{xy}$).

To investigate the effect of having the energy distribution more strongly biased towards the smaller scales, with respect to the critical wavelength, a parallel control integration was run on the same grid, with similar initial conditions, but with the mid-channel Coriolis parameter reduced to $f_0 = 10^{-5}$ s$^{-1}$. This has the effect of doubling the critical wavelength, as defined by Eq. (26), to $8.88 \times 10^6$ m. As before, the control integration was run for two days with an Euler-backward timestep; the observed geopotential field, together with the wind components derived from the linearized balance equation, was used as input to an initialization experiment exactly parallel to Experiment 2B. The r.m.s. wind error during this experiment, labelled 2F, is shown in Fig. 10, together with the results of Experiment 2B for comparison. Convergence takes place more slowly; in fact Experiment 2F was run for a total of 50 cycles (equivalent to 550 timesteps), at which stage the r.m.s. wind error was still decreasing, but extremely slowly. With respect to the new critical wavelength, the proportion of energy on large scales ($\lambda_{xy} > \lambda_c$) at Day 2 of this second control integration was only 28 per cent.

8. Further Experiments

The upshot of the foregoing analysis is that, in any dynamic initialization scheme, provision ought to be made for the wind field to adjust to the mass field, as well as vice versa. In an operational situation, this approach is after all the most reasonable, since neither field is known precisely from the available observations, but we will normally have useful information on both fields. So we return to our dynamic initialization experiments, in particular those involving averaging schemes, and seek to incorporate this approach. A simple way of doing so would be to carry out a number of initialization cycles, meaning $u$ and $v$ while restoring the control $\phi$ field, as before; then to perform a second sequence
in which $\phi$ would be meanded while the $u$ and $v$ fields obtained at the end of the first sequence would be restored after each iteration.

Applying the second part of this scheme to the experiments with artificially generated test data, we are no longer seeking to force the wind field to converge to the control wind field; rather, we are trying to obtain a balance by allowing the geopotential to adjust to the initialized wind field. The natural way of measuring the success of the scheme is to apply the continued integration test as was done previously.

The combined scheme was tested using the basic scheme of Experiment 2B as a starting point; recall that this involved the averaging method with an iteration amplitude of $6\Delta t = 1$ hr. The wind fields obtained after ten iteration cycles were extracted and used as input to a second sequence of ten cycles, in which these wind fields were restored after each cycle, while the geopotential field was meanded. The fields thus obtained after a total of 20 cycles were used as initial conditions for a 24-hr integration using centred timesteps, and as usual the r.m.s. mass divergence was monitored. The results are shown in Fig. 3, where they are referred to as Experiment 3. The new set of initial conditions has provided a much smoother integration than those obtained after 25 cycles of Experiment 2B; in fact, it is almost as smooth as the integration using the observed fields as initial data, and it is clear that we have effectively 'solved the initialization problem' in this particular case. For the parallel integration with $f_0 = 10^{-4}$ s$^{-1}$, the same initialization procedure and subsequent testing were carried out, with comparably satisfactory results.

These experiments have demonstrated the benefits of allowing both the mass and the wind fields to take part in the adjustment process implied in a dynamic initialization scheme, though we have not determined the best way of combining the two procedures; it is probably a function of the distribution of energy amongst the various scales, as discussed in Section 7. The amount of computing time spent on initialization for Experiment 3 is still equivalent to that required for a forward integration of 36 hours; it seems likely that by a judicious combination of mass-to-wind and wind-to-mass adjustment this amount could be further reduced.

9. Concluding remarks

The main result of this study has been the proposal of a new dynamic initialization scheme, and the demonstration of some of its advantages when applied to a relatively simple primitive equation model. The next step is clearly to test the scheme on a more complex
and realistic model, incorporating several levels in the vertical. From the point of view of the adjustment process, the main difference will be that such a model contains the slower internal gravity-wave modes in addition to the external mode common to both free-surface and multi-level models. Also, the incorporation of secondary physical effects into the model will have to be investigated.

Research is currently proceeding along these lines, and it is hoped to publish the results at a later date.

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REFERENCES


