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Comments on: 'The possible use of Laguerre polynomials for representing the vertical structure of numerical models of the atmosphere' by P. E. Francis

By B. J. Hoskins

In a recent paper, Francis (1972) described the technique of using finite series of orthogonal functions to describe the vertical structure of meteorological variables. Laguerre polynomials were used and a linear stability analysis of the two-dimensional advection equation then suggested a severe computational penalty in the use of this representation.

A general development in terms of orthogonal functions, analogous to that performed by Francis may be written:

Let \( \sigma \) be any vertical coordinate running between the fixed limits \( a \) and \( b \), and let the functions \( \phi_r(\sigma) \) be a complete orthonormal set

\[
\int_a^b \phi_r \phi_i f(\sigma) d\sigma = \delta_{ri}
\]

where \( f(\sigma) \) is the weighting function.

Consider the two-dimensional advection equation

\[
\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} = 0,
\]

with

\[
\theta(x, \sigma, t) = \sum_r \theta_r(x, t) \phi_r(\sigma), \quad u(\sigma) = \sum_r u_r \phi_r(\sigma).
\]

Then

\[
\frac{\partial \theta_r}{\partial t} = - \sum_q U_{rq} \frac{\partial \theta_q}{\partial x}, \quad \text{for each } r,
\]

where

\[
U_{rq} = \sum_p u_p \int_a^b \phi_p \phi_q f(\sigma) d\sigma.
\]

The eigenvalues of the matrix \( U \) are the characteristic velocities associated with the corresponding \( u \) profile. As described by Francis, the eigenvalue of maximum absolute value determines the largest phase speed and thus the largest time step possible in a numerical integration.

One possibility (Machenauer 1972) is to use

\[
\sigma = \frac{2p}{p_s} - 1,
\]

where \( p_s \) is the surface pressure. Then

\[
a = -1 \quad (p = 0), \quad b = 1 \quad (p = p_s),
\]

and the Legendre polynomials are suitable orthogonal functions. Truncating the series expressions after the first 3 terms as was done by Francis, we may represent a velocity profile similar to his:

\[
u = -9(\sigma^2 + \sigma - 2).
\]

Here \( u \approx 0 \) at \( p \approx 1,000 \text{ mb} \), increases to a maximum of \( 20-25 \text{ ms}^{-1} \) at \( p \approx 250 \text{ mb} \) and decreases to \( 18 \text{ ms}^{-1} \) at \( p = 0 \text{ mb} \). The eigenvalues are \( 5.034 \text{ ms}^{-1} \), \( 16.535 \text{ ms}^{-1} \) and \( 19.316 \text{ ms}^{-1} \). Since these are all within the range of \( u \), the same timestep can be chosen as in the finite difference case. The real question is whether the time taken in dealing with the integrals of products of vertical profile functions is worth the extra accuracy obtained.

The Laguerre polynomials used by Francis each tend to infinity more quickly than the previous one as the top of the atmosphere is approached. Thus the simple velocity profile he used has a local maximum of \( 19.2 \text{ ms}^{-1} \) at \( p \approx 200 \text{ mb} \), but tends to minus infinity as \( p \) tends to \( 0 \text{ mb} \). It is therefore quite consistent that one of the eigenvelocities should be \( -200 \text{ ms}^{-1} \).

If we now consider the primitive equations using polynomial representations in the vertical, there is one apparent difficulty. Using \( p \) as vertical coordinate, the hydrostatic relation may be written

\[
\frac{\partial \Phi}{\partial p} = -RT/p,
\]

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where $\Phi$ is the geopotential. Any polynomial representation of $T$ must lead to a logarithmic term, in the representation of $\Phi$, and thus a polynomial expansion for both is not possible. However, the horizontal gradient of $\Phi$ is

$$\nabla \Phi = \nabla \Phi_0 + \oint_{\rho} - R \frac{\nabla T}{\rho} \, dp.$$  

As pointed out by Machenauer (1972), since horizontal pressure gradient forces are finite at the top of the atmosphere, this integral must converge. Hence $\nabla T$ must tend to zero at least as fast as $\rho$ as the top of the atmosphere is approached. Therefore in any polynomial representation of $T$, the linear term is not dependent on the horizontal co-ordinates. The logarithmic singularity in $\Phi$ is therefore also independent of horizontal position, and a polynomial representation of the horizontal gradients in $\Phi$ is valid.

**REFERENCES**


University of Reading, Department of Geophysics, Whiteknights, Reading, Berkshire 10 October 1972.

**REPLY**

By P. E. Francis

I agree with Dr. Hoskins that the use of Legendre polynomials to represent vertical profiles leads to a situation where a constraint has to be applied to the predicted temperature profile. The choice of Laguerre polynomials in my paper is dictated by the desire to avoid such a constraint, i.e. the requirement is to obtain an analytic representation that is exact subject only to the truncation of the polynomial expressions. Unfortunately, as indicated in the paper, stability considerations decide against using such a representation. I note Dr. Machenauer's work with great interest.


**THE DRY-LINE OF NORTHERN INDIA AND ITS ROLE IN CUMULONIMBUS CONVECTION**

By B. N. Desai

In a paper on the above subject, Weston (1972) has designated as 'dry-line' the surface of separation between the dry westerly to north-westerly and moist southerly airmasses over north-east India during the premonsoon months, and considered its role in the development of cumulonimbus giving rise to thunderstorms known as 'Nor' westers'. The inversion between the lower moist and upper dry airmasses is penetrated by convection set up by insolation first on the western side where the moist layer is shallow; cumulus formed due to convection develops into