The development of a dry inversion-capped convectively unstable boundary layer

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SUMMARY

A model is proposed for the development of a dry convectively unstable boundary layer capped by a stable layer; included are the effects of time-dependent surface heating, the stability of the capping layer, subsidence and any degree of turbulent interfacial mixing. A differential equation describing the evolution indicates quantitatively the relative importance of each factor and, in particular, indicates the importance of the entrainment process. The model is applied to the 1953 O’Neill boundary layer data in an attempt to assess to what degree interfacial mixing is realized in the atmosphere. Realistic boundary layer developments are obtained.

1. Introduction

There is a real need to investigate and understand the physical and dynamical processes which govern the evolving and spatially varying atmospheric boundary layer. A realistic knowledge of the development of the layer is required, for example, when estimating the dispersion of concentrations of atmospheric pollutants and also for incorporation in atmospheric, numerical forecasting and general circulation models.

A scheme for inserting such an evolution into large-scale models was proposed by Charnock and Ellison (1967) who, from a study of radio-sonde ascents over the north-east Atlantic, classified the boundary layer into a small number of different types and observed that the dry convectively unstable boundary layer capped by a deep non-turbulent stable layer accounted for about fifty per cent of the available cases. The importance of parameterizing the variations in the depth of the boundary layer for use in general circulation models was also emphasized in a comprehensive study by Deardorff (1972a) who provided an alternative approach to those of Clarke (1970a, b) and Delsol, Miyakoda and Clarke (1971) in which several fixed layers in the first few kilometres are used to resolve the boundary layer’s vertical structure and the distribution of the fluxes.

The first specific study of the development of the dry, convectively unstable layer capped by a stable layer was that of Ball (1960), whose theoretical results and those of subsequent workers, notably Lilly (1968) and Deardorff, Willis and Lilly (1969), require certain restrictive assumptions about the nature of the turbulent mixing at the interface between the convectively unstable and the quiescent stable layers and leave two important questions unanswered:

(i) How important is any degree of turbulent interfacial mixing to the rate of entrainment of the stable layer into the convectively unstable boundary layer?

(ii) To what degree is interfacial mixing realized in the atmosphere during the typical development of a convectively unstable boundary layer?

The present study provides a simple, but realistic, model of such a developing boundary layer, including the effects of a continuous, time-dependent, surface sensible heat flux, a synoptic scale vertical velocity field (generally taken to be a subsidence field), the gradient of potential temperature in the stable layer and, most importantly, the effect of sensible heat brought into the layer through entrainment of the capping stable layer into the boundary layer. The results indicate the relative importance of each of the above parameters and provide an answer to question (i). The second question is tackled by applying the model to the 1953 O’Neill boundary layer observations (Letttau and Davidson 1957) which imply several distinct phases in the boundary layer’s evolution.
2. The diurnal variation of the boundary layer depth

Fig. 1, derived and discussed in an earlier study by Carson (1971), illustrates the diurnal variation of the mean boundary layer thickness, \( <h(t)> \), and the surface sensible heat flux, \( <H(0, t)> \), observed in strongly convective daytime conditions in August-September, 1953 over the flat prairies at O'Neill. Lettau's estimated heat fluxes were used and although the absolute magnitudes may be significantly in error, the relative magnitudes and hence the diurnal pattern are thought to be realistic. \( <H(0, t)> \) is small, negative and effectively steady throughout the night-time period but after changing sign about an hour after sunrise varies markedly, almost sinusoidally, with a maximum value close to midday, before changing sign again about an hour and a half before sunset.

![Figure 1](image-url)  
*Figure 1. The mean boundary layer thickness, \( <h(t)> \), and the sensible heat flux at the surface, \( <H(0, t)> \), deduced for the O'Neill data and plotted with standard errors as functions of time of day, \( t \), in Mean Solar Time.*

From the discontinuities in \( <h(t)> \) in the neighbourhoods of sunrise and sunset we distinguish between the relatively shallow night-time inversion layer in which buoyancy and viscous effects suppress any mechanically generated turbulent motions and the eventually deeper, daytime, well-mixed layer. We note the evolution of the nocturnal boundary layer, which, although not so pronounced as that of the daytime convectively unstable layer, is nonetheless very important in pollutant dispersal problems and has been studied by Deardorff (1972b). The present study is restricted to the unstable phase of the diurnal evolution and for detailed qualitative descriptions see, for example, Ball (1960) and Plate (1971).

It is worth noting that Hanna (1969) tested steady state (and in some cases neutral) formulae against the O'Neill data and their inability to account for the evolutionary nature of the situation is undoubtedly a major reason for the large scatter obtained in his comparisons of theoretically derived against profile-estimated depths of the boundary layer. Following Hanna, a preliminary study of the same data by Carson (1971) showed that formulations based on steady state similarity theory, e.g.

\[
h = \frac{u_*}{f} S(\mu)\]

where

\[
\mu = \frac{ku_*}{fL}
\]
is a stability parameter, $u_*$ is the surface friction velocity, $L$ the surface layer Monin-Obukhov length, $f$ the Coriolis parameter, $k$ the von Kármán constant and $S$ is a function of $\mu$, are totally inadequate and that time dependent models are essential. The same conclusion was stated by Deardorff (1972b) when discussing the growth rate of the nocturnal boundary layer.

3. A simple model

Fig. 2 gives a schematic representation of the adopted potential temperature, $\theta$, profile in the developing convectively unstable boundary layer. The shallow superadiabatic layer ($\ll$ few tens of metres) with its large vertical shears of wind and temperature is assumed to incorporate the regions of forced and mixed convection (Townsend 1962; Deardorff and Willis 1967) where heat is predominantly transported by mechanically induced turbulent motions. We take this layer as the surface layer where flux-gradient or Jacobs-type formulae could be applied to give estimates of the surface fluxes; henceforth it is neglected. The thorough turbulent mixing in the free convection layer above the surface layer is buoyancy dominated and is assumed to produce a $\theta$-profile virtually independent of height (Webb 1958), although it is often observed to be slightly stable, particularly in the upper region of the layer.

The laboratory experiments of Deardorff, Willis and Lilly (1969) show the unstable-stable interface to be a highly contorted, almost indefinable surface due to the physical overshoot into the stable layer of energetic convective elements which originate in the surface layers and continually bombard the interface. Within this strongly agitated region the temperature field indicates marked spatial, perhaps discontinuous, variability, the net effect being a change across the layer from the free convection layer value to the value of $\theta$ at the base of the yet undisturbed stable profile. Interfacial layers will vary in depth and character, however, to facilitate the analysis, they are represented by a step discontinuity.
\[ \Delta \theta, \text{ in the } \theta\text{-profile at } z = h, \text{ the nominal top of the convectively unstable boundary layer.} \]
The level \( \zeta \) is defined as the height at which the stable profile, extrapolated downwards, intersects the unstable profile and the term 'overshoot' is used to denote the depth
\[ \sigma = h - \zeta. \]  
(1)

The extent of profile overshoot is clearly proportional to \( \Delta \theta \) which in turn reflects the degree of physical overshoot and entrainment.

The unstable layer grows due to the effects of the external solar heating of the surface and the internal redistribution of heat arising from the entrainment of the capping stable air into the boundary layer (Ball 1960), a possible subsidence arising from synoptic scale convergence being the only counter effect included. Fig. 3 shows schematic \( \theta\)-profiles separated by time \( \delta t \) and we note the change in the stable gradient due to the subsidence field.

\[ \text{Figure 3. Adopted potential temperature profiles as a function of } z \text{ and } t \text{ and a representation of the parameters } h, \zeta, \Delta \theta, \gamma, \theta_0 \text{ and } \theta_c. \]

Advection, radiation and evaporation are not considered although in certain circumstances each or all of these processes can be important. For example, advection processes rather than diurnal variation will mostly control the development of the boundary layer over the sea. Also, although it is probably safe to neglect radiation effects in relatively cloudless daytime conditions (Elliott 1964), the study of moist cloud-topped mixed layers under a strong inversion (Lilly 1968) requires radiative cooling from the cloud layer top as the principal mechanism for inducing convective mixing beneath the inversion and entrainment across the interface.

The potential temperature profile is defined by,
\[ \theta(z,t) = \begin{cases} \theta_c(t) & z < h, \\ \theta_s(z,t) & z > h, \end{cases} \]  
(2)
\[ \theta_s(z,t) = \theta_0 + \gamma(t)z, \]  
(3)
where $\theta_0$ is effectively the near surface temperature when $h(t) = 0$ and $\gamma(t)$ is the vertical gradient of $\theta$ in the capping stable layer.

The heat balance equation in convective situations has been shown by the intensive theoretical treatments of Boussinesq type approximations by Ogura and Phillips (1962), Calder (1968) and Dutton and Fichtl (1969) to be simply,

$$\frac{\partial H}{\partial z} = - \rho c_p \frac{d\theta}{dt} = - \rho c_p \left[ \frac{\partial \theta}{\partial t} + w(z) \frac{\partial \theta}{\partial z} \right],$$  \hspace{1cm} (4)

where $H(z,t)$ is the sensible eddy heat flux, $w(z)$ is the vertical velocity field, $\rho$ is the mean air density in the boundary layer and $c_p$ is the specific heat of air at constant pressure. We assume that there is no appreciable turbulent sensible heat flux in the stable layer, and so

$$H(z,t) = 0 \quad z > h.$$  \hspace{1cm} (5)

In the stable layer Eq. (4) takes the form

$$\frac{\partial \theta_s}{\partial t} = - w(z) \frac{\partial \theta_s}{\partial z} \quad z > h,$$  \hspace{1cm} (6)

which from Eq. (3) implies

$$w(z) = - \beta z$$  \hspace{1cm} (7)

and

$$\gamma(t) = \gamma(0) \exp(\beta t)$$  \hspace{1cm} (8)

where $\beta$ is a constant convergence parameter, more aptly termed the subsidence parameter since we will consider only $w(z) \leq 0$. Thus the model implies a subsidence field linear with height which will increase the stability of the capping layer according to Eq. (8).

In the free convection layer,

$$\frac{\partial H}{\partial z} = - \rho c_p \frac{d\theta_c}{dt} \quad z < h,$$  \hspace{1cm} (9)

implying a sensible heat flux profile, linear in $z$,

$$H(z,t) = H(0,t) - \frac{z}{h} \left( H(0,t) - H(h,t) \right), \quad z < h.$$  \hspace{1cm} (10)

and, as $z \to h-$,

$$H(h,t) = H(0,t) - \rho c_p h \frac{d\theta_c}{dt}.$$  \hspace{1cm} (11)

Finally, by treating $\theta(z,t)$ as a generalized function we can integrate across the interface from $h - \epsilon$ to $h + \epsilon$ and take the limit as $\epsilon \to 0$ to obtain

$$H(h,t) = - \rho c_p \left( \frac{dh}{dt} - w(h) \right) [\theta_s(h,t) - \theta_c(t)]$$  \hspace{1cm} (12)

and if we write

$$\frac{dh}{dt} = w(h) + w_e(t)$$  \hspace{1cm} (13)

then $w_e(t)$ is the rate at which stable air is entrained into the boundary layer and Eq. (12) reduces to

$$H(h,t) = - \rho c_p w_e(t) \Delta \theta,$$

$$\Delta \theta = - \rho c_p w_e(t) \gamma(t) \sigma(t).$$  \hspace{1cm} (14)

Eq. (12) was derived with the inclusion of a radiation flux by Lilly (1968) and in a different but equivalent, form by Ball (1960). Eq. (14) shows that if there is no overshoot then there
is no contribution of heat into the boundary layer through mixing at the interface. However, when $\sigma$ is zero the boundary layer can still develop since $w_0$ may be non-zero.

The profile geometry gives

$$\gamma \xi = \phi(t) - \phi_0$$  \hspace{1cm} (15)

which implies, from Eqs. (1) and (11), that

$$\frac{d(yh)}{dt} = \frac{d(\gamma \phi)}{dt} = \frac{H(0,t) - H(h,t)}{\rho c_p h}$$  \hspace{1cm} (16)

Further, Eqs. (7), (8) and (13) allow Eq. (14) to be rewritten

$$\frac{d(yh)}{dt} = -\frac{H(h,t)}{\rho c_p \sigma}$$  \hspace{1cm} (17)

To obtain $h(t)$ from Eqs. (16) and (17) it is necessary to eliminate the overshoot $\sigma(t)$ which we have noted is closely related to the degree of mixing at the interface.

Consideration of the local turbulent kinetic energy balance led Ball (1960) to propose that the heat gained by entrainment from above is equal to the heat supplied at the base; however his suggestion that viscous dissipation is responsible for the destruction of a small part of the turbulent kinetic energy only is open to question, particularly in the light of the experimental results of Deardorff, Willis and Lilly (1969) and the recently obtained profiles of the mean rate of molecular dissipation throughout the boundary layer for different stability classes (Rayment 1972a, b). Ball’s hypothesis does, however, provide a maximum possible entrainment criterion for models in which the interfacial mixing is driven solely by the surface heating (see Lilly 1968).

To avoid questionable manipulation of the energy balance equation we shall simply assume that the heat brought into the boundary layer due to entrainment is proportional to the surface sensible heat flux, thus

$$H(h,t) = -A H(0,t)$$  \hspace{1cm} (18)*

This allows the possibility of any degree of interfacial mixing including the extreme cases, $A = 1$ (Ball’s hypothesis) and $A = 0$ (Lilly’s practical minimum entrainment criterion).

Dividing Eq. (16) by (17) and applying condition (18) we obtain

$$\frac{d(\gamma \phi)}{d(yh)} + \left(\frac{1 + A}{A}\right) \frac{\sigma}{h} - 1 = 0$$  \hspace{1cm} (19)

which, after integration, yields

$$\sigma = \alpha h$$  \hspace{1cm} (20)

where

$$\alpha = \frac{A}{1 + 2A} = \frac{H(h,t)}{2H(h,t) - H(0,t)}$$  \hspace{1cm} (21)

if $\sigma = 0$ when $h = 0$ and $\gamma \neq 0$. These solutions require the overshoot to be a constant fraction $\alpha$ of the depth of the evolving boundary layer where $\alpha$ is related to the degree of interfacial mixing through Eq. (21).

With $\sigma$ replaced by $A h$ Eq. (16) reduces to

$$\frac{d(yh)}{dt} = \frac{H(0,t) - 2H(h,t)}{\rho c_p h}$$  \hspace{1cm} (22)

which can be expressed in a variety of forms,

* The closure Eq. (18) was proposed independently by A. K. Betts in his paper, 'Non-precipitating cumulus convection and its parameterization', *Quart. J. R. Met. Soc.*, (1973) 99, pp. 178-196, which has been published since the submission of the present paper.
\[
\frac{dh^2}{dt} + 2Bh^2 = \frac{2[H(0,t) - 2H(h,t)]}{\rho c_p \gamma(t)}, \quad (23)
\]
\[
= \frac{2(1 + 2A) H(0,t)}{\rho c_p \gamma(t)}, \quad (24)
\]
\[
= \frac{2H(0,t)}{(1 - 2\alpha) \rho c_p \gamma(t)}. \quad (25)
\]
Integration of Eq. (23) gives
\[
h^2(t) = h^2(t_0) \exp[2B(t_0 - t)] + 2\exp(-2Bt) \int_{t_0}^{t} \exp(2B\tau) \frac{[H(0,\tau) - 2H(h,\tau)]}{\rho c_p \gamma(\tau)} d\tau \quad (26)
\]
for the evolution of the depth of the convectively unstable boundary layer. The corresponding evolutionary expressions for \(w_e(t)\), \(\Delta\theta(t)\) and \(\theta_e(t)\) are,
\[
w_e(t) = \frac{d}{dt} w(h) = \frac{1}{\gamma} \frac{d(\gamma h)}{dt} = \frac{H(0,t) - 2H(h,t)}{\rho c_p \gamma h}, \quad (27)
\]
\[
\Delta\theta(t) = \gamma \sigma = \gamma \alpha h = \frac{A \gamma h}{1 + 2A}, \quad (28)
\]
and
\[
\theta_e(t) = \theta_e(h,t) - \Delta\theta(t) = \theta_0 + \left(\frac{1 + A}{1 + 2A}\right) \gamma h. \quad (29)
\]

4. Results from the Simple Model

There are several points of immediate interest.

(i) We note the term \(2H(h,t)\) in Eq. (22). In contrast to the externally controlled \(H(0,t)\), \(H(h,t)\) is internal to the system. This flux not only reflects, and is derived from, the growth of the mixed layer into the stable layer but can then be utilized to generate further growth in the same sense as does \(H(0,t)\). Thus \(H(h,t)\) plays a doubly effective role in boundary layer growth.

(ii) Ball’s hypothesis, \(A = 1\) implying \(\alpha = \frac{1}{2}\), with \(\beta = 0\) and \(h(0) = 0\), gives
\[
h^2(t) = 6 \int_{t_0}^{t} \frac{H(0,\tau)}{\rho c_p \gamma(\tau)} d\tau. \quad (30)
\]
The maximum growth rate is therefore \(\sqrt{3}\) times the growth rate with no mixing across the interface. Eq. (30) corrects Plate’s (1971) expression for the case assuming Ball’s hypothesis.

(iii) With \(\beta = A = 0\) and a constant surface heat flux, \(H_0\), Eq. (26) reduces to
\[
h^2(t) = \frac{2H_0 t}{\rho c_p \gamma(0)}, \quad (31)
\]
the expression derived by Deardorff, Willis and Lilly (1969).

(iv) A simple sinusoidal heat flux is more realistic than constant surface heating when modelling the growth of the atmospheric convectively unstable boundary layer (Fig. 1) and provides simple analytical expressions for \(h(t)\), \(w_e(t)\), \(\Delta\theta(t)\) and \(\theta_e(t)\).

Let \(H(0,t) = H \sin(\Omega t)\),
\[
\text{(32)}
\]
a good approximation close to an equinox, where \(\Omega\) is the Earth’s angular rotation and \(h(0) = 0\), then integration of Eq. (24) gives
\[
h^2(t) = \frac{2(1 + 2A)H}{\gamma(0)(\beta^2 + \Omega^2) \rho c_p} \exp(-2\beta t) \left[\exp(\beta t)(\beta \sin \Omega t - \Omega \cos \Omega t) + \Omega\right]. \quad (33)
\]
The effects of each parameter are best studied by considering the non-dimensional depth
\( h^*(t; \beta) \) where

\[
h^*(t; \beta) = \frac{h^2 \gamma(0) \Omega \rho c_p}{2(1 + 2A) \bar{H}} \]

\[
= \frac{\Omega \exp(-2\beta t)}{\beta^2 + \Omega^2} [\exp(\beta t)(\beta \sin \Omega t - \Omega \cos \Omega t) + \Omega].
\]

Fig. 4 illustrates \( h^* \) as a function of time for various values of \( \beta \). In convective conditions \( \beta \) is typically about 0.6 x 10^{-5} \text{ s}^{-1} \) (Ball 1960; Lilly 1968) implying a subsidence velocity of 0.6 cm s^{-1} at a height of 1 km. Not only does the subsidence suppress the growth of the unstable layer but if sufficiently intense will eventually outweigh the entrainment rate and cause the boundary layer to decrease in depth even although heat may still be entering the boundary layer from below and possibly above. Fig. 5 provides an alternative representation of \( h^* \) as a function of \( \beta \) and the non-dimensional heat flux \( \dot{H}^* = \dot{H}(0,t)/\bar{H} = \sin \Omega t \).

\[
\begin{align*}
&\text{Figure 4. The non-dimensional depth } h^* \text{ as a function of time and various values of the subsidence parameter } \beta. \text{ Sinusoidal heat flux model.}
\end{align*}
\]

The effects of different degrees of interfacial mixing, capping stability and surface sensible heat input are obtained by altering the \( h^*(t; \beta) \) profile proportionally according to Eq. (34). Fig. 6 shows the development of \( h(t) \) for various degrees of interfacial mixing for typical values of \( \gamma(0) \) and \( \bar{H} \) in the cases of no subsidence and typical subsidence. The importance of interfacial mixing to the development of the layer is emphasized in the ultimate difference (1,200 m) between the extreme cases.

Fig. 7 illustrates the non-dimensional entrainment rate

\[
w^*(t; \beta) = \left[ \frac{2 \rho c_p \gamma(0)}{(1 + 2A) \bar{H}} \right]^{1/2} w_e = \frac{\exp(-\beta t) \sin \Omega t}{h^*(t; \beta)}.
\]

\[ (35) \]
which is a maximum at the start of the development ($\lim_{t \to 0} \omega^*(t; \beta) = \sqrt{2}$) and then decays steadily to zero.

The importance of the simple model is not so much that it is capable of giving the broad evolutionary features of the developing convectively unstable boundary layer but that it enables us to judge the relative importance of each of the contributing factors. In particular we note the importance to the development of heat flow into the boundary layer from above but to date we have no reliable estimate of the degree of interfacial mixing that can be achieved in the atmosphere. The answer to this problem must come from combined observational and theoretical study of interfacial mixing layers and the dynamics of the thermal convection which induces and maintains such processes.

In the meantime we shall consider the simple model in relation to the 1953 O'Neill data and the typical boundary layer development illustrated in Fig. 1. We note immediately that the graph of $<h(t)>$ does not have the simple shape suggested in Fig. 6.

5. Application of the model to the 1953 O'Neill data

The development of the unstable boundary layer depends on the four parameters $\beta$, $A$, $\gamma(t)$, $H(0,t)$. In the following discussion the subsidence parameter will be neglected, bearing in mind, however, the suppressive role it can play. Also the mean O'Neill surface sensible heat flux, not being a simple sine wave, will be integrated graphically.

In the simple model $\gamma(0)$ and $A$ have been assumed constant throughout any particular development. A study of actual temperature profiles shows that $\partial \theta / \partial z$ varies with height and so strictly the gradient of potential temperature in the stable layer immediately above
the mixed layer is a function not only of time but of the depth $h(t)$. A detailed study of the O'Neill profiles suggests a minimum requirement of two distinct stages. In the early stages of the development the unstable layer is entraining the nocturnally established surface inversion which has a typical gradient of about $18 \times 10^{-3}$ K m$^{-1}$. We note from Fig. 1 that these inversions are typically about 400 m deep and will take between 2-4 hours to be eroded from below after which the unstable layer is capped by a stable, though not necessarily inverted layer, with gradient typically about $6 \times 10^{-3}$ K m$^{-1}$. The factor three between the stable $\theta$-gradients which characterize each stage will obviously have marked effects on the nature of the development.

There remains our treatment of the parameter $A$ which effectively measures the degree of interfacial mixing about which we have little direct knowledge. Strictly speaking our differential equation applies to any period when the ratio $H(h(t))/H(0,t)$ is constant and the overshoot is a constant fraction of the depth of the boundary layer. Values of $A$ at the O'Neill ascent times were inferred from Eq. (24) with $\beta = 0$ and $h$, $dh/dt$, $H(0,t)$ and $\gamma(t)$ estimated from the profiles. Although there was a great deal of scatter the values obtained for $A$ implied at least three stages in the development of the convectively unstable boundary layer. For the first half of the development $A$ is close to zero, this is followed abruptly by a period of about four hours where $A$ has a distinct non-zero value in the neighbourhood of $A$, and finally there follows a period with $A$ again close to zero.

If we combine the stages suggested in Fig. 1 and those suggested by the variations in time of $\gamma$ and $A$ we have five distinct Phases to represent the diurnal evolution of the land boundary layer in clear sky conditions with marked daytime insolation. In reality the
development will proceed continuously through Phases 1-4, the strict discontinuities being between Phases 4 and 5 and Phases 5 and 1.

**Phase 1.** This begins about one hour after sunrise when the surface sensible heat flux becomes positive and begins to erode the nocturnally established surface inversion. Thermal penetration into the stable layer is quickly suppressed by the marked stable stratification. There is no significant interfacial mixing and the unstable boundary layer develops slowly through encroachment of the stable layer.

**Phase 2.** The nocturnal inversion has been eroded and the marked change in the stability of the capping layer and the strengthening of the thermals cause the development to proceed more rapidly. Interfacial mixing is still effectively slight but increasing.

**Phase 3.** The surface sensible heat flux is close to maximum value, thermal penetration of the stable layer is now at its peak and a significant interfacial entrainment layer is established. Heat from above will be playing a significant role in the boundary layer’s development.

**Phase 4.** The surface sensible heat flux remains positive but is decreasing and the weakening thermals are no longer able to maintain the thorough mixing throughout the established deep convection layer or their penetration of the stable layer. Interfacial mixing has decayed and subsidence and mechanical effects begin to dominate the evolution. The depth of the boundary layer remains steady or even begins to decrease.

**Phase 5.** About an hour or so before sunset the heat flux changes sign and the nocturnal inversion re-establishes itself from the surface. The boundary layer remains a shallow, slowly evolving, stable layer until about an hour after sunrise.

Phases 1-3 can be adequately described by Eq. (24) if we allow discontinuous changes in the values of $A$, and hence $\alpha$ and the entrainment rate, at the changeover times between...
the phases. The depth during Phase i, i = 1,2,3, defined by \( t_{0,i} \leq t \leq t_{1,i} \), is given in the usual notation by

\[
h^2(t) = h^2(t_{0,i})\exp[2\beta(t_{0,i} - t)] + \frac{2(1 + 2A_i)\exp(-2\beta t)}{\gamma_i(0)}\int_{t_{0,i}}^{t} \exp(\beta \tau) \frac{H(0,\tau)}{\rho c_p} \, d\tau,
\]

(36)

where \( h^2(t_{0,i}) = h^2(t_{1,i-1}) \), and the entrainment rate by

\[
w(t) = \frac{(1 + 2A_i)\exp(-\beta t) \cdot H(0,t)}{\rho c_p \gamma_i(0) h(t)}
\]

(37)

with corresponding expressions for \( \Delta \theta \) and \( \theta_0(t) \).

With the multi-phase model an important application is forecasting the time of breakdown of the nocturnally established inversion. In the presence of subsidence the height, \( \eta \), to the top of the inversion is given by

\[
\eta(t) = \eta(0)\exp(-\beta t)
\]

(38)

\( t = 0 \) coinciding with \( H(0,t) = h(t) = 0 \), \( \frac{\partial H(0,t)}{\partial t} > 0 \), whereas the depth of the developing boundary layer is from Eq. (36),

\[
h^2(t) = \frac{2(1 + 2A_i)}{\gamma_i(0)}\exp(-2\beta t)\int_{0}^{t} \frac{\exp(\beta \tau)H(0,\tau)}{\rho c_p} \, d\tau
\]

(39)

At breakdown, \( h(t) = \eta(t) \) and so from Eqs. (38) and (39),

\[
\int_{0}^{t_{1,i}^{-1}} \frac{\exp(\beta \tau)H(0,\tau)}{\rho c_p} \, d\tau = \frac{\gamma_i(0)\eta^2(0)}{2(1 + 2A_i)}
\]

(40)

**Figure 8.** The non-dimensional height, \( \eta^*(0) \), of the top of the nocturnally established surface inversion layer as a function of the inversion breakdown time, \( t_{1,i} \), and subsidence parameter \( \beta \). Sinusoidal heat flux model.
The time of breakdown is found by solving Eq. (40) for \( t_{1.1} \).

With the simple sinusoidal heat flux model Eq. (40) can be expressed in terms of a non-dimensional inversion depth

\[
\eta^*(0)[t_{1.1}; \beta] = \gamma(0) \left[ \frac{\rho c \beta \eta(0)}{2(1 + 2A_1)H_f} \right]^4 = \exp(\beta t_{1.1}) h^*(t_{1.1}; \beta) \tag{41}
\]

which must be solved for \( t_{1.1} \). Fig. 8 shows \( \eta^*(0) \) as a function of \( t_{1.1} \) and \( \beta \) and with \( \beta = A_1 = 0 \), \( \eta(0) = 400 \) m, \( \gamma(0) = 18 \times 10^{-3} \) oK m\(^{-1}\) and a maximum heat flux of 30 mwatt cm\(^{-2}\), \( t_{1.1} \) is about 3 hours in good agreement with observations at O’Neill.

Fig. 9 illustrates the rate of entrainment in a simple two-phase model with first phase as described above and \( \gamma_2(0) = 6 \times 10^{-3} \) oK m\(^{-1}\) in the second phase. The first phase, lasting 3 hours, has no interfacial mixing and the subsequent development in the second phase is given for several values of \( A_2 \). We note the rapid acceleration in the entrainment rate immediately after breakdown of the nocturnal inversion even when \( A_2 = 0 \) and the change is due entirely to the change from \( \gamma_1 \) to \( \gamma_2 \).

![Figure 9. The entrainment rate \( w_0(t) \) for a two-phase model characterized by the given parameters. Sinusoidal heat flux model.](image)

Fig. 10 compares the mean observed depth of the O’Neill boundary layer with several theoretical evolutions based on a three-phase model assuming \( \beta = 0 \) and the mean O’Neill heat flux. Values adopted for the governing parameters are \( \gamma(0) = 18 \times 10^{-3} \) oK m\(^{-1}\), \( \gamma_2(0) = 6 \times 10^{-3} \) oK m\(^{-1}\), \( h(t_{1.1}) = \gamma(t_{1.1}) = \eta(0) = 400 \) m, \( h(t_{1.2}) = 850 \) m, \( A_1 = A_2 = 0 \) and \( A_3 \) takes several values in the range 0 to 1. Interfacial mixing has been delayed until the third phase and we see here the evidence for suggesting \( A_3 = \frac{1}{2} \). Phase 4 requires a model which can account for the decay of the convective mixing within the boundary layer. With \( A_3 = \frac{1}{2} \) the correlation coefficient between the predicted and observed mean
depths is almost 1 and the standard error of estimation of the mean depth is about 70 m (the observed depths, however, are in themselves accurate only to within about 100 m).

Fig. 11 represents the evolution of $h(t)$ as a function of $H(0, t)$ for the adopted three-phase model allowing for heat flux curves of different amplitude but similar shape to the mean O'Neill curve. Contours are drawn for different values of $H$, the maximum surface sensible heat flux, and also for the time taken to evolve to a given stage.

When applied to individual O'Neill cases the mean model produced a correlation coefficient between predicted mean depths and observed actual depths, $h_0(t)$, of 0·86 with a standard error of estimation about the 45° line of 345 m.

The actual heat flux estimates at O'Neill could be in error by as much as 50 per cent and so it was not desirable to replace the mean heat flux values by actual values. However, in Fig. 12 we note the importance of choosing $\gamma(t)$ appropriate to the occasion. With $\gamma(t)$ estimated from individual profiles the three-phase model produces the correlation between observed and predicted depths of the convectively unstable boundary layer shown in Fig. 13. The correlation coefficient is 0·93 and the standard error of estimation about the line $h_0(t) = h(t)$ is 255 m. Period 5 produced the worst fit and indeed was the most difficult period for estimating $h_0$ due to marked advection effects (Hanna 1969; Carson 1971). The rogue point in Period 2 implies that we have arrived at Phase 3 about an hour too soon. Bearing in mind the errors in estimating $h_0(t)$ (Hanna 1969; Carson 1971), the results from applying the modified simple model to actual data are very encouraging.

6. Conclusions

A model is proposed which describes the evolution of the convectively unstable boundary layer accounting for the effects of a diurnal pattern in the surface sensible heat flux, a subsiding, capping stable layer and entrainment of the stable air into the boundary
Figure 11. A three-phase model representation of $h(t)$ as a function of $H(0,t)$, assuming heat flux curves of different amplitudes but similar shape to the mean O'Neill curve. The broken contour lines give the time in hours to evolve to a particular stage.

Figure 12. The dependence of the evolution of $h(t)$, in a typical three-phase model, on the choice of $\gamma_2$ and $\gamma_3$. 
layer due to interfacial mixing which results in a heat flow into the boundary layer at its top. Eq. (22) allows us to assess the relative importance of each of the contributing factors in the boundary layer’s development and, in particular, we have noted the twofold effect of interfacial mixing.

No attempt has been made to model the dynamics of the important interfacial mixing but modification of the simple model based on the 1953 O’Neill data leads to a multi-phase model which prompts the provisional proposal that interfacial mixing is most important for a period in the early afternoon following the time of maximum surface heating. During this period at O’Neill we estimate \( \frac{1}{2} \) for the ratio of the heat flux entering the boundary layer at the top to that entering at the surface. There is no evidence in the O’Neill data to support Ball’s hypothesis that the ratio is near unity, indeed for a large part of the development the suggestion is that the interfacial mixing contribution is effectively zero, particularly in the early stages when the capping layer is the markedly stable, nocturnally established inversion. These deductions, being model dependent, are naturally speculative and we must look to the future development of studies such as those by Readings, Golton and Browning (1972) and Browning, Starr and Whyman (1972) for our answers. The part played by wind shear in the ultimate breakdown of the convoluted interface is a primary concern.

The multi-phase model is capable of producing realistic boundary layer development (Figs. 10 and 13) and a provisional attempt to include such non-steady features in a practical scheme for estimating the vertical dispersion of pollutants has already been outlined by Smith (1972).

The model should strictly be limited to the dry, convectively unstable boundary layer in virtually clear-sky, non-advective conditions. Important extensions to non-steady, fog layers or moist cloud-topped mixed layers will require extensive treatment of complex radiative and evaporative processes (see for example Lilly 1968).
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