The evolution of droplet spectra and large droplets by condensation in cumulus clouds

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SUMMARY

The paper describes a relatively simple model of a non-precipitating cumulus cloud that grows by the ascent of successive spherical thermals through the residues of their predecessors, mixing with the surroundings being determined by the relative velocity and the radius of the thermal. The model appears to account for several important features of the cloud structure and the droplet-size spectrum that are not produced in models of a single parcel or plume. In particular, it produces modest clouds in 30-40 min in which the vertical velocity, cloud depth and liquid-water content agree quite well with observations. The computed droplet spectra closely resemble measured spectra and reproduce the bimodal structure observed by Warner. In model maritime clouds containing small concentrations of droplets, the spectra broaden quite rapidly and produce droplets of r = 25μm by condensation on nuclei of m = 10^{-11} g in concentrations of order 100 m^{-3} within half an hour, beyond which size they may continue to grow rapidly by coalescence to precipitation size. However, in continental clouds containing droplet concentrations of a higher order, it is difficult to produce a significant number of droplets of even 20μm radius.

1. INTRODUCTION

One of the outstanding problems in cloud microphysics is to account for the observed size distributions of droplets in non-precipitating, non-freezing cumulus clouds, their tendency to show a bimodal character, and the growth of sufficient droplets to a radius (∼25 μm) beyond which they can continue to grow rapidly by coalescence to precipitation size. Discussion of these problems, their background and importance, appear in Mason (1969, 1971) and need not be repeated here. We need recall only the conclusions of Mason and Chien (1962) and of Warner (1969b) that condensation on a realistic spectrum of soluble nuclei in an ascending air parcel cooling adiabatically without mixing with its surroundings produces much narrower cloud droplet distributions than are actually observed, and cannot account for the high concentrations of small droplets (r < 5 μm) which often appear at all levels in the cloud because the simple theory predicts that all these should surpass this size after an ascent of only some 100 m above cloud base. Moreover such simple models cannot account for the development of bimodal spectra, nor for the production of droplets of r > 20 μm unless one assumes the presence of giant condensation nuclei. Mason and Chien (1962) demonstrated that much broader droplet spectra could be produced in a rising air parcel that mixes with its surroundings, exchanging heat, momentum, moisture, and droplets for fresh unactivated nuclei but, again, they could not explain the formation of droplets of r > 20 μm except by condensation on giant salt nuclei of m > 10^{-10} g. Moreover, Warner (1973), in making calculations very similar to those of Mason and Chien, has recently pointed out that although their model produces a broadening of the droplet spectrum, it does not predict the shape of spectra observed in natural clouds, nor the observed bimodality which tends to be more pronounced at higher levels in the cloud and when the environment becomes increasingly unstable.

In this paper we extend the computations of Mason and Chien, which dealt with only a single rising thermal, to study the effect on the evolution of droplet spectra of allowing the thermal, after reaching its maximum height, to sink back as a result of evaporative cooling
following the entrainment of dry environmental air, and then to become mixed with a second thermal rising through it. Even such a simple model of a building cloud, in which mixing is assumed to occur uniformly across a thermal and to be determined only by the vertical velocity and radius of the thermal, predicts relationships between the height, updraught, liquid-water content and droplet-size spectra that are much more realistic than for a single isolated thermal. Moreover, although the great majority of the droplets in the decaying and subsiding thermal may evaporate, the largest ones may survive to be caught up and continue their growth in a succeeding thermal. In this way it is possible to produce droplets of \( r = 25 \, \mu m \) on nuclei of \( m = 10^{-11} \, g \) inside a maritime cumulus cloud \(<2 \, km \) deep within half an hour but, in a continental cloud containing much higher concentrations of nuclei and droplets, it is very difficult to produce droplets of even \( 20 \, \mu m \) radius. The maritime cloud model also produces droplet spectra very similar in shape to those observed by Warner with bimodal peaks at droplet radii of about \( 5 \, \mu m \) and \( 15-20 \, \mu m \).

The presence of clear patches largely devoid of droplets deep in the cloud interior are also accounted for by the evaporation and subsequent descent of protuberances formed on the surfaces of emergent cloud towers.

The model therefore appears capable of accounting for the main features of the structure and constitution of non-precipitating cumulus clouds.

2. **Turbulent mixing of a cloudy thermal with its surroundings**

We treat a rising cloudy thermal as an expanding spherical mass of radius \( R \), mass \( M \), mixing with its surroundings by turbulent motions generated within the thermal. Its rate of increase of mass is then given by

\[
\frac{dM}{dt} = \frac{d}{dt} \left( \frac{4}{3} \pi R^3 \rho \right) = 4\pi R^2 (\phi \rho' - \phi' \rho),
\]

(1)

where \( \phi, \phi' \) are the mixing velocities and \( \rho, \rho' \) the potential densities within the thermal and environment respectively. It is assumed that the inflow and outflow of fluid are determined solely by the r.m.s. turbulent velocities in the thermal and in the environment, i.e.

\[ \phi = a \sigma_{\omega}, \quad \phi' = a \sigma_{\omega}', \]

\( \sigma_{\omega}, \sigma_{\omega}' \) being the r.m.s. turbulent velocities.

Thus

\[
\frac{1}{M} \frac{dM}{dt} = \frac{3a}{R} \frac{(\sigma_{\omega} \rho' - \sigma_{\omega}' \rho)}{\rho}.
\]

If the surroundings are very much less turbulent than the thermal, i.e. \( \sigma_{\omega}' \ll \sigma_{\omega} \)

\[
\frac{1}{M} \frac{dM}{dt} = \frac{3a}{R} \frac{\sigma_{\omega}}{\rho}.
\]

(2)

Telford (1966) determined the 'mixing constant' \( a \) from an analysis of the spreading of plumes of constant buoyancy in laboratory tanks to be

\[
a = \left( \frac{3}{\beta} \right)^{1/2} \frac{5}{6} \beta,
\]

(3)

where \( \beta \) is the half-angle of the plume spreading along a vertical cone. Turner (1956) gives
DROPLET GROWTH IN CUMULUS

a mean value for $\beta = 0.24$, while Richards (1961) gives $\beta = 0.28$. Taking an average value of $\beta = 0.26$ gives

$$a = \frac{1}{6},$$

and

$$\frac{1}{M} \frac{dM}{dt} = \frac{\sigma_{\omega}}{2R}. \quad (4)$$

We now calculate $\sigma_{\omega}$ for a spherical thermal mixing with its surroundings by isotropic turbulent motions in which there is assumed to be no production or dissipation of energy but only inertial transfer to smaller and smaller eddies. Although these conditions are unlikely to be closely obeyed in a growing cumulus in which energy is being released by condensation, these assumptions are unlikely to produce unrealistic estimates of $\sigma_{\omega}$.

According to Smith and Hay (1961), the 3-dimensional energy spectrum appropriate to a cluster of particles having a standard deviation $\sigma$ in their positions about the centre of the cloud is

$$E_{\omega}(k) = E(k)(1 - e^{-\sigma^2 k^2}), \quad (5)$$

where $k$ is the wave number and $E(k)$ the full spectral density given by

$$E(k) = \frac{3}{2} e^{2/3} k^{-5/3},$$

where $\varepsilon$ is the rate of dissipation of turbulent kinetic energy per unit mass. Now

$$\sigma_{\omega}^2 = \frac{2}{3} \int_0^\infty E_{\omega}(k) \, dk$$

$$= e^{2/3} \int_0^\infty k^{-5/3}(1 - e^{-\sigma^2 k^2}) \, dk$$

$$= e^{2/3} \sigma^{2/3} \int_0^\infty x^{-5/3}(1 - e^{-x^2}) \, dx$$

which, with $y = x^2$,

$$= \frac{1}{2} e^{2/3} \sigma^{2/3} \int_0^{\sigma^2} y^{-4/3}(1 - e^{-y}) \, dy$$

$$= \frac{9}{4} \Gamma\left(\frac{5}{3}\right) e^{2/3} \sigma^{2/3}$$ \quad (6)

or

$$\sigma_{\omega} \simeq \frac{3}{2} e^{1/3} \sigma^{1/3}. \quad (7)$$

Thus Eq. (4) becomes

$$\frac{1}{M} \frac{dM}{dt} = \frac{3 e^{1/3} \sigma^{1/3}}{4 R}, \quad (8)$$

which can be evaluated if we assume that $R = 2\sigma$ and specify $\varepsilon$. 
From observations on emerging cumulus towers Saunders (1961) finds that, on average, the ratio of their radii to their height is $\beta' = 0.2$
so
$$\frac{1}{M} \frac{dM}{dz} = \frac{3\beta'}{R} = \frac{0.6}{R}$$
and
$$\frac{1}{M} \frac{dM}{dt} = \frac{0.6U}{R} = \frac{\mu U}{R},$$
(9)
where $U$ is the rate of ascent of the thermal cap, and $\mu$ is the mixing parameter.
Comparison of Eqs. (8) and (9) allows us to write
$$\frac{3}{4} \varepsilon^{1/3} \sigma^{1/3} = 0.6U$$
or
$$\varepsilon = (0.8)^2 \frac{U}{\sigma} \approx \frac{U^3}{2\sigma} = \frac{U^3}{R},$$
(10)
which for a cumulus tower of $R = 0.5$ km gives the following values of $\varepsilon$:
$$U = 1 \quad 2 \quad 3 \quad 5 \quad \text{m s}^{-1}$$
$$\varepsilon = 20 \quad 160 \quad 540 \quad 2,500 \quad \text{cm}^2 \text{s}^{-3}$$
that lie in the range frequently quoted for growing cumulus towers.

We shall adopt Eq. (9) to compute the rate of dilution of a cloudy thermal mixing with its surroundings noting that the rate of mass dilution is proportional to the vertical velocity and inversely proportional to the radius of the thermal.

The time scale of the mixing, defined as the average time taken for a particle to move from the centre to the edge of the thermal is $\tau \approx 2\sigma/\varepsilon \sigma = \frac{3}{8} \sigma^{2/3} \varepsilon^{-1/3}$ which, with $2\sigma = R = 500$ m and $\varepsilon = 500$ cm$^2$ s$^{-3}$, gives $\tau = 150$ s, which is short compared with the life-time of the thermal and suggests that the assumption of uniform mixing is reasonable.

3. THE CLOUD MODEL AND GOVERNING EQUATIONS

We consider a spherical cloudy thermal possessing horizontal homogeneity ascending from the condensation level and mixing with its surroundings so that its mass increases at the rate given by Eq. (9). Mixing results in the exchange of heat, momentum and moisture, and is assumed to transfer droplets from the cloud to the drier environment where they evaporate and are replaced by fresh condensation nuclei of the same mass. The governing equations for the vertical motion of the thermal, and for the rates of change of the temperature, supersaturation, and liquid-water content are essentially the same as those derived by Mason and Chien (1962).

The equation of vertical motion is
$$\frac{dU}{dt} = \left( \frac{T - T'}{T} - w \right) g - \frac{\mu U^2}{R},$$
(11)
where $U$ is the vertical velocity relative to the surroundings, $T, T'$ the virtual temperatures of the thermal and the environment, $R$ the radius of the thermal, $w$ its liquid-water mixing ratio, and $\mu = 0.6$. 
The rate of change of temperature of the thermal is

$$\frac{dT}{dt} = \frac{1}{C_p} \left[ \frac{L}{dt} \left( T - T' \right) + \frac{L}{C_p} (x - x') \right],$$

(12)

where $L$ is the latent heat of condensation, $C_p$ the specific heat of dry air at constant pressure, $x$, $x'$ the humidity mixing ratios of the cloud and environment respectively.

The rate of change of supersaturation $\sigma$ is

$$\frac{d\sigma}{dt} = \frac{1}{x_s} \left[ \frac{dw}{dt} + \frac{\mu U}{R} (x - x' + w - w') \right] - (1 + \sigma) \left[ \frac{\varepsilon L}{R_s T^2} \frac{dT}{dt} + \frac{g U}{R_s T} \right],$$

(13)

where $x_s$ denotes the saturation humidity mixing ratio, $\varepsilon$ the ratio of the densities of water vapour and dry air and $R_d$ the gas constant for dry air.

The rate of expansion of the thermal is given by

$$\frac{dR}{dt} = 0.2U,$$

or

$$R = R_0 + 0.2z.$$  

(14)

The cloud droplet spectrum is assumed to consist of a number of sets of droplets containing $n_{ij}$ droplets per unit mass of air with radius $r_i$ and nucleus mass $m_{ij}$. The rate of change of liquid-water mixing ratio is

$$\frac{dw}{dt} = 4\pi \sum_i \sum_j n_{ij} r_i^2 \frac{dr_i}{dt},$$

(15)

where

$$\frac{dr_i}{dt} = \left[ \sigma - \frac{2\gamma}{\rho_L R_w Tr_i} + \frac{8.6m_{ij}}{W r_i^3} \right] \left[ \frac{L \rho_L}{K T} - 1 \right] + \frac{R_w T \rho_L}{D e_s}.$$  

(16)

in which $\gamma$ is the surface tension and $\rho_L$ the density of water, $R_w$ the gas constant for water vapour, $W$ the molecular weight of the nucleus (assumed to be NaCl), $K$ the thermal conductivity of air and $D$ the diffusion coefficient of water vapour in air and $e_s$ the saturation vapour pressure of water at temperature $T$. Finally we have an equation to express the assumption that droplets are transferred to the cloud boundaries, where they evaporate and are replaced by nuclei of the same mass entrained from the surroundings, at the rate

$$\frac{d}{dt} n_{ij} = -\frac{\mu U}{R} (n_{ij} - n'_{ij}).$$

(17)

so that the total concentration of nuclei plus droplets in the thermal remains unchanged with some removed from each set and additional sets created by the entrainment of new nuclei at each time step.

Given initial values for the radius $R_0$, vertical velocity $U_0$, and temperature excess $(T - T')_0$ of the thermal at the condensation level, the lapse rate and humidity of the environment, and the initial nucleus spectrum, as listed in Table 1, the seven equations (11)-(17) are integrated to give $R$, $U$, $T$, $w$, $\sigma$, $n_{ij}$, $r_{ij}$ as functions of time and height $z = \int_0^t U dt$ above cloud base.

The first thermal is assumed to rise through and mix with clear air of constant relative humidity and lapse rate in which the condensation-nucleus spectrum is assumed to be
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature and pressure at condensation level</td>
<td>10°C, 900 mb</td>
</tr>
<tr>
<td>Initial radius of thermal</td>
<td>$R_0$ 350 m</td>
</tr>
<tr>
<td>Initial vertical velocity of thermal</td>
<td>$U_0 1,0 \text{ m s}^{-1}$</td>
</tr>
<tr>
<td>Initial excess temperature</td>
<td>$\Delta T_0$ 0.1°C (maritime), 0.6°C (continental)</td>
</tr>
<tr>
<td>Lapse rate of environment</td>
<td>$\alpha$ 7°C km$^{-1}$</td>
</tr>
<tr>
<td>Humidity of environment</td>
<td>$H'$ 85 per cent</td>
</tr>
<tr>
<td>Mixing parameter</td>
<td>$\mu$ 0.6</td>
</tr>
<tr>
<td>Mass of nuclei (g of NaCl)</td>
<td>$5 \times 10^{-16}$ 10$^{-15}$ 3 $\times$ 10$^{-14}$ 10$^{-13}$ 3 $\times$ 10$^{-12}$ 10$^{-12}$ 3 $\times$ 10$^{-12}$ 10$^{-11}$ 3 $\times$ 10$^{-12}$ 10$^{-11}$ Total Conc.</td>
</tr>
<tr>
<td>Concentration (per g of air)</td>
<td>Mar $3.8 \times 10^4$ 1.9 $\times$ 10$^4$ 6 $\times$ 10$^3$ 1.9 $\times$ 10$^3$ 6 $\times$ 10$^2$ 1.9 $\times$ 10$^2$ 60 19 6 1.9 6.6 $\times$ 10$^4$/g.</td>
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<tr>
<td></td>
<td>Con $3 \times 10^5$ 1.5 $\times$ 10$^5$ 4.8 $\times$ 10$^4$ 1.5 $\times$ 10$^4$ 4.8 $\times$ 10$^3$ 1.5 $\times$ 10$^3$ 480 150 48 15 5.3 $\times$ 10$^5$/g.</td>
</tr>
</tbody>
</table>
always the same as that existing at cloud base. This spectrum normally consists of 10 classes with nuclear masses ranging from $5 \times 10^{-16}g$ to $10^{-11}g$ although the upper limit was increased in some calculations to $3 \times 10^{-10}g$. The concentrations of nuclei are assumed to vary inversely as the nuclear mass. The radius of the nucleus droplets before they enter the thermal are assumed to be those given by Mason and Chien but very similar results are obtained if the nuclei are carried up from a level well below cloud base where they are assumed to be in equilibrium with a relative humidity of 78 per cent. In order to reduce the number of sets of droplet sizes generated by the model, Eq. (17) was integrated in steps of 5 s rather than 0.05 s for the other equations. That such an approximation did not materially affect the results was checked by altering the time interval and is consistent with the time scale of the mixing process being much longer than that of the condensation process. A useful check on the numerical technique was obtained by setting $1/M dM/dt$ independent of $U$ when it was possible to reproduce the cloud development predicted by Mason and Emig (1961).

When the first thermal reaches the level at which it loses its buoyancy and comes to rest, calculations are continued to follow its subsequent behaviour as it cools and subsides through mixing with its drier surroundings. Nearly all the droplets, except the few largest, evaporate before the thermal comes to rest in the vicinity of the condensation level.

The next stage is to follow the development of a newly rising thermal that crosses the condensation level with the same initial size, velocity, and temperature excess over the cloud environment as the first thermal but now mixing with the residue of this first thermal and, after passing through it, continuing to mix with the drier environment above. The second thermal naturally achieves a greater buoyancy, height, vertical velocity and liquid-water content than its predecessor, produces a broader droplet spectrum, and also some relatively large droplets by allowing continued condensation on the largest drops that have grown in the first thermal and survived its subsidence and evaporation.

4. THE DEVELOPMENT OF MODEL CLOUDS

Eqs. (11) to (17) have been integrated for several different values of the parameters listed in Table 2 to produce a variety of model clouds but full details and discussions will be

<table>
<thead>
<tr>
<th>$U_0$</th>
<th>$\Delta T_0$</th>
<th>$a$</th>
<th>$H'$</th>
<th>$R_0$</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$U_1$</th>
<th>$U_2$</th>
<th>$w_1$</th>
<th>$w_2$</th>
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<tr>
<td>(a) Maritime cloud</td>
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<tr>
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<td>7.0</td>
<td>85</td>
<td>350</td>
<td>330</td>
<td>1610</td>
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<td>0.3</td>
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<td>550</td>
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<td>350</td>
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<td>3500</td>
<td>5.7</td>
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<td>0.1</td>
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<td>7.0</td>
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<td>340</td>
<td>3010</td>
<td>2.1</td>
<td>6.2</td>
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<tr>
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<td>230</td>
<td>270</td>
<td>950</td>
<td>1.1</td>
<td>2.3</td>
</tr>
<tr>
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<td>7.0</td>
<td>85</td>
<td>350</td>
<td>340</td>
<td>3137</td>
<td>1.1</td>
<td>2.6</td>
<td>0.2</td>
<td>0.7</td>
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<tr>
<td>(b) Continental cloud</td>
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</table>

$h_1, U_1, w_1$ are the maximum height, vertical velocity and liquid-water content of the first thermal

$h_2, U_2, w_2$ the corresponding quantities for the second thermal
confined to only two examples, one intended to typify a small maritime cumulus containing a low concentration of nuclei/droplets, the other being typical of a continental cloud containing a much higher concentration of droplets. The computed values of vertical velocity, liquid-water content and supersaturation as functions of height above condensation level and of elapsed time are shown in Figs. 1 and 2, and the corresponding droplet spectra in Figs. 3 and 6.

For the model maritime cloud, Fig. 1(a) shows that the centre of the first thermal, crossing the condensation level with an initial radius of 350 m, initial velocity of 1 m s⁻¹ and temperature excess of 0.1 deg C, attains a maximum height of 0.33 km before coming to rest, a maximum vertical velocity of only 1.01 m s⁻¹ at a height of 30 m, and a maximum liquid-water content of 0.3 g/kg or 0.38 of the adiabatic value. On attaining the level of zero buoyancy, after 9 min, the thermal now mixes with the surrounding air of 85 per cent humidity, the progress of its evaporation and descent being depicted in Fig. 1(b). It attains

![Graph showing the development of successive thermals in a maritime cloud.](image)

Figure 1. The development of successive thermals in a maritime cloud. Condensation level 900 mb, 10°C: \( h_0 = 350 \text{ m}, u_0 = 1 \text{ m s}^{-1}, \Delta T_0 = 0.1 \text{°C}, \) lapse rate \( a = 7 \text{ °C} \text{ km}^{-1}, \) environmental humidity \( H' = 85 \text{ per cent} \). The vertical velocity \( U, \) liquid-water content \( W, \) and the ratio of \( W \) to the adiabatic value \( W_s, \) are plotted as functions of height.

(a) The ascent of the first thermal in clear air.
(b) The decay and descent of the first thermal.
(c) The ascent of a second thermal through the residue of the first.

The figures on the right of each diagram show the elapsed time (min) since the start of each stage. The progress of the supersaturation during the growth of the second thermal is shown \( \ldots \ldots \ldots \) and was obtained for a nucleus population of 66 per mg of air.
a maximum downward velocity of 1.2 m s⁻¹ before coming to rest near the original condensation level after a total elapsed time of 17 min by when all but the largest droplets have evaporated. At this stage, a new thermal is assumed to cross the condensation level with the same initial properties as its predecessor but it now mixes with the residue of the earlier thermal assumed to be quiescent and just saturated. On emerging from the old thermal, it continues to mix with the drier environmental air, soon reaches its maximum vertical velocity of 2.4 m s⁻¹ and thereafter decelerates to reach a maximum height of 1.6 km, a maximum radius of 670 m, the total elapsed time being 35 min. The discontinuity at the maximum of the liquid-water profile is the consequence of a sudden transition as it emerges from the old one and, as such, is rather artificial. Over the upper two-thirds of the ascent the liquid-water concentration ranges from 0.6 g/kg to 0.9 g/kg or about 0.35 of the adiabatic value, and supersaturation reaches a peak value of 0.4 per cent.

The extent to which the development of such a model cloud is affected by changes in the initial and environmental parameters is indicated in Table 2, where the sensitive factors are seen to be the lapse rate and the humidity of the environment, and the rate of mixing as represented by the parameter μ/R, the initial buoyancy and vertical velocity of the thermal being less critical. Thus decreasing the environmental humidity from 85 to 80 per cent practically halves the vertical velocity and liquid-water content of the cloud, whereas increasing the humidity from 85 to 90 per cent more than doubles these parameters and
also the vertical velocity. Increasing the environmental lapse rate from 7 deg C/km to an unrealistically high value of 8 deg C/km throughout the layer produces an explosive growth of the model cloud. Reducing the initial radius of the thermal from 350 to 250 m halves the maximum height and liquid-water content.

The development of a continental cloud growing under the same environmental conditions and starting with the same initial radius and vertical velocity as the maritime cloud, but with an excess temperature of 0.6 deg C and a total nucleus population of 530/mg of air instead of only 66/mg, is shown in Fig. 2. This cloud attains a maximum updraught of 6.1 m s⁻¹ and a height of 3.45 km within a total time of 44 min. The liquid-water content in the upper half of the cloud now averages about 1.5 g/kg, again about 0.3–0.4 of the adiabatic value. The sensitivity of the development to the prescribed parameters is similar to that described for the maritime cloud as shown in Table 2.

5. Evolution of the droplet spectra

The evolution of the droplet spectrum in the maritime cloud is shown in Fig. 3 where the size distributions are shown at various heights in histograms with the number concentrations on a logarithmic scale. These spectra contain both droplets and unactivated nuclei (smallest size groups), the numbers inside the histograms representing the numbers

![Figure 3](image-url)
of droplets per mg of air. Extension of the nucleus spectrum to include nuclei as small as $m = 10^{-16}$ g produced no noticeable change in the evolution of the droplet spectrum.

During the ascent of the first thermal, droplets growing on the largest nuclei ($m = 10^{-11}$ g) attain a maximum radius of 16$\mu$m in concentrations of about 1/litre. During the subsequent descent and evaporation of the thermal most of the droplets evaporate below their critical radius, so that when the thermal comes to rest, the droplet concentration is only about 1 cm$^{-3}$ the largest of which are 8$\mu$m in radius. Fig. 3(c) shows that, during the ascent of the second thermal, the spectrum develops a bimodal structure with a marked peak at $r = 15-25\mu$m, the concentration of large drops increasing very rapidly with height. Thus near the top of the ascent, droplets of $r \geq 25\mu$m appear in concentrations of about 300 m$^{-3}$ and those of $r \geq 20\mu$m in concentrations as high as 10 cm$^{-3}$. When the cloud

![Figure 4](image)

Figure 4. A comparison of a droplet spectrum predicted by the model after the ascent of a second thermal 150 m above the base of a maritime cloud now assumed to contain 130 nuclei/mg, with a spectrum (dashed line) measured by Warner at a similar height in a similar sized cloud where the total droplet concentration was 180 cm$^{-3}$.
was allowed to develop further by the ascent of a third thermal the spectrum was extended to larger drop sizes and the bimodality was preserved.

In order to facilitate comparison with some measured spectra published by Warner (1969a, b), some computations were made using the model of Fig. 3 but with the concentrations of nuclei doubled to match the observed total droplet concentrations. Computed spectra for levels near cloud base and at 1,400 m above it are shown by the histograms of Figs. 4 and 5 while the dashed curves show spectra obtained by Warner at these levels in a cloud of similar size. The general agreement between computed and observed spectra is good, the observed bimodality with a second peak at \( r = 15-20\mu m \) being well predicted by the model. Moreover, the most unrealistic feature of the earlier computations by Mason and Chien (1962) and of Warner (1973), namely the existence of a persistent plateau at the small end of the spectrum, is absent from the present results.

Figure 5. Comparison of the computed spectrum at 1.4 km above the condensation level in the maritime cloud of Fig. 1(G) and Fig. 4, with the mean of two spectra observed by Warner near the top of a cloud 1.4 km deep in which the droplet concentrations were 143 and 116 cm\(^{-3}\).
Figure 6. Evolution of the droplet spectrum in the continental cloud of Fig. 2. Representation is the same as in Fig. 3.
The evolution of droplet spectra in the model continental cloud is shown in Fig. 6 where, even in the last stages of ascent of the second thermal, the largest droplets do not exceed 20\(\mu m\) in radius and the concentration of 19-20\(\mu m\) droplets is only 30 m\(^{-3}\). The bimodal structure is still evident but less marked than for the maritime cloud.

6. FORMATION OF HOLES BY PENETRATIVE DOWNDRAUGHTS

Patchiness in the distribution of liquid-water content along a horizontal traverse, and the appearance of holes largely devoid of droplets deep in the interior of cumulus clouds have been frequently reported, notably by Squires (1958) who suggested that they are produced by the entrainment of pockets of dry air into the tops of cloud towers. We have used our model of mixing to examine this hypothesis by assuming that the process starts with the appearance of protuberances of order 100 m in diameter on the caps of emerging cloud towers which trap similar sized pockets of relatively dry environmental air from above the cloud and that these then sink into the cloud mass as a result of evaporative cooling. The descent is sustained by mixing of the parcel with the surrounding cloud, chilling by the rapid evaporation of the entrained cloud droplets being more than sufficient to offset warming due to adiabatic compression.

As an example, we consider a spherical parcel of radius 50 m starting at the summit of the cloud depicted in Fig. 2(c), having the environmental humidity of 85 per cent and an initial downward velocity of 0.5 m s\(^{-1}\), and follow its subsequent descent through the cloud using the same equations and mixing parameter as in calculating the descent of the thermal of Fig. 2(b) through clear air. The computations show that if the updraught in the cloud is assumed to have the profile shown in Fig. 2(c), then the parcel penetrates only to a depth of about 200 m below the cloud summit before coming to rest. Its relative humidity is then 96 per cent and it contains no droplets. However, the penetration is critically dependent on the updraught in the cloud because the rate of mixing is proportional to \(U^2/R\), where \(U\) is the velocity of the parcel relative to its surroundings. If the cloud updraught is weak the same descending parcel penetrates all the way down to cloud base. Thus in one example, with the cloud updraught reduced to zero, the parcel reaches cloud base with a relative humidity of 99 per cent and a temperature deficit of 0.4 deg C within 32 min, its maximum downward velocity being 2.5 m s\(^{-1}\). Although small parcels of air are eroded more rapidly and penetrate less deeply than larger ones, the penetration is not very sensitive to the size of the parcel if the cloud updraught is weak because the dilution is slow and the penetration large in any case.

Thus we would expect to find deeply penetrating downdraughts and holes mainly in inactive or decaying clouds where they will introduce inhomogeneities and reductions in the liquid-water content and hasten the dissipation of the cloud.

7. DISCUSSION

We have developed a relatively simple model which envisages a cumulus cloud to grow by the ascent of successive spherical thermals through the residues of their predecessors, mixing with the surroundings being determined only by the relative velocity and the radius of the thermal.

This model appears to account for several important features of the cloud structure and the droplet-size spectrum that have not been reproduced in previous models that treat only a single buoyant parcel or plume. In particular, the new model produces modest cumulus clouds in 30-40 min in which the vertical velocity, cloud depth and liquid-water content accord quite well with measurements whereas, as pointed out by Warner (1970), the
usual steady-state one-dimensional models with a single mixing parameter cannot simultaneously predict values of vertical depth and liquid-water content that agree with observations. Even with the over-simplified assumption that mixing occurs uniformly throughout the thermal, the predicted liquid-water concentrations average only about 0.3–0.4 of the corresponding adiabatic values and the ratio falls off with increasing height in the observed manner. If the maritime cloud of Fig. 1 is allowed to develop a stage further by the ascent of a third thermal, the value of \( w/w_0 \) decreases to only about 0.2 in the upper part of the cloud where the updraught barely exceeds \( 3 \text{ m s}^{-1} \). Further dilution of the liquid water and inhomogeneities in its horizontal distribution may be caused by the entrainment and descent of small pockets of dry air from the cloud top.

The model has also produced, for the first time, computed droplet spectra that closely resemble measured spectra and, in particular, reproduces the bimodal structure observed by Warner (1969a, b). In model maritime clouds containing small concentrations of droplets, the spectra broaden quite rapidly and produce droplets of \( r = 25 \mu \text{m} \) by condensation on nuclei of \( m = 10^{-11} \text{g} \) in concentrations of order 100 \( \text{m}^{-3} \) within half an hour. Thus we appear to have solved the long-standing problem of how to account for the production by condensation of droplets of this size, beyond which they may then continue to grow rapidly by coalescence to precipitation size, without invoking the presence of giant hygroscopic nuclei. However, the computations show that in continental clouds, containing droplet concentrations of a higher order, it is difficult to produce a significant number of droplets of even 20\( \mu \text{m} \) radius. This confirms the general impression that continental cumulus are more stable and produce showers less readily than maritime clouds of similar size and vigour.

The model calculations suggest that the microstructure of a cumulus cloud is related to its size, life-history (in particular the number of successive thermals involved in its building), its age, and the intensity of the mixing processes. They point to the desirability of relating aircraft measurements of droplet size distributions, concentrations of large droplets, liquid-water content and updraught structure to the size, growth and evolution of the cloud and, in particular, to the development of individual cloud towers, as determined by careful ground-based visual, photogrammetric and radar observations.

References


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