A three-dimensional primitive equation model of cumulonimbus convection

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SUMMARY

The model is described and the results of three integrations presented. Its main novel features are the inclusion of non-hydrostatic terms in a pressure co-ordinate system, and the treatment of flow through the lateral boundaries. The results of the integrations, one with an ambient vertical wind shear and two without, indicate that the cloud dynamics is quite sensitive to assumptions made regarding the cloud microphysics, particularly at the stage when the downdraught is produced.

I. INTRODUCTION

This paper describes a three-dimensional primitive equation numerical model of the atmosphere designed to simulate deep convective systems. The main respects in which it differs from synoptic scale primitive equation models is in the inclusion of non-hydrostatic terms in the vertical momentum equation, and the use of a cloud physics parameterization which computes explicitly the concentrations both of cloud water droplets and falling water drops. The model is of a limited volume of the atmosphere and, in order to be able to simulate convection in shear, boundary conditions have been specified in such a way as to enable flow to be maintained through the lateral boundaries.

Attempts to mathematically model cumulus convection systems have been made using both analytic and numerical methods. With the assumption of steady, two-dimensional inviscid flow, of Richardson number of order unity, finite amplitude convective overturning in shear consistent with Browning and Ludlam's (1962) observational model, was studied by Green and Pearce (1962) and more recently by Moncrieff and Green (1972). Green and Pearce examined the dynamical feasibility of Ludlam and Browning's model. Moncrieff extended the work to derive transfer properties of this model in terms of large-scale parameters, as required for convective parameterizations in large-scale models. Their type of analytical study provides a complementary approach to that described in this paper.

Many numerical models have been developed, the first being that of Malkus and Witt (1959). Earlier models were of shallow convection and two-dimensional, either of dry thermals in a neutral environment or of moist convection in which only the latent heat release associated with the condensation process was modelled: notable amongst these are the models of Ogura (1962, 1963), Lilly (1962), Murray and Anderson (1965), Nickerson (1965) and Orville (1965). These early models were either in rectangular Cartesian co-ordinates
(x, z) (i.e. with no y dependence), or axially-symmetric cylindrical co-ordinates (r, z) (i.e. with no azimuthal dependence). Ogura endeavoured to compare models in both co-ordinate systems. More recently, Murray (1970) and Soong and Ogura (1973) have described similar, more thorough, comparisons with interesting results with respect to the intensities of the convection in the two co-ordinate systems; although differing in certain respects both studies underline the importance of the areal ratio of upward and downward motions.

Also among the earlier models was one of a squall-line, the work of Ogura and Charney (1962). This was unique in its use of the primitive equations rather than a vorticity equation.

Many problems of a computational nature were encountered in these early models, although problems connected with lateral boundaries were for the most part avoided by the 'closed box' assumption, i.e. no flow was allowed through the boundaries. This of course precluded the use of non-zero ambient windfields. More recently some authors, notably Takeda (1971) and Orville (1968), have described two-dimensional models with flow through boundaries.

The last few years have seen research work directed mostly towards the sophistication of existing models, and their extension to the full depth of the troposphere, with cloud physics parameterization. It is arguable, however, that the basic inadequacies of the existing models have made some of these more complex additions redundant.

The basic equations are those defining conservation of mass, momentum, energy and water substance. Nearly all models have in common the use of the Boussinesq or anelastic equation set (in which density changes appear only in association with gravity). A vorticity equation can readily be derived and, with the assumption of two-dimensionality, a stream function (ψ) can be defined; this, together with the vorticity and other conservation equations, forms a comparatively simple modelling system in two dimensions. Using this system a Poisson-type equation is solved for ψ. If the pressure field is required explicitly, a second Poisson-type equation then has to be solved. (The pressure could alternatively be obtained by integrating a Bernoulli equation along each streamline, using Lagrangian techniques.)

The extension into three dimensions of such a vorticity system with reasonable vertical resolution is an almost impossible task and the alternative approach using a model based on the primitive equations is therefore pursued. The use of primitive equations on smaller convective scales is so far minimal, the previously-mentioned model of Ogura and Charney being noteworthy. They carried out a frequency analysis of the perturbation equations and considered ways of simplifying the set in a practical and consistent fashion. This analysis was done in height co-ordinates, although the primitive equations are more often used in pressure co-ordinate form on the synoptic scale. Two studies in three dimensions have recently been published. Steiner (1973) describes a model of a dry thermal using a horizontal grid length of 200 m. Pastushkov (1973) describes a model of deep moist convection based on the primitive equations.

The choice between height or pressure as a vertical co-ordinate is determined in part by the nature of the study being undertaken. The standard use of pressure co-ordinates (pressure surface representation) on the large scale is prompted to some extent by the neatness of handling the air density in the momentum and continuity equations (although the availability of data at pressure levels is the main reason).

For theoretical work, height co-ordinates are used almost exclusively, forming a true Cartesian set: by suitable manipulation they can be used to handle the density conveniently also. The mathematical imperfections of pressure co-ordinates are, however, small and rarely introduce approximations that are significant. Convection models have, without exception, been based on a height co-ordinate system mainly to allow the stream function with vorticity equation formulation (although a stream function formulation can be used in pressure co-ordinates with approximations). This use of gridpoints fixed in space introduces one
practical disadvantage when modelling deep convection. Inconsistencies occur in computing the condensation rate of vapour to liquid water if the departures of pressure (both hydrostatic and non-hydrostatic) are not taken into account; however, to do so introduces complex, implicit relationships which have been adequately handled only with some difficulty by modellers to date. The use of gridpoints in pressure surfaces circumvents this problem very conveniently. The results of Williamson and Ogura (1972) have however suggested that, at least in numerical models, the pressure perturbations are small enough to make their inclusion in the thermodynamic calculations less critical.

Pressure is used as vertical co-ordinate in the model described here. The use of these co-ordinates in a non-hydrostatic system, however, is novel, and consistent vertical momentum and balance equations have had to be derived (see Section 3(a)). An analysis of the linearized equation-set in pressure co-ordinates has been carried out by Miller (1974), and is referred to in the next Section describing the mathematical formulation of the model.

Any model simulating a cumulonimbus must be large enough to contain all stages of the cloud's life-cycle, i.e. unrealistic boundary effects such as might result, for example, from the impingement of the cloud on a rigid upper boundary, must not interfere. As a result, the model was designed to span a depth of atmosphere greater than the depth of the tropical troposphere – up to 100 mb. In view of computer limitations, the horizontal area was limited to 15 km × 15 km. While this is sufficient to contain all but the very largest clouds, it certainly will not contain all the accompanying motion field. This areal limitation was with a horizontal grid length of one kilometre, a rather coarse grid. Because of this limitation on horizontal area, the model was designed with open lateral boundaries, and the boundary conditions formulated to model in a simple manner that part of the environment outside the main updraught and adjacent parts of the downdraught.

The vertical grid-spacing was irregular, defined by seven pressure surfaces (1000, 850, 700, 500, 300, 200 and 150 mb) with a free-surface boundary condition applied at 100 mb. The physical variables were computed on a grid staggered in space but not in time.

The deficiencies introduced by the coarseness of the grid (such as large implicit smoothings associated with truncation errors) are technological in so far as they can be minimized by finer grids and improved finite-differences, and it is to the development of a functioning model that the main effort has been directed.

2. Notation

\( u \)  \( x \)-component of velocity

\( v \)  \( y \)-component of velocity

\( \omega \)  \( \frac{Dp}{Dt} \)

\( w \)  vertical velocity \( \frac{Dh}{Dt} \)

\( \psi \)  \( gh' + \frac{1}{2}(u^2 + v^2) \)

\( \zeta \)  vertical component of absolute vorticity

\( h \)  height of isobaric surfaces

\( \rho \)  air density

\( l \)  total mass of liquid water per unit mass of air

\( l_c \)  mass of cloud water per unit mass of air

\( l_r \)  mass of rainwater per unit mass of air
\( g \)  
acceleration due to gravity

\( g_\ast \)  
\( g(1 + l) \)

\( \theta \)  
potential temperature (°K)

\( T \)  
temperature (°K)

\( Q \)  
sink of specific humidity due to condensation

\( F \)  
initiating heat source

\( F_x, F_y \)  
acceleration components associated with subgrid scale momentum transfer

\( c_p \)  
specific heat of air at constant pressure

\( L \)  
latent heat of condensation of water

\( q \)  
specific humidity

\( p \)  
pressure

\( q_{sat}(T) \)  
saturation specific humidity

\( e_{sat}(T) \)  
saturation vapour pressure

\( V_t \)  
terminal velocity of raindrops

\( \alpha, \beta, \gamma \)  
cloud physics parameters

\( K_p, K_\theta, K_q \)  
diffusion (smoothing) coefficients

Suffix \( _0 \) denotes surface (1000 mb) value

Suffix \( _s \) denotes initial (standard) value

A prime denotes deviation from standard

\( \Delta \) denotes an increment.

3. Mathematical formulation

(a) Basic equations

These are expressed in pressure co-ordinates. The horizontal momentum equations are taken as

\[
\frac{\partial u}{\partial t} = -\frac{\partial \psi}{\partial x} + v' - \omega \frac{\partial u}{\partial p} + F_x
\]

and

\[
\frac{\partial v}{\partial t} = -\frac{\partial \psi}{\partial y} - u' - \omega \frac{\partial v}{\partial p} + F_y
\]

where

\[
\psi = gh' + \frac{1}{2}(u^2 + v^2)
\]

The vertical momentum equation is written in the form

\[
\frac{1}{gg_\ast} \frac{D}{Dt} \left( \frac{\omega}{\rho_s} \right) + g_\ast \frac{\partial h}{\partial p} + 1 = 0
\]

where \( g_\ast = g(1 + l) \), \( l \) being the liquid water content; water drops are assumed to be falling at their terminal velocity. This equation is derived from the vertical momentum equation in \( z \) co-ordinates:

\[
\frac{Dw}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial z} + g_\ast = 0
\]

by rewriting it in the form

\[
-\rho \frac{\partial h}{\partial p} = \frac{1}{g_\ast} \left( 1 + \frac{1}{g_\ast} \frac{Dw}{Dt} \right)^{-1} \approx \frac{1}{g_\ast} \left( 1 - \frac{1}{g_\ast} \frac{Dw}{Dt} \right)
\]

and then replacing \( \frac{Dw}{Dt} \) by \( \frac{D}{Dt} \left( -\frac{\omega}{\rho_s} \right) \). The approximation involved in the binomial
expansion truncation is obviously well justified, since \( \frac{1}{g} \frac{\partial w}{\partial t} \ll 1 \), but the approximate replacement of \( \frac{\partial w}{\partial t} \) needs further justification; this is discussed by Miller (1974) and shown to be a valid approximation. The virtual temperature effect is assumed to be included in calculations of \( \rho \) and \( \rho_v \).

The mass continuity equation is

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0
\]

and the thermal energy equation is taken as

\[
\frac{\partial \theta}{\partial t} = - \frac{\partial}{\partial x} (u \theta) - \frac{\partial}{\partial y} (v \theta) - \frac{\partial}{\partial p} (\omega \theta) + \frac{\theta}{T} \left( \frac{LQ}{c_p} + F \right)
\]

where \( \frac{LQ}{c_p} \) defines the latent heat source and sink distribution and \( F \) defines a heat source used to initiate the convection. The equation expressing continuity of water vapour is

\[
\frac{\partial q}{\partial t} = - \frac{\partial}{\partial x} (uq) - \frac{\partial}{\partial y} (vq) - \frac{\partial}{\partial p} (\omega q) - Q
\]

The quantity \( Q \) in Eqs. (6) and (7) is calculated according to a procedure derived below (sub-section (c)). Eq. (4) may be used to derive an approximate form of thickness equation by subtracting from it the (hydrostatic) equation defining the undisturbed atmosphere:

\[
g \frac{\partial h}{\partial p} + 1 = 0
\]

and neglecting the acceleration term and terms including products of \( h' \), \( p' \) and \( l \). The resulting thickness equation is

\[
\frac{\partial h'}{\partial p} = \frac{\partial h_s}{\partial p} \left( \frac{\theta^*}{\theta_s^*} - 1 \right)
\]

where \( \theta^* \) is the potential temperature defined in terms of the virtual temperature \( T^* \).

The deviation height field is calculated in this non-hydrostatic system by solving the Poisson-type equation obtained by operating on Eq. (1) by \( \frac{\partial}{\partial x} \), Eq. (2) by \( \frac{\partial}{\partial y} \) and on Eq. (4) by \( \frac{\partial}{\partial p} \) (after first subtracting Eq. (8) and multiplying by \( g \rho_s \rho_v \)); the resulting three equations are added to give a diagnostic equation for \( h' \), essentially a form of ‘balance’ equation:

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) h' + g^2 \frac{\partial}{\partial p} \left( \rho_s \frac{\partial h'}{\partial p} \right) = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial p}
\]

where

\[
g G_1 = - \frac{1}{2} \frac{\partial}{\partial x} (u^2 + v^2) + v^2 \frac{\partial \omega}{\partial p} + F_x
\]

\[
g G_2 = - \frac{1}{2} \frac{\partial}{\partial y} (u^2 + v^2) - u^2 \frac{\partial \omega}{\partial p} + F_y
\]

and

\[
g G_3 = - u \frac{\partial \omega}{\partial x} - v \frac{\partial \omega}{\partial y} - \omega \rho_s \frac{\partial}{\partial p} \left( \frac{\omega}{\rho_s} \right) - g^2 \rho_s \theta^* \left( \frac{\theta^*}{\theta_s^*} - 1 \right)
\]
(b) Boundary conditions

The upper boundary of the model is taken at 100 mb, and a free surface condition adopted there, i.e.

$$\omega = 0 \text{ at } p = 100 \text{ mb} \quad . \quad . \quad . \quad (11)$$

This condition is used to calculate $\omega(p)$ from Eq. (5). In calculating $h'$ from Eq. (10), an upper boundary value of $\frac{\partial h'}{\partial p}$ is obtained from Eq. (9), using the value of $\theta'$ obtained from Eq. (6) (with $l = 0$).

The lower boundary condition used in the solution of Eq. (10) is that $h'$ for $p = 1000$ mb ($h_0$) is given by

$$\frac{\partial h_0'}{\partial t} = -u_0 \frac{\partial h_0'}{\partial x} - v_0 \frac{\partial h_0'}{\partial y} + \frac{\omega}{g \rho_0} \quad . \quad . \quad . \quad (12)$$

which results from the kinematic condition

$$w = 0 \text{ at } p = 1000 \text{ mb}. \quad . \quad . \quad .$$

The lateral boundary conditions were the subject of much study and experimentation, since no completely satisfactory theory exists for the correct specification of boundary conditions for the primitive equations on lateral boundaries through which fluid is assumed to flow. Charney (1955) proposed on the basis of an analysis of a simple one-dimensional system of continuous equations that the normal velocity component should be specified as data at all boundary points, and the potential vorticity at inflow points; at outflow points the potential vorticity should be calculated by continuity with the interior values. However, it was found that instabilities arose in regions of changeover from inflow to outflow when attempts were made to implement these conditions. The reason seems to be that in a numerical model, large gradients of vorticity arise in these regions of changeover, since there is a change in the numerical method used to calculate the vorticity; such instabilities are avoided if the same method of calculating the vorticity is used at all boundary points; as a numerical procedure it is necessary to adopt boundary conditions which lead to a stable numerical algorithm, even though these conditions may apparently contradict in some respects criteria based on the analysis of the continuous system of equations, the finite-difference analogues of which are used to define the numerical model.

The lateral boundary conditions adopted for the model described here are to specify the normal and tangential velocity components at all boundary points; heights of isobaric surfaces are calculated by solving the three-dimensional balance equation (Eq. (10)) for $h'$, using values of $\frac{\partial h'}{\partial n}$ on these boundaries from the horizontal momentum equations (Eqs. (1) and (2)) (horizontal advection terms being omitted in shear flow); temperature deviations on the boundaries are obtained from the thickness equation (Eq. (9)). Values of $q$ are specified on the boundary as data at inflow points, and by continuity with interior values at outflow points; since virtually no feedback into the dynamical equations takes place, any locally large gradients of $q$ do not lead to computational instabilities.

Clearly the most satisfactory procedure would be to choose lateral boundaries at a sufficient distance from the main disturbance centre (i.e. the cumulonimbus updraught) for the disturbance of the velocity field on the boundary to be zero for the period of convective activity from, say, the time of initiation of the disturbance until the cloud disperses. Both velocity components would then be kept constant at their initial values on the lateral boundaries throughout the integration. This, however, would strictly involve taking the boundaries at something like 1000 km from the centre, the distance travelled by external
gravity waves in an hour. Even the internal gravity oscillations may travel up to 100 km in this period. Thus, either a very large number of regularly-spaced gridpoints must be used, or it becomes necessary to use some kind of graded mesh or resort to some other device in order to be able to define a boundary condition of this kind for the problem. A regular grid length of 1 km was adopted for the model, which, with only 15 gridpoints in each horizontal direction, implies a lateral boundary at 7 km or so from the centre of the grid area. It would clearly impose too unrealistic a constraint on the system to assume a zero perturbation velocity throughout the integration on such a boundary (although in cases of zero initial shear such a condition was imposed for some experimental runs with the model – effectively simulating complete reflection of gravity waves at the lateral boundaries – without any computational difficulties arising).

The method adopted for dealing with this problem is to specify an outer boundary several grid lengths outside the boundary of the main computation area, on which it is assumed that the perturbation velocity is zero, and to calculate divergences adjacent to the grid boundary, assuming that the perturbation velocities on the grid boundary (Δv) are linear interpolants between the perturbation velocities at gridpoints just inside the boundary and the zero values on the implied outer boundary. Effectively this assumes a uniform value of ω in each isobaric surface on radial lines joining the gridpoints just inside the boundary and corresponding points on the implied outer boundary. First attempts to implement this principle resulted in the generation of spurious gravity waves which travelled round the boundary, and it was soon realized that these arose from the corners of the rectangular area, i.e. they were associated with the singular nature of the implied divergence in the corner regions. The method was then modified to eliminate the corner singularities by converting boundary velocity components to polar co-ordinate form, with origin at the centre of the grid area, and carrying out the interpolation process for Δv on a polar co-ordinate net (it is not strictly necessary to calculate Δv values explicitly as long as the boundary divergences are obtained).

The horizontal divergence D for a boundary grid square was computed in the following manner. Referring to Fig. 1, radial lines are drawn from the centre of the grid region, bisecting the sides (EF, GH, etc.) of the boundary grid squares. Increments of velocity components Δw and Δθ parallel and perpendicular to these radial lines are computed at points (a), (d), etc., by simple interpolation from the gridpoint increments. Radial and

![Figure 1. Grid geometry for boundary divergence calculations.](image-url)
tangential parts of the divergence increments ($\Delta D_r$ and $\Delta D_\theta$) are then computed for the trapezium ABCD as follows:

Define $\overline{\Delta v_r} = \frac{1}{2}[\Delta v_r(a) + \Delta v_r(d)]$ as an approximation to the mean radial incremental velocity component across AD, and $\sigma \overline{\Delta v_r}$ as that across BC; then

$$\Delta D_r = \frac{1}{r} \frac{\partial}{\partial r} (r \Delta v_r) \approx (\sigma - 1) \frac{\Delta v_r}{\Delta r_i} + \frac{(\sigma + 1)}{2} \frac{\Delta v_r}{r_i}$$

(13)

where $\Delta r_i = \frac{\text{area of trapezium}}{(r \Delta \theta)_i}$ and the value of the suffix $i$ ($i = 1, 2, \ldots, N$) identifies the particular trapezium.

Similarly approximate

$$\overline{\Delta v_\theta(ab)} = \frac{(1 + \sigma')}{2} \Delta v_\theta(a)$$

and

$$\overline{\Delta v_\theta(cd)} = \frac{(1 + \sigma')}{2} \Delta v_\theta(d)$$

as the mean tangential incremental velocity components across AB and CD respectively; then

$$\Delta D_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} (r \Delta v_\theta) \approx \frac{[\Delta v_\theta(ab) - \Delta v_\theta(cd)]}{r \Delta \theta}$$

(14)

where $r \Delta \theta = \Sigma (r \Delta \theta)_i / N$. The use in Eq. (14) of $r \Delta \theta$ and not $(r \Delta \theta)_i$ ensures that $\Delta D_\theta$ summed round the boundary is identically zero. The use of $\sigma$ and $\sigma'$ as the interpolation factors between the points just inside the boundary and the boundary points for the implied outer boundary; thus $\sigma' = 0$ models a zero tangential perturbation boundary component, and $\sigma$ ($0 \leq \sigma \leq 1$) essentially models the distribution of $\Delta D$ between the inner and implied outer boundary. The computed divergences $\Delta D_i = \Delta D_{ri} + \Delta D_{\theta i}$ are then assumed to represent the mean value for the boundary grid square of the Cartesian grid. To complete the boundary computation, $\Delta v_r$ on the boundary is computed at the gridpoints F, H, etc., by linear interpolation from the radial velocity values assumed at points (b), (c), etc., i.e. approximated as $\sigma \Delta v_r(a)$, $\sigma \Delta v_r(d)$, etc.

(c) Condensation, evaporation and cloud physics formulation

(i) Condensation and evaporation calculations. $Q$ in Eqs. (6) and (7) is calculated by an adjustment procedure. Increments $\Delta \theta_1$ (and hence $\Delta T_1$) and $\Delta q_1$ are first obtained using the equations:

$$\frac{D \theta}{D t} = 0$$

and

$$\frac{D q}{D t} = 0$$

i.e. assuming no mixing by subgrid scale processes. No explicit formulation of subgrid scale mixing is in fact included other than that of the 'smoothing' terms (see Section 4).

If $q + \Delta q_1 < q_{sat}(T + \Delta T_1)$ and no cloud water is present, no further adjustment is made, i.e.

$$Q = 0$$

and

$$\Delta T = \Delta T_1, \quad \Delta q = \Delta q_1.$$
If \( q + \Delta q_1 > q_{sat}(T + \Delta T_1) \), \( \Delta T \) is obtained from the truncated Taylor expansion

\[
q - \left\{ q_{sat}(T) + \Delta T q'_{sat}(T) + \frac{(\Delta T)^2}{2} q''_{sat}(T) + \ldots \right\} = \frac{c_p \Delta T}{L}
\]

where \( \frac{\partial}{\partial T} \), \( q_{sat}(T) = 3.8 \times 10^{-1} \left( \frac{aT}{T+0.5} \right) \) and \( q'_{sat} = \frac{q_{sat}L_v}{R_vT^2} \);

and for \( T \) (in degrees C) \( \geq 0 \); \( a = 7.5 \), \( b = 237.5 \); for \( T < 0 \); \( a = 9.5 \), \( b = 265.5 \). In this case \( \Delta q \) is taken as \( q_{sat}(T + \Delta T) - q \). A similar procedure is carried out for the evaporation of cloud water.

(ii) Cloud physics. The cloud physics parameterization follows closely a scheme used by Liu and Orville (1969) based on the ideas of Kessler (1969) and Srivastava (1967), using the equations

\[
\frac{D}{Dt} (q + l_c) = -\text{PROD} \quad \ldots \quad \ldots \quad (15)
\]

and

\[
\frac{D}{Dt} l_r = \text{PROD} - g \frac{\partial}{\partial p} (\rho l_r V_t) \quad \ldots \quad \ldots \quad (16)
\]

where \( l_c \) denotes cloudwater and \( l_r \) rainwater in \( \text{gm kg}^{-1} \); \( V_t(\text{m s}^{-1}) \) is the mean terminal velocity of rainwater drops relative to the air, parameterized as \( V_t = 5.32 l_r^{0.5} \).

The rainwater production term \( \text{PROD} \) is written as

\[
\text{PROD} = P_1 + P_2 + P_3
\]

with \( P_1 \) (evaporation of raindrops) taken as

\[
P_1 = \beta (q - q_{sat}),
\]

\( \beta \) a parameter, typically of order \( 10^{-3} \text{ s}^{-1} \), \( P_2 \) (accretion of \( l_c \) by \( l_r \)) taken as

\[
P_2 = \gamma l_c l_r^{0.95},
\]

\( \gamma \) a parameter, typically of order \( 10^{-2} \) to \( 10^{-3} \text{ s}^{-1} \) (the larger the value of \( \gamma \) the larger the rate at which the cloud rains out), and \( P_3 \) (conversion of \( l_c \) to \( l_r \)) taken as

\[
P_3 = \alpha (l_c - l_{crit}),
\]

\( \alpha \) being a parameter typically of order \( 10^{-3} \text{ s}^{-1} \) and \( l_{crit} \) a cloud water value below which there is no auto-conversion. Thus until \( l_c \) exceeds \( l_{crit} \), no rainwater is produced and thereafter the rate of production is defined largely by the value of \( \gamma \). The coalescence and ice-accretion processes are not each parameterized explicitly in this formulation.

(d) The finite-difference equations

The finite-difference representations of the above set of continuous equations are given in the Appendix. These are related to the staggered grid shown in Fig. 2. The scheme is a modification of one originally developed for a hydrostatic system of equations used to model synoptic-scale systems using a staggered grid. It does not completely conserve energy or mean square vorticity. The main departures from energy conservation are of the order of magnitude of the advection of kinetic energy. The scheme used at present is quite economical and efficient, but is not regarded as in any sense an optimum one for this scale.
of modelling, and it would clearly be desirable eventually to use a scheme which at least conserves energy.

4. RESULTS OF COMPUTATIONS

The results of 3 runs are presented (Figs. 5 to 9). In Runs 1 and 2 there is no ambient wind; the only difference is that the accretion parameter $\gamma$ is larger in Run 2 than in Run 1 by a factor of 10; in Run 3 the initial flow is assumed to be sheared. The initial temperature
Figure 4. Initial u-component of windfield in Run 3.

and dew-point distributions are shown in Fig. 3, typical of a tropical atmosphere. The convection was initiated by raising the specific humidity 1 gm kg\(^{-1}\) above the undisturbed value over four adjacent gridpoints at 850 mb at the centre of the grid area, and, at the same time, applying heating at the rate of 1.2 deg C min\(^{-1}\) for 3 min at the same gridpoints; this produced a disturbance of an amplitude just sufficient to start and maintain the release of latent heat. In all the runs the cloud physics parameter \(l_{crt}\) is taken as \(5 \times 10^{-4}\).

(a) Run 1. Figs. 5 and 6

(No ambient wind; Coriolis parameter zero; cloud physics parameters:
\[ \alpha = 10^{-3} \text{s}^{-1}; \quad \beta = 10^{-3} \text{s}^{-1}; \quad \gamma = 10^{-3} \text{s}^{-1}. \])

Cloud first forms at about 7\(\frac{1}{2}\) min when the maximum updraught speed is 5 m s\(^{-1}\). These early development stages are characterized by low pressure (negative deviation height field) under the updraught (inflow) and high pressure (positive deviation height field) at outflow. A gravity oscillation in the vertical is also present. Rainwater first forms at 775 mb at 15 min (Fig. 5) with cloud water values of 4 gm kg\(^{-1}\); at about this time the largest vertical velocities and temperature excesses are found, 10 m s\(^{-1}\) at 700 mb and +1.9 deg C at 600 mb. From 15 min very rapid changes occur under the dominating influence of the liquid water load and rainwater evaporation; the fields at 20 min (Fig. 5) show this effect clearly with evaporative cooling of the air coming into the updraught, and the reversal of the vertical deviation pressure gradient. Inflow previously at both the 1000 and 850 mb levels soon ceases at 1000 mb with inflow largest at 700 mb thereafter. By 20 min only the top of the cloud contains air rising at more than 2 m s\(^{-1}\), being free of significant water loading; the cloud is at its maximum size at this time. The period 20–25 min sees the development of a deep downdraught with heavy rainfall down to the surface. This downdraught extends from 500 mb to the surface with, at 22.5 min, as much as 2 deg C temperature deficit through the lowest 200 mb. A maximum height deviation at 100 mb of 8 m (\(\approx 0.8\) mb) is produced with the low-level outflow attaining 10 m s\(^{-1}\). Most of the downdraught air originates from the 500–700 mb layer, where there is a pressure minimum and the minimum \(\theta_w\) of the sounding, a result consistent with the concept of downdraught air having middle-level origins (Ludlam 1963). By 30 min (Fig. 6), the downdraught has reduced to 2 m s\(^{-1}\) with cloud water at upper levels only (\(\approx 0.5\) gm kg\(^{-1}\)).
Figure 5. Computed fields for Run 1 (vertical section through axis of cloud). Solid lines denote positive values, dotted lines denote negative values (magnitude only indicated). Units are: $w$ in m s$^{-1}$, $h'$ in m, $\theta$ in $^\circ$K, $l_c$ and $l_r$ in gm kg$^{-1}$. Horizontal intervals are 1 km, vertical intervals are 100 mb (surface at 1000 mb).
Figure 6. Computed fields for Run 1 (continued from Fig. 5).
below this cloud the warming of air due to descent now exceeds any evaporative cooling as the cloud has rained out, and positive potential temperature deviations are apparent except at the lowest level. About 5 mm of rainfall occurs in the period 18–30 min.

The results of Run 1 clearly show the dominating control of the cloud physics, particularly in the later stages; in view of this, closer analysis of the relationship in time and space of the cloud and rainwater concentrations was made. Although both liquid water variables contribute to the dynamics through the drag term in the equations, the cloud water is assumed to advect with the flow only, while the rainwater has a terminal velocity component as well as flow advection. The changing proportions of both liquid variables are therefore important to the overall liquid distribution in the cloud. Run 1 showed the onset of rain to be delayed with the transformation of cloud to rain, allowing values of cloud water in excess of 2 gm kg\(^{-1}\) to exist until after 23 min with a maximum value of 5 gm kg\(^{-1}\) in the earlier stages. This has the effect of spreading the water load over a deep layer eventually resulting in the deep downdraught.

The dominating coalescence process was therefore accelerated (by an exaggerated amount) in Run 2. The equivalent parameterization used by Kessler (1969) suggested a larger value (for deep convection) of the accretion parameter \(\gamma\).

As far as the evaporation parameter (\(\beta\)) was concerned, the modelled surface temperature deficits would appear to be reasonable, although less evaporative cooling would reduce the large downdraught speeds.

The values of \(\alpha, \beta\) and \(\gamma\) chosen for these experiments are no more than crude estimates of realistic atmospheric values. Further refinement of the cloud physics parameters was felt to be unjustified with a model of such low spatial resolution.

(b) Run 2. Fig. 7

(As Run 1 but with the accretion parameter \(\gamma = 10^{-2}\) s\(^{-1}\).) Until rain forms, there are of course no differences from the results of Run 1; results are therefore presented from 20 min. The differences from this point are quite substantial; as expected, much larger rainfall concentrations are produced at low levels with cloud water values \(\sim 1\) gm kg\(^{-1}\); buoyancy as ascent is maintained above 700 mb up to 30 min. The development of a downdraught is again marked, but limited to the lowest 200 mb; although containing a slightly larger temperature deficit (\(\sim 2.5\) deg C) the downdraught is somewhat smaller in magnitude, principally through its smaller vertical extent. In both Runs 1 and 2, a ring of 'cold' air spreads out at the lowest levels. The rainfall duration is considerably longer in Run 2, resulting in three times as much total rainfall as in Run 1.

(c) Run 3. Figs. 8 and 9

(Ambient wind (\(u\)-component) as in Fig. 4; Coriolis parameter zero; cloud physics parameters: \(\alpha = 10^{-3}\) s\(^{-1}\); \(\beta = 10^{-3}\) s\(^{-1}\); \(\gamma = 2 \times 10^{-3}\) s\(^{-1}\).) All figures, except one in Fig. 9, represent fields through the line of symmetry along the mean wind.

Some difficulties were encountered at the first attempts to include environmental shear, mainly in the early growth stages of the convection. The initial 'forced' cumulus tendency to grow only very slowly or to dissipate under the influence of the shear and smoothing terms in the equations. A further problem was the propagation and advection of the cloud across the model grid resulting (after some time) in convection near the lateral boundaries and consequent computational problems. The shear configuration was chosen with a view to minimizing this problem – with some success – although this run still had to be terminated before the cloud had completed its life-cycle. In view of these difficulties in the early stages,
Figure 7. Computed fields for Run 2 (as Fig. 5).
the first 20 min are treated as an initialization period, and the main interest focused on later times. It would be possible to compress this first twenty minutes period into a shorter time, making a total simulation time given in the diagrams an overestimate.

As previously mentioned, the developing convection was found to be sensitive to the magnitudes of the smoothing terms; the following smoothing coefficients were used in the run reproduced here: \( K_v = 500 \text{ m}^2 \text{s}^{-1} \), \( T > 5 \text{ min} \): i.e. the velocity smoothing has the form \( K_v v^2 \); \( K_\theta = K_\theta = 0 \) for \( T < 25 \text{ min} \) and, for \( T > 25 \text{ min} \), a maximum effective \( K_\theta \) and \( K_v \approx 500 \text{ m}^2 \text{s}^{-1} \) where \( K_\theta \) and \( K_v \) are defined from a \( K' |V|^2 |V|^2 \) formulation, i.e. the actual smoothing is expressed as \( K_\theta V^2 \theta \) or \( K_\theta V^2 q \) where \( K_\theta = K' |V|^2 \theta \) and \( K_v = K' |V|^2 q \).

The initialization stages (\( T < 20 \text{ min} \)) produce a small cloud with updraught speeds \( 3 \text{ m s}^{-1} \), and cloud water values \( 2 \text{ g m}^{-1} \). This cloud extends to 500 mb by 30 min and a circulation is apparent with the ascending air forming cloud and warming through latent heat release, while on the left-hand side of the cloud, ‘cold’ air descends. This descending branch is part of the wave-like form of the convective system and is strengthened by continuous evaporation of cloud water on the left-hand side with temperature deviations \( \pm 2 \text{ deg C} \) in the two branches.

This convective system exhibits only slow changes between 27\( \frac{1}{2} \) min and 40 min while moving in the direction of the low level wind with a well-defined speed of about \( 3 \text{ m s}^{-1} \) corresponding to the 850 mb level environment wind speed. The slow changes that occur during this period are mostly related to the formation of rainwater. Rain first forms at 32 min and reaches the ground at 37\( \frac{1}{2} \) min, extending the chilled region down to the surface. By 42\( \frac{1}{2} \) min the downdraught branch of the circulation is strengthening, and ‘cold’ air spreads out at low levels. This divergence of air could have some important implications which are discussed below. The cloud leans down the shear in the later stages. Beyond 45 min the proximity of the lateral boundary spoiled the computation. The structure of the convection at right angles to the mean flow (Fig. 9) has a symmetric form very similar to that of the unsheared cases discussed previously.

The nature of this convective system and its observed quasi-steady behaviour poses several questions, particularly about the large cloud-water evaporation. If the model retains too much water in cloud water form, the evaporative cooling is exaggerated, for cloud water is assumed to evaporate during a timestep (3 sec) until either saturation is reached or the cloud water has disappeared. The simulated values of 1–3 g m\(^{-1}\) do not however appear unduly higher than those observed (Mason 1962). Excessive explicit mixing of cloud and clear air might also exaggerate the evaporation of cloud water, but this is not the case here, where smoothing terms have been kept as small as possible.

The convective system modelled in Run 3 represents one ‘mode’ in which a cumulonimbus can exist for a given initial environment. The development of a downdraught of cold air spreading out at low level produced regions of low-level convergence; these regions do not, however, reinforce existing updraughts. The system therefore does not have a ‘long’ lifetime although attaining a quasi-steady state before the rainwater evaporation becomes dominant. For a long-lasting convective system with the ‘steady-overturning’ character observed in severe storm studies, there exists a requirement for the downdraught to stimulate and strengthen convective development on the downshear side (Ludlam 1963).

To summarize, all runs are characterized by an initial development of a saturated updraught surrounded by induced gravity oscillations. The updraught extends into the upper troposphere and then, with the onset of rain, rapidly transforms into an evaporative downdraught which spreads out at the surface. This downdraught is shallower in Run 2 where the rain is assumed to be produced from the cloud droplets more rapidly than in Run 1. With initial ambient shear, development is somewhat slower than in Runs 1 and 2, and is characterized by a period during which the cloud exists in the upward moving branch.
Figure 8. Computed fields for Run 3 (as Fig. 5). (Vertical section through axis of cloud along the ambient wind.)
Figure 9. Computed fields for Run 3. (Left-hand side figures are a vertical cross-section through A as indicated on Fig. 8; right-hand side figures as in Fig. 8.)
of a propagating mid-tropospheric circulation. Eventually, however, sufficient rainwater accumulates to produce a strong evaporative downdraught.

5. SUMMARY AND CONCLUDING REMARKS

A numerical model capable of simulating deep convection has been described, together with examples of simulations in an unsheared and sheared environment. Some limitations of earlier models have been removed, thus realizing a more flexible and adaptable research tool for future work.

The main advantages gained are as follows:
(i) full three-dimensionality in space
(ii) accurate computations of the thermodynamics of the vapour-liquid transitions
(iii) open lateral boundaries and
(iv) the inclusion of the spatial variations of density.

The model also extends to the 100 mb level, thus virtually eliminating upper boundary effects inherent in some models. Inevitably, these improvements and extensions have been gained at the expense of other factors although in general without introducing any serious disadvantages.

The extension to three dimensions in space made heavy demands on computing facilities and programming, and, when combined with open lateral boundaries, introduced boundary condition formulation problems. This aspect of the model proved to be a difficult problem and one which has not yet been satisfactorily or completely solved; further work is needed on a consistent set of dynamical boundary conditions for fluid flowing across a gridpoint model boundary. The improved thermodynamic computations and the inclusion of the density variations are directly the result of using pressure as the vertical co-ordinate. These advantages were gained by reformulating the modelling equations in pressure co-ordinates.

Despite the coarse resolution and relative crudity of the parameterization of liquid water, the sensitivity of the simulated convection to this parameterization is an encouraging feature – for the substantial role of the liquid water phase in the cloud’s life-cycle, while showing the importance of the parameterization, emphasizes the essential response of the model to physical influence. The rapid changes that occur at certain times in the model-cloud’s history are governed primarily by the assumed cloud physical processes. The parameterization deficiencies have been referred to previously, but it is clear that an improved handling of the spectrum of raindrop fall-speeds is desirable.

No diagnostics have been presented here, although preliminary computations have been conducted. The demands on computer time and storage of the present model were such as to make it impracticable at the time. A diagnostic program is now being developed, and it is intended later to present further results with, in particular, full energy and moisture budget details.

More recent experiments have been conducted using models with improved horizontal resolution and equal pressure intervals of 50 or 100 mb. These runs have in general supported the results shown here while indicating certain deficiencies also. Further experiments are in progress.

It is proposed to use the model in the study of the basic dynamics of convection. A series of experiments can be carried out in which the characteristics of the convection are studied in relation to particular initial atmospheric conditions. The convective systems discussed in Section 4 provide examples of the types of flow studied as a dynamical problem by Moncrieff and Green (1972), and support the ideas developed there; these ideas in turn
provide a basis for further modelling investigations particularly with regard to the transfers of momentum, heat and moisture.

The modelling of storms, using initial windfields and soundings similar to those occasions for which descriptive models have been developed from detailed observation (e.g. Browning and Ludlam 1962; Browning 1965), is a practical possibility with a larger version of the model. It should be possible to construct three-dimensional flow patterns and compare with those of descriptive models. Such studies could well prove useful in the context of the convective parameterization problem by enabling vertical heat and moisture transfers associated with the life-cycles of large convective storms to be calculated under various conditions of shear and stability.

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APPENDIX: THE FINITE-DIFFERENCE EQUATIONS

(a) Time-differencing scheme

This consists of alternate timesteps, the first with the calculation of temporary values of $\phi$ (where $\phi$ may be $u$, $v$, $\theta$, $q$, $l^c$, $l$, or $h_0$) and the second without.

Odd timestep:

$$\phi^*(t + \Delta t) = \phi(t) + \Delta t \frac{\partial \phi(t)}{\partial t},$$

$$\phi(t + \Delta t) = \phi(t) + \Delta t \frac{\partial \phi^*}{\partial t} (t + \Delta t) = \phi(t) + \Delta \phi(t).$$

Even timestep:

$$\phi(t + 2\Delta t) = \phi(t + \Delta t) + \Delta t \frac{\partial \phi}{\partial t} (t + \Delta t) = \phi(t + \Delta t) + \Delta \phi(t + \Delta t).$$

Starting with the fields at time $t$, first the $\Delta u$ and $\Delta v$ fields are computed, and then $\omega(t + \Delta t)$ by integration of the continuity equation; it is this value of $\omega$ rather than $\omega(t)$ which is used, together with $u(t + \Delta t)$ and $v(t + \Delta t)$, to form $h_0(t + \Delta t)$ and $\theta(t + \Delta t)$. At the next timestep $\omega(t + 2\Delta t)$ is used, together with $u(t + 2\Delta t)$ and $v(t + 2\Delta t)$ to form $h_0(t + 2\Delta t)$ and $\theta(t + 2\Delta t)$.

This scheme is stable for the linearized system when $\Delta t \leq \Delta x/c$ where $c$ is the speed of external gravity waves. Stability of the meteorological wave for the non-linear system is ensured by taking an occasional (approx. 1 every $10^4$) odd timestep formulation at the even timestep.

(b) Space-differencing scheme

Using the notation $\phi$ (i.e. without a suffix) to denote $\phi(lx, my, p_k)$ and $\phi_k$ to denote $\phi((l + \frac{1}{2})x, (m + \frac{1}{2})y, p_{k+1})$ the pressure levels $p_k$ (1000, 850, 700, 500, 300, 200 and 150 mb) corresponding respectively to $k = 7, 6, \ldots, 1$, the space derivatives in Eqs. (1), (2), (6) and (7) are replaced to give
\[
\frac{\partial u}{\partial t} = -\delta_x \bar{v}^y + v_x^{sy} - \bar{\omega}^{sy} \delta_x \rho
\]  
(A1)

\[
\frac{\partial \bar{v}}{\partial t} = -\delta_y \bar{v}^x - u_x^{sy} - \bar{\omega}^{sy} \delta_y \rho
\]  
(A2)

\[
\frac{\partial \bar{\theta}^*}{\partial t} = -\delta_x (\bar{u}_*^{xy} \bar{\theta}_*^{xy}) - \delta_y (\bar{v}_*^{xy} \bar{\theta}_*^{xy}) - \delta_x (\omega \bar{\theta}_*^{xy}) + Q^*_x
\]  
(A3)

\[
\frac{\partial \bar{q}_*}{\partial t} = -\delta_x (\bar{u}_*^{xy} \bar{q}_*^{xy}) - \delta_y (\bar{v}_*^{xy} \bar{q}_*^{xy}) - \delta_x (\omega \bar{q}_*^{xy}) + Q''_x
\]  
(A4)

where \( Q^*_x \) and \( Q''_x \), the finite difference analogues of the last terms in Eqs. (6) and (7) respectively, are computed as described in Section 3.

and

\[
\psi = gh \{ (l + \frac{1}{2}) \Delta x, (m + \frac{1}{2}) \Delta y, p_k \} + \frac{1}{2} ((\bar{u}^y)^2 + (\bar{v}^x)^2)
\]  
(A5)

where

\[
\bar{v}^x = \frac{1}{2} [\phi \{(l + \frac{1}{2}) \Delta x, m \Delta y, p_k \} + \phi \{(l - \frac{1}{2}) \Delta x, m \Delta y, p_k \}]
\]

\[
\bar{v}^y = \frac{1}{2} [\phi \{(l \Delta x, (m + \frac{1}{2}) \Delta y, p_k \} + \phi \{(l \Delta x, (m - \frac{1}{2}) \Delta y, p_k \}]
\]

\[
\delta_x \phi = \frac{1}{2} [\phi \{(l + \frac{1}{2}) \Delta x, m \Delta y, p_k \} - \phi \{(l - \frac{1}{2}) \Delta x, m \Delta y, p_k \}]
\]

\[
\delta_y \phi = \frac{1}{2} [\phi \{(l \Delta x, (m + \frac{1}{2}) \Delta y, p_k \} - \phi \{(l \Delta x, (m - \frac{1}{2}) \Delta y, p_k \}]
\]

and \( \delta_x \phi \) and \( \delta_y \phi \) in Eqs. (A1) and (A2) is a quadratic interpolant through values at three adjacent levels. The expression \( \delta_x \phi \) and \( \delta_y \phi \) in Eqs. (A3) and (A4) is:

\[
[\phi \{(l + \frac{1}{2}) \Delta x, (m + \frac{1}{2}) \Delta y, p_{k+1} \} - \phi \{(l + \frac{1}{2}) \Delta x, (m + \frac{1}{2}) \Delta y, p_k \}]/(p_{k+1} - p_k)
\]

and the value of \( \zeta \) from which \( \bar{\zeta}^{xy} \) values are formed is:

\[
\delta_x \bar{v}^y - \delta_y \bar{u}^x.
\]

The finite-difference analogue for Eq. (5) integrated w.r.t. pressure is given by the trapezium rule:

\[
\omega(p_k) = \sum_{i=1}^{k} \frac{1}{2} \{D(p_{i-1}) + D(p_i)\}(p_i - p_{i-1})
\]  
(A6)

where

\[
D = \delta_x \bar{u}^y + \delta_y \bar{v}^x.
\]

The space derivatives in Eq. (12) are replaced to give (at 1000 mb)

\[
\frac{\partial \bar{h}'}{\partial t} = -\bar{u}^y \delta_x \bar{h}^x - \bar{v}^x \delta_y \bar{h}^y + \omega \frac{\omega}{g \rho_s}
\]  
(A7)

The finite-difference formulation of Eq. (10) (the balance equation) is consistent with that of its constituent equations. The advection equations for rainwater and cloud water are treated by a method analogous to that used in Eq. (A4).