CORRESPONDENCE AND NOTES

THE ROLE OF POTENTIAL VORTICITY IN SYMMETRIC STABILITY AND INSTABILITY

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1. SUMMARY

In the zonal flow of an inviscid, adiabatic atmosphere, instability to symmetric disturbances is possible only if the potential vorticity is negative. In the absence of diabatic or frictional effects, the latter is a conserved property. Thus these effects are necessary for the generation of symmetric instability in a previously stable atmosphere. In a stable situation the 'conserved' quantity potential vorticity is the product of the squares of the maximum and minimum frequencies of symmetric disturbances.

The purpose of this note is to draw attention to some simple properties of the stability of a baroclinic circular vortex to symmetric disturbances (Eliassen and Kleinschmidt 1957; Ooyama 1966), and to point out the relevance of these properties to discussions of symmetric circulations in fronts and jet streams. Some of the properties the author has not been stated explicitly in previous work and one of the results (Eq. (3) below) is believed to be new.

For simplicity we neglect curvature effects and consider the stability of a flow \((\phi, V(x, z), \theta)\), potential temperature \(\theta(x, z)\) on an f plane to motions independent of \(y\). We also make the Boussinesq approximation and suppose that the flow is in geostrophic balance:

\[ f \partial V/\partial z = (g/\theta_0) \partial \theta/\partial x, \]

\(\theta_0\) being a constant reference potential temperature. In the absence of \(y\) dependence, continuity implies that the perturbation velocity may be written

\[(u, v, w) = \left(-\frac{\partial \psi}{\partial z}, v, \frac{\partial \psi}{\partial x}\right).\]

Linearizing the inviscid, adiabatic equations and eliminating \(v\), the perturbation potential temperature, and the perturbation pressure gives (see e.g. Ooyama 1966):

\[
\frac{\partial^2}{\partial t^2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \psi + \left( N^2 \frac{\partial^2}{\partial x^2} - 2s^2 \frac{\partial^2}{\partial x \partial z} + f^2 \frac{\partial^2}{\partial z^2} \right) \psi = 0
\]

Here \(N^2 = (g/\theta_0) \partial \theta/\partial z\) (positive for convective stability)
\(s^2 = (g/\theta_0) \partial \theta/\partial x = f\partial V/\partial z\) (positive for \(x\) pointing towards warmer air)
\(\zeta = f + \partial V/\partial x\) (the vertical component of absolute vorticity)

have all been considered constant.

In an unbounded domain we may seek solutions

\[ \psi \propto e^{i\sigma t} e^{i(k \sin \phi + z \cos \phi)}, \]

where \(\phi\) is the disturbance orientation measured from the horizontal.

Then \(\sigma^2 = N^2 \sin^2 \phi - 2s^2 \sin \phi \cos \phi + f^2 \cos^2 \phi \)

\[= \cos^2 \phi (N^2 \tau^2 - 2s^2 \tau + f^2) \quad (\tau = \tan \phi) \quad . . . \quad . \quad . \quad . \quad (1)\]

(i) Unstable: Let \(q = f^2 N^2 - s^4\). Since \(q\) is the discriminant of the quadratic expression in Eq. (1), there is stability (\(\sigma^2\) positive for all \(\phi\)) if \(q\) is positive. However, if \(q\) is negative, there exist orientations \(\phi\) for which the motion is unstable (\(\sigma^2\) negative). Since the absolute vorticity on an isentropic surface is

\[\zeta_\theta = \zeta - \frac{\partial V \partial \theta}{\partial z} \frac{\partial \theta}{\partial z} = q(fN^2), \quad . . . \quad . \quad . \quad (2)\]

the instability criterion is the usual \(\zeta_\theta\) negative (if \(f\) and \(N^2\) are positive).

A further interpretation of the instability criterion is given by noting that

\[q = \frac{g f}{\theta_0} \left[ \left( f + \frac{\partial V}{\partial x} \right) \frac{\partial \theta}{\partial z} - \frac{\partial V \partial \theta}{\partial z \partial x} \right]\]

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is proportional to the potential vorticity of the basic flow (Charney 1973). It is, therefore, a conserved quantity in the absence of friction and diabatic effects.

In a convectively stable state of no motion,

\[ q = \frac{gf^2 \, d\theta}{\theta_0 \, dz} \]

which is positive. This means that the unstable basic zonal flow considered above cannot be generated from this state of no motion by any frictionless, adiabatic motion whatever.

**Thus, frictional and heating effects are needed to generate instability to symmetric motions in a previously stable atmosphere.**

(ii) **Stable.** In the stable situation, as shown by Ooyama (1966),

\[ 2\sigma^2 = N^2 + f^2 \zeta - A \cos 2(\phi - \phi_i) \]

where

\[ A = \frac{1}{2} \left[ (N^2 + f^2 \zeta)^2 + 4s^4 \right]^{1/2}, \]

\[ \sin 2\phi_i = 2s^2/A, \cos 2\phi_i = (N^2 - f^2 \zeta)/A. \]

Usually, in the atmosphere, we have

\[ f^2 \zeta \ll s^2 \ll N^2. \]

Then

\[ \sigma_{\text{min}}^2 = \frac{1}{2} (N^2 + f^2 \zeta - A) \simeq f^2 \zeta. \]

This corresponds to inertial oscillations approximately along isentropic surfaces. Also,

\[ \sigma_{\text{max}}^2 = \frac{1}{2} (N^2 + f^2 \zeta + A) \simeq N^2, \]

corresponding to gravitational oscillations in the vertical. From the above equations it is easily verified that an exact relation is

\[ q = \sigma_{\text{max}}^2 \sigma_{\text{min}}^2. \]

Thus, in a quantitative way, potential vorticity is a measure of the 'stiffness' of the baroclinic fluid with respect to symmetrical disturbances. Any inviscid, adiabatic rearrangement of the unidirectional flow must preserve the product of the minimum and maximum frequencies for symmetric disturbances.

(iii) **Relevance to fronts and jet streams.** Atmospheric fronts and their associated jet streams can be formed via inviscid, adiabatic motions as has been shown theoretically by Hoskins and Bretherton (1972). During such a frontogenetic process, the static stability increases in the front, so that \( \sigma_{\text{max}} \) increases there. Then if potential vorticity is conserved Eq. (3) shows that \( \sigma_{\text{min}} \) decreases. That is, the stability to vertical motions is increased but that to motion along isentropes is decreased. At this point it would be possible that turbulent diffusion could destabilize the front either by permitting \( q \) to change sign, or by relaxing the inertial or gravitational constraint (McIntyre 1970). However, it is more likely that symmetric instability would be triggered by the effective decrease in static stability due to latent heat release.

If we introduce a Richardson number

\[ Ri = \frac{g}{\theta_0} \left. \frac{\partial \theta}{\partial z} \left( \frac{\partial V}{\partial z} \right)^2 \right|, \]

from Eq. (2),

\[ \zeta_0 = \zeta - f/Ri. \]

Thus the condition for symmetric instability is

\[ Ri < f/\zeta. \]

For flows with no horizontal shear this becomes \( Ri < 1 \) (Stone 1966). However in frontal regions, the Richardson number can be less then unity without symmetric instability being possible.

It is possible that the rain bands observed by Browning et al. (1973) may be a manifestation of symmetric instability triggered by latent heat release. It is hoped to study this possibility in the future.

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