A comparison of grid-point and spectral methods in a meteorological problem

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SUMMARY

Integrations of the shallow water equations on the sphere using a second order finite difference method and a spectral method are compared. By increasing the resolution, a good estimate of the exact solution may be made, thus allowing an estimate of the accuracy of each integration. The particular initial conditions used are a Rossby wave of zonal wavenumber 4 which moves with little change in shape or amplitude and a Rossby wave of zonal wavenumber 8 which undergoes large changes within 5 days. All the models perform reasonably for the wavenumber 4 integration. A $5^\circ \times 3^\circ$ grid for the finite difference simulation is insufficient to resolve the breakdown in wavenumber 8 despite there being 9 points per zonal wavelength. A spectral model with a truncation at wavenumber 16 uses less storage and takes the same computing time as the grid-point model. However, it is able to predict quite accurately the breakdown.

1. INTRODUCTION

The standard numerical method for integrating the meteorological equations has been to represent the dependent variables by their values at specified points. Derivatives are then determined from these grid-point values. In the spectral method, on the other hand, the functions of interest are represented by truncated expansions in terms of suitable base functions, as in the usual Fourier analysis. Exact derivatives of these may be obtained directly. Historically, this method has not been competitive with the finite difference method, but with developments in technique (Orszag 1970; Eliassen, Machenhauer and Rasmussen 1970) the superiority of the latter became less clear. To examine the relative accuracy of the two methods, Orszag (1971) studied numerically the two dimensional convection of a passive scalar with a conical distribution by a uniform rotation, and also a three-dimensional vortex decay problem. From these experiments, and some analysis, he concluded that, for given accuracy, 4th and 2nd order finite difference schemes require respectively at least $3$ and $5$ times the degrees of freedom for each dimension required by the spectral method. Following such work, it is of interest to extend the comparative study of spectral and grid-point methods to situations of more direct meteorological significance.

To this end, in this paper we consider integrations of the shallow water equations on the sphere. These equations describe the motion of a homogeneous, incompressible, inviscid fluid with a free surface and are sometimes referred to as the divergent barotropic equations. As has frequently been done previously to test numerical schemes, we perform integrations using Rossby–Haurwitz wave initial conditions. The Rossby–Haurwitz wave is an exact solution of the non-divergent barotropic vorticity equation on the sphere, but is not an exact solution of the equations with divergence. The barotropic instability of a large amplitude Rossby wave was first examined, using the non-divergent equations on a $\beta$-plane, by Lorenz (1972) and subsequently by Hoskins and Hollingsworth (1973). The instability of the Rossby–Haurwitz wave on the sphere was examined in detail, using the divergent and non-divergent equations, in Hoskins (1973), hereafter referred to as H.

In H, a theory based on severely truncated spectral expansions suggests that waves of zonal wavenumber greater than 5 will be unstable if their amplitude is sufficiently large,
and confirmation is provided by spectral numerical integrations of the shallow water equations and barotropic vorticity equation on the sphere. We describe integrations for two cases. In the first, the initial disturbance is in zonal wavenumber 8, and breaks down completely over about five days, producing a marked modification of the zonal flow. In the second, the initial disturbance is in wavenumber 4, and the wave shows relatively small changes in form.

An underlying principle of numerical integration of the meteorological equations is that, for given initial conditions and a specified period of interest, the solutions for increasing resolution converge. Further, it is assumed that this limit is the exact solution of the continuous equations for those initial conditions. When using two entirely different methods of integration, we may estimate the comparative errors involved in low resolution runs by increasing the resolution until the methods agree to any required accuracy.

Our first set of 'low resolution' experiments utilized a grid-point model having second order accurate finite differences on a $3^\circ$ latitude by $5^\circ$ longitude mesh, and a spectral model truncated to include modes up to zonal wavenumber 16, with a similar meridional resolution. These models, which are described in Section 2, require a similar amount of computing time, with the spectral model using less storage owing to its smaller number of degrees of freedom. Qualitative comparison of height field contours (Section 3) and quantitative comparison of the principal spectral components of the motion (Section 4) revealed a reasonable agreement for wavenumber 4, but marked differences in the results for the unstable wavenumber 8. These differences were largely resolved by performing a second set of integrations with higher resolution. The comparisons to be presented illustrate a close agreement between the high and low resolution spectral integrations, and the failure of a $3^\circ \times 5^\circ$ grid to adequately describe an active disturbance in wavenumber 8.

Viewed in the spectral framework, the instability of wavenumber 8 depends crucially on the phase differences between two waves of the same zonal wavelength but different meridional structure. The low resolution grid point model treats the instability poorly because of its inaccurate representation of zonal phase velocities, despite there being 9 points per zonal wavelength.

2. DESCRIPTION OF MODELS  

(a) The spectral model  

The spectral model is based on that described by Bourke (1972). The shallow water equations are utilized in their vorticity and divergence forms. The dependent variables are $\zeta$, the absolute vorticity; $D$, the divergence; and $\phi$, the deviation of the free surface from some mean value; and they are represented by truncated series of spherical harmonics:

$$\zeta(\mu, \lambda, t) = \sum_{m=-M}^{M} \sum_{n=|m|}^{|m|+1} c_n^m(t) P_n^m(\mu) e^{im\lambda}, \text{ etc.} \tag{1}$$

where $\mu = \sin \theta$, $\theta$ = latitude, $\lambda$ = longitude, and $P_n^m(\mu)$ is the associated Legendre function. We note that a spectral component $(m, n)$ has zonal wavenumber $m$. It also has $n - |m|$ zeros between the poles. Thus $n - |m| + 1$ is a pseudo-latitudinal wavenumber. Linear operations are carried out in phase space, and products evaluated by transforming to physical space. For the most part, a semi-implicit time scheme is used.

The integrations described in this paper are initiated with a super-rotation of the atmosphere upon which is superimposed a Rossby–Haurwitz wave in the stream function. The divergence is set to zero, and the height field obtained by solving the inverse balance equation to make the initial tendency of $D$ zero. Grid-point values of the stream function
and height field are calculated, and provide initial conditions for the grid-point model. Parameter values and the initial amplitudes are based on those used by Phillips (1959), and are identical with those described in H. The fluid depth is taken as 8km on a sphere whose radius, \(a\), and rotation rate, \(\Omega\), are those of the earth. The initial stream function is of the form

\[
\psi = -\omega u - \frac{K}{n(n+1)} P^m_n(\mu) \cos m\lambda
\]

where \(\omega\) is such as to give a super-rotation of 50 m s\(^{-1}\) at the equator. The vorticity amplitude \(K\) is taken as \(-0.8776\) for both wavenumber 4 \((m = 4, n = 5)\) and wavenumber 8 \((m = 8, n = 9)\).

The truncation used in the 'low resolution' experiments is \(M = 16, J = 15\). Thus zonal wavenumbers 0, 4, 8, 12 and 16 are retained in the integrations initiated with wavenumber 4, and 0, 8 and 16 for these begun with wavenumber 8. In the 'high resolution' experiments, the truncation is \(M = 32, J = 31\). Symmetry about the equator is imposed.

Results will be presented for time steps of 1/4 hour for both low and high resolution spectral integrations. A time step of one hour proved unstable using the semi-implicit scheme at high resolution. The low resolution wavenumber 8 integrations have been carried out using the semi-implicit scheme with 10 minute, 1/4 hour and one hour time steps, a two hour time step proving unstable, and using an explicit scheme with a 10 minute time step. Differences after 7 days between the 10 minute runs amounted to about one part in \(10^5\) for the principal spectral components of the motion, whilst relative errors of the order of 1% were found between the 10 minute and 1/4 hour integrations, and 5% between the 1/4 hour and one hour integrations.

(b) The grid-point model

The grid-point model uses the flux form of the horizontal momentum equations on the sphere:

\[
\frac{\partial}{\partial t} (\psi u) + \frac{1}{a \cos \theta} \left[ \frac{\partial}{\partial \lambda} (\psi u^2) + \frac{\partial}{\partial \theta} (\psi u v \cos \theta) \right] = -\phi u \left( 2\Omega \sin \theta + \frac{u}{a} \tan \theta \right) + \frac{\phi}{a \cos \theta} \frac{\partial \phi}{\partial \lambda} = 0
\]

\[
\frac{\partial}{\partial t} (\psi v) + \frac{1}{a \cos \theta} \left[ \frac{\partial}{\partial \lambda} (\psi u v) + \frac{\partial}{\partial \theta} (\psi v^2 \cos \theta) \right] + \phi u \left( 2\Omega \sin \theta + \frac{u}{a} \tan \theta \right) + \frac{\phi}{a} \frac{\partial \phi}{\partial \theta} = 0
\]

and the equation of conservation of mass:

\[
\frac{\partial \phi}{\partial t} + \frac{1}{a \cos \theta} \left[ \frac{\partial}{\partial \lambda} (\phi u) + \frac{\partial}{\partial \theta} (\phi v \cos \theta) \right] = 0
\]

where \(u\) and \(v\) are the eastward and northward velocity components and \(\phi\) is the height of the free surface multiplied by \(g\). Symmetry is assumed about the equator. The model uses a regular latitude-longitude grid with no staggering, all variables being defined at each grid-point. The finite difference scheme is second order and a straightforward adaption of that used in our 5-level model, which is described in GARP Pub. No. 13. The scheme conserves mass, kinetic energy in the advection terms and total (potential + kinetic) energy. Absolute angular momentum is not formally conserved (as indeed it is not in the spectral model) but a monitoring of its value suggests that this is not a serious deficiency.
The grid-point integrations to be described subsequently also utilize a semi-implicit time scheme. This is a direct adaptation of the method of Kwizak and Robert (1971) to the flux form of the equations. The finite difference form of the Helmholtz equation which results for the height field at time \( t + \Delta t \) is solved exactly, to ensure mass conservation, by a Fourier transform in the zonal direction and a Gaussian elimination in the meridional direction. To prevent the splitting of even and odd time steps, a time filter (Robert 1965) is used with a value \( \nu = 0.005 \).

In the 'low resolution' integrations a 3° latitude by 5° longitude grid is used, giving 18 points per 90° longitude and 28 points from equator to pole. The line of points nearest the pole represent triangular sectors extending from the pole to 81°N with an aperture of 5°. To allow use of the fast Fourier transform, the grid lengths are not simply halved for the 'high resolution' runs. Instead, we use 32 points per 90° longitude and \( \Delta \theta = 1\frac{1}{2}° \) (the line of points nearest the pole representing sectors extending to 84°N) which gives 57 points between equator and pole.

The results to be presented were obtained with time steps of \( \frac{1}{4} \) hour in the low resolution experiments, and 10 minutes at the high resolution. Instability was encountered when these time steps were doubled. The low resolution integrations have been compared with an explicit integration using a 10-minute time step and a Fourier smoothing following Arakawa (1971) north of 69°N. Differences after 7 days were typically less than 1% in the wind and height fields. Smaller discrepancies were found between semi-implicit 10 and 15-minute integrations at high resolution. Overall, in the comparisons to be made in following sections, errors due to time truncation appear to be no more than about 1%.

A summary of the number of degrees of freedom for each variable in the subsequent experiments is given in Table 1. The low resolution models require similar computing time, but the spectral model uses less storage.

| Table 1. |
|-------------------------------|------------------|------------------|
|                               | spectral         | grid point       |
|                               | low              | high             | low              | high             |
| wavenumber 4                  | 72               | 272              | 504              | 1824             |
| wavenumber 8                  | 40               | 144              | 252              | 912              |

3. A SYNOPSIS COMPARISON

Contours of the height fields given by the spectral model have been drawn using linear interpolation between values calculated at the grid-points used in the spectral integrations. In this section we compare the resulting maps with those produced from the grid-point integrations.

The evolution of the height fields for an initial disturbance in wavenumber 8 is shown on Fig. 1. For this example the behaviour of the low resolution spectral model (Fig. 1(a)) has been discussed in detail in H. The wave moves eastward at about 30° per day. It soon develops a tilt which feeds angular momentum equatorwards, thus weakening the zonal flow in middle latitudes. Energy is fed from the initial wave into the zonal flow, and into shorter scales mostly with the same zonal wavelength as the initial disturbance. By day 5 the original wave has clearly broken down. A significant amount of energy has reached the highest wavenumber components by this time, and the validity of any longer integration with this resolution is in doubt.

A marked difference is seen in the height fields resulting from the low resolution grid-
point model (Fig. 1(b)). At day 5, the wavenumber 8 component of the flow remains of significant amplitude, and the integration does not exhibit the tilting of trough and ridge, and rapid breakdown observed in the spectral model. Although some breakdown eventually does occur in the grid-point model, the low resolution integrations are so different that, taken by themselves, they give little confidence in either model. Appeal to energy diagnostics is of no help as in both integrations the sum of kinetic and potential energy is conserved to about 2 parts in $10^4$.

The height fields of the high resolution spectral and grid-point integrations are shown on Fig. 1(c) and (d) respectively. Comparison with Fig. 1(a) reveals a remarkable similarity between the results from the two runs with the spectral model, and this is also seen in a study of individual spectral components which is discussed in the following section. This similarity suggests that Fig. 1(a) and (c) represent close approximations to the true solution against which the performance of the grid-point model may be compared. The grid-point integration with the higher resolution (Fig. 1(d)) is seen to be much closer to the spectral integrations than was the grid-point integration with low resolution. The tilt and breakdown of the wave are well reproduced, and the only appreciable difference appears to be a delay of about $\frac{1}{2}$ day in the breakdown as described by the grid-point model.

Results from integrations with a wavenumber 4 disturbance do not exhibit the marked variations, and favouring of the spectral method, apparent from the results for wavenumber 8. The height fields from the low and high resolution runs are presented on Fig. 2 for the period from 0 to 7 days. All the integrations describe the essential stability of wave-
number 4 as it moves eastward at about $10^\circ$ per day. Only minor changes occur in the form of the height field. A weak tilting of the flow pattern transfers angular momentum so as to weaken the zonal flow in middle latitudes, and a closed low appears. This is deeper in the low resolution spectral model than in the corresponding grid-point model, and the cut-off is more marked, appearing in the contouring of Fig. 2. Increasing the resolution deepens the low in the grid-point model, and slightly weakens it in the spectral model. The cut-off becomes less pronounced in the high resolution spectral integration. Comparison of spot values gives no conclusive evidence as to which of the two models is closer to the true solution.

4. Comparison of Spectral Coefficients

Using finite differences, vorticity fields have been calculated from the wind fields given by the grid-point model. Truncated spectral expansions of the form (1) have then been derived for these vorticity fields, and the corresponding height fields, using a least squares method (cf. Hildebrand 1956, p. 261). We compare the principal spectral components thus calculated with those components computed directly by the spectral method.

(a) Wavenumber 8

The development in time of the amplitudes of some components of the vorticity field are shown, for the wavenumber 8 integrations, on Fig. 3. Initially, the vorticity of the wave
Figure 3. The variation with time of some principal spectral components of the vorticity in the wave-number 8 integrations.

is concentrated in the (8, 9) component illustrated on Fig. 3(a). Looking first at the low resolution spectral integration (solid curve) the breakdown evident on the maps of the height fields is clearly seen, with the amplitude decreasing rapidly to attain a minimum at day 5. A similar fast breakdown, of slightly smaller amplitude, is seen in the high resolution spectral integration. Subsequently, the spectral model at both resolutions describes an increase of energy in the (8, 9) component, although differences between high and low resolution grow significantly by day 7. Discrepancies are to be expected by this time since all vorticity components in the low resolution integration have reached a similar amplitude by day 5, giving little confidence in further results with this truncation. In the high resolution experiment, the amplitudes of the higher wavenumber components remain smaller at day 7 than those of the principal modes by a factor of around ten.
The behaviour of the (8, 9) vorticity component in the low resolution grid-point integration (dashed curve) is seen on Fig. 3(a) to be quite different in form. In place of a decrease in amplitude to a sharp minimum at day 5, there is a gradual decay over the whole seven day period. The initial vorticity amplitudes are not identical. Initial winds in the grid-point model are calculated from the specified stream function using finite differences. A second use of finite differencing gives the vorticity field, and initial differences of the order \( \left( \frac{\sin 8\Delta \lambda}{8\Delta \lambda} \right)^2 \) are thus seen. It is not this, however, which accounts for the differences apparent on Fig. 3(a), since a low-resolution spectral integration with an initial vorticity field identical with that of the grid-point integration again produced a rapid breakdown of the (8, 9) component between days 3 and 5, and a subsequent increase in amplitude, much as in the example illustrated.

In contrast with the low resolution integration, the high resolution grid-point integration gives results similar to those from the spectral integrations, with a fast breakdown, pronounced minimum, and secondary maximum in the amplitude of the (8, 9) component. The minimum occurs about \( \frac{1}{2} \) day later than in the spectral integrations, but by day 7 there appears to be better agreement between the two high resolution integrations than between the two spectral integrations.

Accompanying the decrease in amplitude of the (8, 9) vorticity component is an increase both of zonal and of other wave components. The behaviour of the (8, 11) and (0, 5) coefficients is shown on Fig. 3(b) and (c). The low and high resolution spectral integrations and high resolution grid-point integrations again give similar results, with more gradual changes to be seen in the grid-point model at low resolution. A slight tendency of the low resolution spectral model to overestimate changes in amplitude again occurs, as does a delay in achieving maximum amplitudes in the high resolution grid-point model, although there is evidence for these components that maxima in the low resolution spectral model are reached somewhat early.

Significant differences are seen in the phases of the wavenumber 8 components of the vorticity. On Fig. 4(a) we plot the longitude of the wave associated with the (8, 9) component. Up to day 4, this is the dominant component in wavenumber 8, and it moves much as predicted by non-divergent barotropic theory. The exact treatment of spatial derivatives in the spectral model is such as to give excellent agreement between low and high resolution over this initial period, while the movement of the wave is retarded in the grid point model. The acceleration of the (8, 9) component when it is of small amplitude near day 5 is similar in the low and high resolution spectral, and high resolution grid-point, integrations. The wave slows down in the low resolution grid-point integration.

On Fig. 4(b) we plot the important difference in phase between the (8, 9) and (8, 11) components. As discussed in H, a difference in phase between the (8, 11) component and the main wave produces a tilted pattern and feed of angular momentum, and the theoretical analysis given therein describes how such a phase difference may be 'locked' by a tendency of non-linear interactions to counter the dispersive effect of the differing Rossby wave speeds of the components. Over a period of four days, Fig. 4(b) shows that, in all but the low resolution grid-point integration, the two components possess a phase difference which varies little from \(-90^\circ\), and the feed of angular momentum is close to a maximum. In contrast, there is no initial locking of the phase difference in the low resolution grid-point integration, and although the difference appears to become fixed subsequently, the angle of \(-30^\circ\) gives a much weaker feed of momentum, and a more gradual breakdown of the wave.

Comparison of the principal spectral components of the height fields again emphasizes the similarity of all but the low resolution grid-point integration.
Figure 4. Aspects of the phase behaviour in the wavenumber 8 integrations. (a) The position on the sphere of the (8, 9) component of the vorticity as a function of time. (b) The difference in phase between the (8, 9) and (8, 11) components of the vorticity as a function of time.

(b) Wavenumber 4

The amplitudes of some components of the vorticity in the wavenumber 4 case are shown in Fig. 5. Changes from the initial state are small compared with the wavenumber 8 example. There is clearly convergence for increasing resolution. Whether the low resolution spectral or grid-point model is nearer this limit depends on which coefficient is considered. As in the initial stages for wavenumber 8, the phase of the main vorticity component (Fig. 6) is described significantly better by the spectral model. In that model, after 7 days, the differences between high and low resolution amount to $\frac{1}{4}$° in the position of the wave on the sphere.
Figure 5. The variation with time of some principal spectral components of the vorticity in the wave-number 4 integrations.

5. SUMMARY AND CONCLUSIONS

Performing experiments with second order finite difference and spectral methods for two different resolutions has given a good indication of the exact solutions. This is particularly true because the two methods approach the limit differently. The spectral model with low resolution tends to overestimate changes, while the finite difference model underestimates. The former effect must be connected with the inclusion of the modes which are important in the instability but limitation of other possible modes. The latter effect is the usual distortion of phase speeds associated with finite difference methods.
The low resolution integrations for the stable wavenumber 4 appear to have sufficient resolution to predict accurately the solution up to day 7 in that experiment. The changes in amplitude that do occur are not clearly dealt with in a superior manner by either method. The phase of the main wave is better represented by the spectral method.

In a barotropic integration, the phase speed of an unperturbed Rossby–Haurwitz wave would be represented exactly in the spectral model apart from errors due to the truncation in time. The wavenumber 4 integrations discussed here, and other integrations initiated with small-amplitude Hough functions*, illustrate the spectral model’s accurate treatment of Rossby wave speeds in the free surface equations, although as these phase speeds differ by only some 10% from those predicted by barotropic theory, the test is not severe.

The spectral method is seen to best advantage in the wavenumber 8 integrations. Close agreement is found in the description of the instability of the wave given by the low and high resolution runs, and a similar instability is observed using the high resolution grid-point model. While the grid-point model at low resolution does describe a breakdown of wavenumber 8, this occurs in a gradual manner, in contrast with the rapid breakdown which characterizes the other integrations discussed here, and others performed both with the shallow water equations, and with the barotropic vorticity equation. The failure of the low resolution grid-point model was not primarily due to a poor treatment of wavenumber 16. The e-folding times for the instability predicted by severely truncated spectral expansions which neglect this component correspond to a breakdown similar to that observed in the other wavenumber 8 integrations, and spectral integrations truncated so as to include only zonal wavenumbers 0 and 8 also produce a sharp breakdown. The inadequacy of the low resolution grid-point integration appears to be in its treatment of wavenumber 8 itself. Over the first two days of the integration, Fig. 3 shows that the amplitudes of the (8, 11) and (0, 5) vorticity components are similar to those in the spectral model, and certainly no more different than those in the high resolution integration. The same is true of other components of the vorticity. The principal discrepancy at this stage is seen on Fig. 4(b) to be in the phase difference between the (8, 9) and (8, 11) component. The low resolution grid-point integration fails to fix this close to 90°, and this gives rise to its subsequent reduced feed of angular momentum, and reduced transfer of energy from the primary wave.

For the wavenumber 8 case, the 5° × 3° grid has 9 points per zonal wavelength. The (8, 9) and (8, 11) modes have approximately 56 and 28 points respectively per wavelength in the meridional direction. Thus in second order finite difference simulation of Rossby wave instability, this resolution is insufficient. The higher resolution with 16 points per zonal

* Integrations initiated with large amplitude Hough functions exhibit stability characteristics similar to the Rossby–Haurwitz wave.
wavelength gives a much improved behaviour, though not obviously superior to the low resolution spectral integrations over the first 5 days.

It would be of interest to subject two other grid-point methods to the quantitative comparison performed here. These are the 4th order finite difference and pseudospectral schemes (Merilees 1973). Our experiments are slightly biased towards the spectral model. All the energy is initially in two spectral modes and the instability is mostly a transfer of energy to many zonal modes and one other wave. It is planned to extend the comparisons presented here to a study of baroclinic instability in multi-level models with grid-point and spectral representations in the horizontal. In such experiments, any bias towards the spectral method should be removed since the developing wave will in general assume a form involving many spectral components. Phase differences are important in determining the stability properties of a baroclinic wave, and it will be of interest to see whether a poor description of phases by the finite difference method again results in a more efficient description being given by the spectral method.

It is now possible to draw up a tentative list of advantages and disadvantages of the spectral versus second order finite difference methods for the dynamical portion of a full general circulation model. It appears from the work described here that a spectral model will need considerably fewer degrees of freedom and less storage. This will be a tremendous advantage with the coming generation of computers in which the speed is increased only when dealing with the fast memory. The semi-implicit algorithm is much easier to apply in a spectral model. Programming a spectral model is more complicated than programming a finite difference model. Having the fields in a spectral representation at every output may be suggestive of significant wave interactions. (In the spectral method used here grid-point values are also available at each timestep.) Since the events at one grid point in a spectral model affect and are affected by the whole atmosphere, the incorporation of physical effects may be difficult. This is a relatively unexplored aspect of spectral models to be resolved before the spectral method can challenge the dominance of the finite difference method in general circulation studies.

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