A comparison of spectral and finite-difference simulations of a growing baroclinic wave

By A. J. SIMMONS and B. J. HOSKINS

U.K. Universities Atmospheric Modelling Group, Department of Geophysics, University of Reading

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SUMMARY

A comparison is made of integrations of the primitive equations on the sphere using a second-order finite-difference model, and spectral models with both triangular and rhomboidal truncation. The initial conditions comprise a baroclinically unstable mid-latitude jet, and a perturbation of small amplitude. Increasing the resolution in each model gives a good estimate of the exact solution, and thus of the errors involved in each integration.

No one method has a superiority in all respects. Spectral integrations with truncations at zonal wavenumber 16 give a more accurate description of amplitudes and phases in the growing wave than does a finite-difference integration using a $5^\circ \times 3^\circ$ grid, but a much poorer description of the fronts that form as the disturbance approaches maturity. Large-scale changes to the zonal-mean state are predicted to greater accuracy using the spectral models, but smaller-scale changes are resolved better by the finite-difference model. The spectral models require less computing time, and less storage. For these experiments, rhomboidal truncation is favoured.

1. INTRODUCTION

Recently, models of the primitive equations suitable both for numerical weather prediction and for general circulation studies have been developed using truncated series in terms of spherical harmonics to represent the horizontal variation of the meteorological fields (e.g. Bourke 1974; Hoskins and Simmons 1975). Following this development, there is need for detailed comparison of these spectral models with the finite-difference models traditionally used for numerical integration. This comparison can be made in a number of ways. One is to use relatively simple initial conditions which nevertheless give rise to significant subsequent development. In the absence of an exact analytical solution, increasing the resolution may indicate a correct solution against which the lower resolution integrations can be compared. Whilst the major test of the spectral method must be in full-scale use, this more idealized approach can give a good indication of the relative merits of the quite different numerical schemes, and in addition may be used to gain an improved understanding of the dynamics of the system under study.

We have adopted such an approach in a previous comparison of spectral and finite-difference simulations of Rossby wave instability in a one-layer fluid (Doron et al. 1974). In that study, which has been complemented subsequently by integrations using a finite element method (Cullen 1974), an inaccurate treatment of phases by the finite-difference method led to a significantly more efficient description of the breakdown of a Rossby wave being given by a spectral model. These results confirmed those of previous comparisons favouring the spectral method (Elsaesser 1966; Orszag 1971), but were perhaps biased in that vorticity was confined initially to just two spectral modes.

In the present paper we compare integrations using multi-level models with identical vertical finite-differencing, but with spectral and second-order finite-difference representations in the horizontal. Results are presented for spectral integrations using both the mathematically more elegant triangular truncation, and the rhomboidal truncation at present chosen for many practical applications (Bourke 1974; Daley, Simmonds and Henderson 1974). As in our earlier study, we present qualitative comparisons of grid-point
fields, and quantitative comparisons of some principal spectral components of the motion.

The case chosen for study is the growth of a baroclinically unstable disturbance to a zonal mid-latitude jet. Correct simulation of the full life cycle of such a disturbance (or parameterization of an ensemble of such disturbances) is a necessary feature of any model of large-scale motion in extratropical latitudes of the earth's atmosphere. The role of baroclinic waves as important transporters of heat and momentum is well known (see e.g. Lorenz 1967), and the frontogenesis that accompanies their growth provides a mechanism whereby energy is transferred to shorter scales. Parameterization of this transfer is a general requirement of numerical models. The present experiments provide a good starting point for a close examination of alternative ways of achieving this.

For this paper, we concentrate on a comparison of the various numerical descriptions of the growth of the baroclinic wave, and make no direct attempt to relate the properties of the wave either to the observed atmosphere, or to more idealized theoretical studies. These properties are nevertheless of considerable intrinsic interest, and will form the subject of a subsequent study.

2. Description of Models

(a) Spectral model

The spectral model used in this study has been described in detail by Hoskins and Simmons (1975). The primitive equations for a dry atmosphere are expressed in $\sigma$-coordinates, and written in vorticity and divergence form. Variables are represented in the horizontal by truncated expansions in terms of spherical harmonics, and used is made of the transform method whereby products are evaluated at points in physical space (Orszag 1970; Eliassen et al. 1970). The grid employed for the latter is such as to avoid aliasing for quadratic terms. Vertical derivatives are represented by finite-differences, using the energy-conserving scheme of Corby, Gilchrist and Newson (1972). Finite-differencing in time is either explicit or semi-implicit. The model includes no parameterization of turbulent diffusion, convection, surface fluxes or radiation.

Results will be presented for two different types of truncation of the spectral expansion

$$\zeta(\lambda, \mu) = \sum_{m=-M}^{M} \sum_{n=|m|}^{N} \xi_n e^{im\lambda} P_n^m(\mu),$$

where $\lambda =$ longitude, $\mu =$ sin (latitude), and $P_n^m$ is an associated Legendre function. These truncations are the triangular, $N = M$, and rhomboidal, $N = |m| + J$. In the triangular truncation favoured in early spectral integrations (e.g. Baer and Platzman 1961), the only length scales retained are all those greater than a fixed scale independent of direction and position on the sphere. For the rhomboidal truncation we take $J = M - 1$, thus keeping $M$ meridional modes for each zonal wavenumber. Following Ellsaesser (1966), this type of truncation has become common (e.g. Bourke 1972; Hoskins 1973), although evidence favouring use of triangular truncation has been put forward by Baer (1972). Use of both truncations in the present series of integrations provides additional evidence as to the accuracy of the solutions, and suggests that truncation most suited to this type of experiment.

For each truncation, integrations have been performed with two different horizontal resolutions. These are defined by $M = 21$ and $M = 42$ for triangular, and $M = 16$, $J = 15$ and $M = 32, J = 31$ for rhomboidal truncation, and these runs will be referred to as T21, T42, R16 and R32 respectively. For most integrations, 5 equally spaced levels are chosen in the vertical. The semi-implicit scheme with a timestep of $1/2$ hour has been used for all cases to be described in detail.
(b) Finite-difference model

The finite-difference integrations employ a regular latitude-longitude grid. The primitive equations are written in flux form, and solved using an energy-conserving second-order accurate spatial finite-difference scheme, with pressure-gradient terms and vertical differencing as in the model of Corby, Gilchrist and Newson. A centred explicit time-scheme is used with a zonal smoothing following Arakawa (1971) applied at all latitudes poleward of $70^\circ$. No other smoothing is applied, either in space or in time.

For this model also, integrations have been performed for two horizontal resolutions. The lower of these utilizes a $3^\circ$ latitude by $5^\circ$ longitude grid. The line of points nearest the pole represents triangular sectors of a polar cap extending to $81^\circ$ latitude, this giving 28 points from equator to pole. A timestep of 10 minutes is used. The grid for the higher resolution integration is $1\frac{1}{2}^\circ$ latitude by $24^\circ$ longitude, with a polar cap from $85\frac{1}{2}^\circ$ latitude, giving 58 points from equator to pole. The timestep used is $4\frac{1}{2}$ minutes.

(c) Comparison of computing requirements

Information regarding timing and storage requirements for the various integrations is given in Table 1. In these experiments the initial perturbation is confined to zonal wavenumber 8, and the figures presented relate to just $\frac{1}{4}$ of the hemisphere. This enables all integrations to be contained within the ‘small core memory’ available on the CDC 7600 used for this study, and there is thus no timing bias against integrations using larger amounts of storage. Timing and storage for the spectral integrations are given for the semi-implicit time-scheme and arbitrary layer-spacing, the finite-difference model being programmed for equally spaced layers. Reductions for a spectral model with an explicit scheme and equally spaced layers are, however, negligible. Care has been taken to make the FORTRAN programming of both models efficient. In addition, the spectral model uses a quite general machine-code Fourier transform package.

For both spectral and finite-difference models, a doubling of the resolution results in an increase by a factor close to 5 in the amount of computer time taken per timestep. A similar factor is found for hemispheric spectral integrations with these resolutions. This is less than the factor of 8 suggested by an asymptotic analysis of the transform method for spectral models (Orszag 1970), and with additional time being taken by the polar smoothing.

<table>
<thead>
<tr>
<th></th>
<th>CP time per timestep (seconds)</th>
<th>Maximum timestep (minutes)</th>
<th>Number of grid points per level</th>
<th>Storage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spectral</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Triangular $M = 21$</td>
<td>0.084</td>
<td>16</td>
<td>128</td>
<td>7K</td>
</tr>
<tr>
<td></td>
<td>$M = 42$</td>
<td>0.420</td>
<td>85</td>
<td>512</td>
</tr>
<tr>
<td>Rhomboidal $M = 16$</td>
<td>0.115</td>
<td>11</td>
<td>50</td>
<td>160</td>
</tr>
<tr>
<td></td>
<td>$M = 32$</td>
<td>0.593</td>
<td>5</td>
<td>640</td>
</tr>
<tr>
<td>Finite-difference</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$5^\circ \times 3^\circ$</td>
<td>0.090</td>
<td>10</td>
<td>252</td>
<td>12K</td>
</tr>
<tr>
<td>$24^\circ \times 1\frac{1}{2}^\circ$</td>
<td>0.455</td>
<td>4</td>
<td>1044</td>
<td>39K</td>
</tr>
</tbody>
</table>
in the finite-difference model, there is no extra timing bias against the spectral model at higher resolution.

Using explicit time-schemes with the largest possible timestep, the finite-difference integrations require about 70% more computing time than the corresponding spectral integrations with triangular truncation, but 20% less than those with rhomboidal truncation. With use of a semi-implicit scheme, timesteps more than 5 times longer are found to be possible in the spectral model, with greater increases in efficiency for rhomboidal truncation and higher resolution. These longer timesteps may be used with negligible increase in computation, and negligible loss of accuracy (Hoskins and Simmons 1975). Implementation of a semi-implicit algorithm in the finite-difference model would involve a significant computational overhead, and is likely to increase efficiency by a factor of about 2½ (Gauntlett and Leslie 1974). Greater increases in efficiency would occur if the models contained time-consuming parameterizations of physical processes, but in such cases the fewer grid points used in the spectral integrations would again give a clear timing advantage. Thus, for the particular resolutions we have chosen, spectral semi-implicit models require less computer time. They also require less storage.

3. Initial Conditions

The initial wind field used for all integrations comprises a zonal jet upon which is superimposed a non-divergent perturbation of small amplitude. The vertical structure of the jet, and of the horizontally averaged temperature field, is illustrated in Fig. 1. These profiles are an idealization of the observed atmosphere, and give an initial balanced temperature field which is everywhere statically stable.

![Figure 1](image.png)

Figure 1. The initial vertical variation of the zonally averaged velocity at 30° latitude (left) and the horizontally averaged temperature (right). The continuous lines denote analytical profiles used for integration with more than five levels.

In the horizontal, winds are specified by analytical functions with no latitudinal bias towards associated Legendre function expansions. The meridional variation of the jet is of the form sin ² 7πμ, and is illustrated in Fig. 2, together with the initial meridional variation of temperature at each level. Zonal winds are a maximum at 30° latitude, vanishing in polar regions and at the equator. A region of marked latitudinal temperature gradient extends from 20° to 60° at the lowest four levels, with a weak reversed gradient at the uppermost 'stratospheric' level.

The perturbation is confined to zonal wavenumber 8. Its initial vorticity is identical at all levels, and is proportional to sin πμ, with amplitude such that the r.m.s. velocity is \( \frac{1}{2} \text{m s}^{-1} \). Balancing gives a temperature wave of maximum amplitude 0.29 deg C, and surface pressure wave of amplitude 0.82 mb.

Initial fields for the finite-difference integrations have been obtained directly from
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Figure 2. The initial meridional variation of the zonally averaged zonal velocity (left) and temperature (right).

balanced spectral components derived using resolution T42 in the manner described in Hoskins and Simmons (1975). Since such fields are not exactly balanced for the finite-difference model, two T21 integrations have been carried out to demonstrate the unimportance of the resulting presence of small-amplitude gravity waves. In the first, only the zonal flow is balanced, with an unbalanced disturbance confined to the vorticity field. In the second the zonal-mean state is unbalanced by setting the upper-level temperature gradient to zero. Gravity waves are generated with larger amplitudes than in the finite-difference integrations, but differences from the standard T21 integration are negligible compared with the truncation errors to be discussed in subsequent sections.

4. COMPARISON OF SPECTRAL INTEGRATIONS

(a) Grid-point fields

Maps of surface pressure and temperature at the $\sigma = 0.9$ level are shown for the spectral integrations in Figs. 3 and 4. We present the initial state, day 0, and the subsequent days 4, 5 and 6. During the first two days of each integration the disturbance adjusts its vertical and horizontal structure to give a form suitable for growth. Thereafter it develops for a while much as described by the linear theory of baroclinic instability such that by day 4 the surface low is 989mb, and situated at 40° latitude. The temperature wave is a maximum at this latitude, and has amplitude 6 1/2 deg C. All integrations are clearly similar at this stage.

The two higher resolution integrations still exhibit very close agreement at day 5. Stronger gradients in surface pressure occur ahead of the low centre, which has moved poleward and deepened to 976mb. The temperature wave shows a pronounced distortion characteristic of the usual warm and cold fronts, and occlusion process. These features are resolved much less well by the lower resolution integrations, rhomboidal truncation proving the more accurate.

The non-local nature of error in the spectral models is illustrated by the surface pressure maps at day 5. Representation of a disturbance confined to middle latitudes inevitably gives rise to some error elsewhere. Here this error is of amplitude only 1 mb, but since the surface pressure in equatorial and polar regions is close to 1000mb, it is clearly shown by the 1000mb contour in Fig. 3.

In the absence of a diffusive mechanism, the sharpening of fronts continues beyond day 5, and by day 6 misrepresentation gives rise to roughness in the contours of the high resolution integrations. Apart from this, however, the two integrations remain in good agreement. Both describe a further deepening, and a change of shape of the surface low,
and the occlusion process continues. In middle latitudes the meridional gradient of zonally averaged temperature has been completely destroyed, with a sharpening of gradients to south and north.

The low resolution integrations are clearly lacking in detail at day 6. Although the low pressure centre continues to deepen, the triangular truncation fails to capture the change of shape of the depression, and this change is only partially represented with rhomboidal truncation. Neither integration accurately represents the frontal regions present in the high resolution experiments. Both, however, are successful in destroying the poleward temperature gradient in mid-latitudes.

(b) Spectral fields

From day 2 to day 4, the largest spectral components of vorticity in the growing wave are \( m = 8, n = 11 \), and \( m = 8, n = 15 \) at the level \( \sigma = 0.9 \). The development in time of the amplitude of these two largest components is shown in Fig. 5. The behaviour of the phase of one component will be discussed subsequently.

Results for resolution T21 are presented for timesteps of \( \frac{1}{4} \) hour and \( 1 \frac{1}{4} \) hour. The similarity between these illustrates the insensitivity of amplitudes to time truncation error. Phases are similarly insensitive. As in the example discussed in greater detail by Hoskins
and Simmons spatial truncation error dominates when using any timestep made possible by the semi-implicit time-scheme.

It is evident from Fig. 5 that the amplitude of the dominant (8, 11) component is predicted accurately to a stage beyond that at which the frontal structure is adequately resolved, with the low resolution rhomboidal truncation proving significantly more accurate than the corresponding triangular truncation. The superiority of the rhomboidal truncation is accentuated by the (8, 15) component, with T21 failing completely to predict the minimum at day 6 attained by all other spectral integrations. A close examination shows this to be related to the failure of T21 to predict the change in shape of the surface low as the disturbance occludes. The additional modes in zonal wavenumber 16 retained in R16 are sufficient to give a marked improvement in the accuracy of the (8, 15) component, a less dramatic success being achieved with the shape of the surface low.

Of interest from the viewpoint of general circulation studies is the net influence of the developing baroclinic wave on the zonally-averaged state. In Fig. 6 we present the amplitude at days 5 and 6 of some zonal coefficients of the vorticity at the level \( \sigma = 0.9 \) where the largest changes take place. Illustrated are all retained modes in the lower resolution experiments, and the lowest 11 modes from the higher resolution runs. The latter two sets of modes are indistinguishable on the scale of Fig. 6 at day 5, and small differences are noticeable in only the highest two modes of the R16 integration. More inaccuracy is apparent in the zonal modes of T21, and in particular the three modes which are not
Figure 5. The amplitudes of the (8, 11) and (8, 15) components of the vorticity at $\sigma = 0.9$ plotted as functions of time. Circles denote values obtained using resolution T21 with a timestep of $1\frac{1}{2}$ hours. Otherwise, a timestep of $\frac{1}{4}$ hour was used.

Figure 6. The amplitudes of zonal components of relative vorticity at $\sigma = 0.9$ for days 5 and 6. The initial distribution is also illustrated.

retained in R16 are totally misrepresented. A broadly similar picture holds at day 6, although error has spread to lower meridional modes. Amplitudes of some higher modes are grossly exaggerated by the T21 integration.

5. VERTICAL RESOLUTION

Prior to discussing the comparison of spectral and finite-difference integrations, we discuss briefly the dependence of our results on vertical resolution. To investigate this, a spectral integration with triangular truncation at total wavenumber 21 has been carried out using an 8-layer model. Since the disturbance itself, and changes to the mean state,
are largest at low levels, the layers were not chosen with equal thickness, but were defined by the half levels

$$\sigma_r = \left( \frac{r}{8} \right) \left( 1 + \frac{56}{95} \left[ 1 - \frac{r^2}{8} \right] \right) \quad r = 1, 2, \ldots, 7$$

Full levels occur at $\sigma = 0.970, 0.9, 0.812, 0.706, 0.580, 0.437, 0.275$ and $0.095$. With this choice, layer thicknesses are everywhere less than in the 5-layer model, with finest resolution close to the ground.

![Image](image_url)

**Figure 7.** The amplitudes of zonal and wavenumber 8 components of relative vorticity at $\sigma = 0.9$, day 5, for horizontal resolution T21 and 5- and 8-layer models.

Direct comparison may be made between 5- and 8-layer results at the level $\sigma = 0.9$, and the amplitudes of zonal and $m = 8$ components of the vorticity are presented for day 5 in Fig. 7. There is clearly very close agreement between the results of the two integrations, even for the higher meridional modes. For these higher modes differences are less than between the various 5-layer integrations, suggesting that for horizontal resolution T21, truncation error in the horizontal is the largest. Conversely, for lower meridional modes vertical truncation error, though small, is the more significant. By day 6, however, the relative magnitude of horizontal truncation error has increased such that for no component is vertical truncation error clearly the larger.

It should be noted that these conclusions do not apply to all fields. Quantitative comparison of temperature and surface pressure fields is prevented by differences in initial balanced components which remain throughout the integration. The development of the vorticity is insensitive to such differences.

6. **Finite-difference integrations**

    **(a)**. Grid-point fields

Maps of the surface pressure at days 4, 5 and 6 are presented in Fig. 8 for the two finite-difference integrations and, for comparison, the T42 spectral integration shown in Fig. 3. The higher resolution finite-difference integration is clearly close to the spectral integration, particularly at days 4 and 5. Minor differences are apparent by day 6, but the intensity and change of orientation of the low pressure centre are similar. Larger differences result from the $5^\circ \times 3^\circ$ finite-difference integration, with growth and movement of the disturbance occurring more slowly than in any other integration. Apart from this, however, smaller scale detail absent in both low resolution spectral integrations is resolved in at least a qualitatively correct manner at days 5 and 6.
Figure 8. Polar stereographic projections of the surface pressure at days 4, 5 and 6. The contour interval is 5 mb.  
(a) Triangular spectral, $M = 42$; (b) Finite-difference, $5^\circ \times 3^\circ$; (c) Finite difference, $24^\circ \times 14^\circ$.

Figure 9. Polar stereographic projections of the temperature at $\sigma = 0.9$ for days 4, 5 and 6. The contour interval is 5 degC. The integrations illustrated are as in Fig. 8.

The superior representation of smaller scales by the finite-difference model is well illustrated in Fig. 9 which shows the low-level temperature field at days 4, 5 and 6. The frontal structure absent in the lower resolution spectral integrations is represented in a generally correct manner by the lower resolution finite-difference model, although errors
in detail become apparent on comparison with higher resolution integrations. Comparing the latter, we again see good agreement to day 5. At day 6, there is overall agreement with regard to the position and orientation of fronts, and the temperature contrasts across them. The finite-difference model has produced much sharper temperature gradients, without the general roughness found using the spectral model. Local roughness occurs in the vicinity of the cold front, and may be due either to a misrepresentation of small-scale instability in this region of low static stability, or to the inability of finite-differences to cope with the large gradients formed by day 6, as in the experiments of Williams (1967).

(b) Spectral fields

Spectral analysis of grid-point data from the finite-difference integrations has been performed as described in the appendix to this paper. In Fig. 10 we present the amplitudes of the dominant (8, 11) and (8, 15) components of the vorticity at \( \sigma = 0.9 \). These clearly illustrate the weaker growth in the \( 5^\circ \times 3^\circ \) finite-difference integration. Also of interest is the failure of this integration to describe the decrease in amplitude of the (8, 15) component as the wave occludes. This was the case with the T21 spectral integration, all other spectral integrations giving a minimum within 2 hours of day 6. A minimum is predicted by the higher resolution finite-difference integration, but this occurs somewhat later.

The superior treatment of phases by the spectral method is evident from Fig. 11, which shows the position on the sphere of the (8, 11) component of vorticity at days 4 and 6 for the triangular spectral and finite-difference integrations. Both spectral integrations with rhomboidal truncation give results similar to those from T42. Apart from the uppermost level, where the amplitude of the disturbance is small, all spectral integrations are in very close agreement at day 4, and thus give an accurate estimate of the exact solution. The finite-difference integrations accurately describe the vertical tilt of the disturbance, but
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Figure 11. The variation with height of the longitude of the maximum of the $(8, 11)$ component of vorticity at days 4 and 6.

exhibit the well-known underestimation of phase-speed, this being generally greater than that suggested by an analysis of linear advection by the correct phase-speed. Phase error in the spectral model, though very small, is more apparent by day 6, with an overestimation by the low resolution integration. These features may be detected in the maps of surface pressure, Figs. 3 and 8.

Considering now the changes to the zonal-mean state induced by the growing wave, the amplitudes of some zonal vorticity coefficients at $\sigma = 0.9$ are presented for days 5 and 6 in Fig. 12. Comparison with Fig. 6 shows the spectral models to give a slightly more accurate treatment of lower-order harmonics, but to be subject to more error in higher modes. Viewed in physical space, the change to the zonal flow comprises a strengthening

Figure 12. The amplitudes of zonal components of relative vorticity at $\sigma = 0.9$ for days 5 and 6.

and sharpening of the jet at low levels, with the production of easterlies to the north and south, in accord with many theoretical studies of the baroclinic wave. A comparison of the structure and strength of the low-level flow, however, is less informative than the above comparison of spectral harmonics. By day 6, absence of some important shorter scales, and error in others, give rise to irregularity in the grid-point values of the zonal flow deduced from the spectral models, and errors of a similar order of magnitude are indicated by differences between the two finite-difference integrations. Thus, particularly near velocity maxima, there is no specific picture of convergence towards an exact value, and no obvious favouring of either method.

Similar considerations apply to the zonally-averaged temperature field. The changes
in amplitude from day 0 of the first three (also the largest three) zonal components at $\sigma = 0.9$ are presented in Table 2 for days 5 and 6. There is clearly convergence of solutions with increasing resolution, and more accurate results from the spectral integrations, despite their poorer fronts. As might be expected, comparison of higher order harmonics shows them to be represented better by the finite-difference model.

7. Conclusions

Performing integrations using spectral and second-order finite-difference models with various resolutions has given a good indication of the exact development of a baroclinic wave up to a stage at which fronts have formed with temperature differences of the order of 10 deg C across the smallest resolved distance, about 200km. In describing this development, no particular model exhibits a superiority in all respects. Instead, some features are described better by one method, and some by the other. The spectral integrations, however, require less computer time and storage to achieve these results, and the spectral method is thus likely to be favoured for many purposes.

The lower order harmonics of the solution are predicted to greater accuracy by the spectral method, results from the finite-difference integration using a $2\frac{1}{4}^\circ \times 1\frac{1}{4}^\circ$ grid proving barely as accurate as those from the much more rapid spectral integration with rhomboidal truncation at zonal wavenumber 16. In particular, the amplitude and phase of the growing wave are represented better by the spectral models. The small phase error that is made is such as to overestimate the speed of the wave, in contrast with the familiar underestimation made using finite-differences.

Smaller-scale features of the disturbance are treated better in the finite-difference model. The lower resolution integration gives a reasonable description of the frontal structure that is almost completely lacking in the lower resolution spectral integrations. It should be noted, however, that this is only a marginal advantage for the models used in this study, since the $5^\circ \times 3^\circ$ finite-difference integration using an explicit time scheme requires an amount of computing time similar to that required by the higher resolution semi-implicit spectral integration with triangular truncation at total wavenumber 42. The latter gives superior results in almost all respects. This emphasizes the value of the semi-implicit time-scheme, which may be implemented with more ease, and more efficiency, in a spectral model.

The models used here give results of similar accuracy as regards changes to the zonally averaged state. Of interest in this respect is the accuracy with which the gradest zonal modes are described by the lower resolution spectral integrations. Although the largest changes occur between days 5 and 6 during which time fronts are a major feature of the synoptic situation, the gross transfer properties of the growing baroclinic wave appear to be relatively
independent of a detailed resolution of the frontal structure. The integration with triangular truncation at wavenumber 21 is particularly efficient in its computational requirements, and, given a parameterization of the transfer of energy to unresolved scales that gives rise to spurious high-wavenumber growth in these experiments, this resolution should prove sufficient for a number of studies of gross features of the general circulation.

Our results are inconclusive with regard to the comparison of triangular and rhomboidal truncation in the spectral model. In these particular experiments involving the growth of a disturbance to a purely zonal jet, the low resolution rhomboidal integration gives a generally more accurate approximation to the solution than does the corresponding triangular integration, which has rather lower computing requirements. Conversely, repetition of the experiments described by Doron et al. (1974), using triangular truncation, shows it to give the better description of Rossby wave instability. For relatively low resolution, choice of truncation is thus quite dependent on the particular nature of the problem under investigation.

In conclusion, this study has suggested that spectral methods possess several advantages over finite-difference methods for use in general circulation studies. For short-term weather prediction, the position is less clear, involving a choice between better large-scale amplitudes and phases in spectral models, and better small-scale features, such as fronts, in finite-difference models. For either application, since the two methods are subject to quite different errors, the availability of results from both models may be of great value in isolating results that are physical rather than numerical in origin.

ACKNOWLEDGMENTS

The finite-difference model used for this study is a modification of one developed by Dr. E. Doron. Part of the routine for performing spectral analysis of the grid-point fields from this model was prepared by Dr. A. Hollingsworth.

REFERENCES


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APPENDIX

THE SPECTRAL ANALYSIS OF GRID-POINT FIELDS

To compare spectral components of fields from the finite-difference model with those computed directly by the spectral method, we have performed spectral analyses of surface pressure, temperature and the components \( U \) and \( V \) defined by

\[
U = \text{zonal velocity} \times \cos(\text{latitude})
\]

\[
V = \text{meridional velocity} \times \cos(\text{latitude})
\]

Spectral components of the relative vorticity, \( \xi \), and divergence, \( D \), are then obtained by inverting the relations

\[
U = (1 - \mu^2) \frac{\partial \psi}{\partial \mu} + \frac{\partial \chi}{\partial \lambda}, \quad V = \frac{\partial \psi}{\partial \lambda} + (1 - \mu^2) \frac{\partial \chi}{\partial \mu}
\]

where the stream function, \( \psi \), and velocity potential, \( \chi \), are such that \( \xi = \nabla^2 \psi \) and \( D = \nabla^2 \chi \). Given a spectral representation of \( U \) and \( V \), this reduces to an algebraic problem.

Performing the spectral analysis for components with zonal wavenumber 8 proved straightforward, results from fitting a number of meridional modes equal to the number of points from equator to pole giving agreement to at least four figures when comparing the lowest four modes with those computed using either a simple trapezoidal approximation to the inversion integral

\[
f_m^n = \int_{-1}^{1} f_m^m(\mu) P_n^m(\mu) d\mu,
\]

or a least-squares fit (cf. Hildebrand 1956, p. 261) of just 11 modes.

More difficulty was encountered in the analysis of zonal components. Fitting the maximum number of meridional modes gave accurate results for the smooth initial conditions used here, but the coefficients of all modes were sensitive to roughness in the fields to be analysed at subsequent times, and results were unreliable. Results are presented in this paper for a least-squares fit of 11 modes. Comparison of fits using 18 and 22 modes shows negligible differences in the lower order harmonics, discrepancies in the numbers presented in Table 2 occurring only in the last figure. Greater uncertainty is found for higher-order harmonics, but this is not sufficient to weaken any of the conclusions reached in this study.