A numerical model of cumulonimbus convection
generating a protected core

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SUMMARY

The theory of buoyant motion in a turbulent environment due to Priestley (1953) is applied to steady-state Cb updraught. An axisymmetric numerical model of columnar convection is modified to allow for radial cloud structure and internal turbulence. In this model the net vertical body force on a fluid element is the buoyancy relative to adjacent elements, and represents a residual of environment-referred buoyancy and non-hydrostatic pressure gradient.

Moisture exchange and lateral transfer of momentum and heat by mass exchange with the environment on the scale of the cloud are disregarded, yet the model gives realistic updraught velocities and cloud depth. It simulates such features as a dome rising from a protected core of the cloud, oscillating motion of cloud tops and penetration of the tropopause. In some respects the predicted behaviour compares favourably with actual measurements.

An incidental result concerns the generating gross buoyancy which is determined by the ambient and pseudo-adiabatic temperature lapse rates. It appears that the values of the latter rate published in Smithsonian Meteorological Tables and incorporated in pseudo-adiabats on aerological diagrams, are in error.

1. INTRODUCTION

Two- and three-dimensional numerical models of cumulonimbus convection are presently being developed for simulating real time growth, precipitation and interaction with precipitation downdraught. A recent such model is that by Miller and Pearce (1974) whose paper appeared after completion of the present work. This is an attempt to simulate features of Cu updraught by means of what may be termed a $1\frac{1}{2}$-dimensional model. Nowadays one might be inclined to regard Cu modelling in less than two-dimensions as perhaps no longer worth while. It is true that it has a powerful basis in the theory of turbulent entrainment which was developed by Priestley and Ball (1955) and Morton, Taylor and Turner (1956). But the application of the theory to models of Cu-type convection notably by Squires and Turner (1962), although successful in simulating updraught speed and cloud depth, failed to simultaneously account for realistic values of liquid water content. This shortcoming of entrainment theory was demonstrated by Warner (1970).

An alternative to turbulent entrainment is the mixing-rate (‘open parcel’) theory of buoyant motion put forward by Priestley (1953, 1954). To date this theory has not been applied systematically to moist convection but Warner adapted it to Squires and Turner’s model to derive the l.w.c. in Cu clouds of small to moderate dimensions. As in the case of entrainment, Warner found that with the exception of small cumuli the numerically derived ratio of l.w.c. to its adiabatic value was much in excess of observed values.

The failure of entrainment and mixing-rate theories to simulate moisture depletion by turbulent exchange with the ambient atmosphere should not, however, condemn them as unsuitable for modelling bulk convection dynamics (as opposed to field-of-motion models).

In the present paper the mixing-rate theory is applied to a Cb model differing significantly in two respects from one-dimensional entrainment models. On the one hand, the constraint to a top-hat profile of velocity, temperature, etc., is removed, non-uniformity
being allowed with restricted degrees of freedom. On the other hand while lateral turbulent transfer of momentum and heat are explicitly allowed for, lateral mass exchange by motion on the scale of the cloud and its effect on momentum and heat are disregarded together with moisture exchange by any mechanism.

The model's top-hat version will be examined first, as it brings out the effects of mixing with the environment when turbulence is explicitly external to the cloud. In the structure model provision will also be made for internal turbulence. The buoyancy forces will have to be re-formulated so that non-hydrostatic pressure is included in the vertical net buoyancy force.

The structure model generates radial differences in velocity, temperature, etc., by turbulent transfers of momentum and heat — and, as a consequence of the differential heat transfer, by conveyance of buoyancy into the interior of the cloud. Owing to these parametrically built-in transfer processes the model is capable of broadly simulating features such as a protected core (Riehl and Malkus 1958) and the familiar dome of broad based, tall cumulo-nimbi. Not being a field-of-motion model it cannot, however, simulate high-level venting which is manifest in anvil-cloud formation. Being restricted to columnar convection, the model also fails to simulate the shape, i.e. vertical change of the cloud's cross-section.

Data for quantitative tests of the model consist of radar observations of ceiling height and tropopause penetration of thundercloud tops taken by the author, and of measurements of updraught speed in Cu-congestus and Cb clouds carried out by others.

2. THE BASIC EQUATIONS OF MOMENTUM AND BUOYANCY

Notation

- $w$: vertical velocity (updraught speed)
- $\dot{w}$: vertical acceleration (= $w \cdot d\dot{w}/dz$)
- $z$: vertical co-ordinate
- $R$: radius of horizontal cross section
- $T$: temperature
- $T^*$: temperature excess over ambient air
- $T_{sb}$: temperature of saturation adiabat pertaining to cloud base
- $\gamma_d$, $\gamma_s$: dry and saturation adiabatic lapse rates respectively
- $p$: hydrostatic pressure
- $p'$: departure from hydrostatic pressure
- $\rho$: density
- $k$: turbulent horizontal mixing rate, dimension $s^{-1}$
- $K$: turbulent horizontal transfer or diffusion coefficient, dimension $cm^2s^{-1}$
- $L$: turbulent horizontal transfer term, dimension $cm s^{-2}$ or $degC s^{-1}$
- $A_i$: model design constants, non-dimensional
- $c$: cross-section shape constant, non-dimensional
- $o$, $e$: subscripts denoting values at cloud base and in ambient air, respectively
- $M$, $H$: subscripts denoting momentum and heat (enthalpy)
- $i$: subscript identifying an element of the column aggregate or the ambient air ($i = 4$ being synonymous with the subscript $e$)

Steady-state one-dimensional models rest on the assumption that the cloud is a buoyancy driven perturbation in a stagnant environment, or in an environment in uniform motion. This implies the absence of windshear in the ambient atmosphere and is not as severe a limitation to the modelling of updraught dynamics in the pre-storm stage as it
might seem; for as long as liquid water or ice are not discharged, or significantly displaced relatively to the air, a tilt of the updraught régime associated with vertical windshear will not seriously affect the release of kinetic energy by buoyancy, unless the tilt were so pronounced as to cause rupture by shearing-off.

Moisture exchange with the environment is disregarded in the present model. It is thought not to be of prime importance for the dynamics of convective updraught in the pre-storm phase, especially in conditions of large convective instability and resultant large gross buoyancy with which we shall be here concerned.

The structure model is essentially an aggregate of open 'columns' exchanging vertical momentum and heat with each other and with the environment. The basic equations will be those for a 'parcel' or 'element' in the original mixing-rate theory which, as its author pointed out, is applicable in principle also to columnar convection. Interaction with the environment occurs through lateral transfer by eddies on a subelemental scale. In columnar convection the horizontal cross section of an element may be, for example, circular.

Organized transfer across the edges of the cloud could be taken into account only by extending the model to at least two dimensions. This is avoided in the theory of continuous plume entrainment because there turbulent mixing is expressed parametrically, through the entrainment constant, by the lateral (radial) velocity component of the flow, this component being in turn eliminated by the mass continuity equation.

In order that we may apply the mixing-rate theory to Cb updraught without essentially altering its concept, the overall effects of lateral mass exchange by motion on the scale of the cloud on vertical momentum and heat, must be assumed small in comparison with those of turbulent transfer. This is a heuristic assumption. The model's capability of simulating properties of the cloud, especially horizontal structure, will be its sole justification. It implies that the continuity equation will not be part of the system of equations to be solved numerically. For this reason the shape of the cloud cannot be simulated as in the case of the entrainment models which can simulate the typical hour-glass shape (Squires and Turner 1962). Finally, the steady-state assumption means that velocity and temperature at any given height are not changing with time (apart from oscillations about the ceiling height, see section 8(b)).

Referring to unit mass of water-vapour saturated air, the relevant equations of motion and buoyancy are:

\[
\begin{align*}
   w \cdot dw/dz &= gT'/T_e + L_M \\ \\
   w \cdot dT'/dz &= -w(dT_e/dz + \gamma) + L_H
\end{align*}
\]  

where

\[
\begin{align*}
   L_M &= -k_Mw = -cK_MR^{-2}w \\ \\
   L_H &= -k_HT' = -cK_HR^{-2}T'
\end{align*}
\]  

(see notation).

In ascending motion, \(L_M\) is a damping term while below the level of thermal equilibrium with the environment (\(T' = 0\)) \(L_H\) represents an additional cooling term. In an unstable tropospheric environment that level occurs at a considerable height above cloud base.

In Eqs. (1) and (2) temperature is strictly the virtual temperature since the positive or negative buoyancy is due to a density deficiency or excess in the cloud. Having disregarded lateral moisture transfer a distinction between actual and virtual temperature is superfluous. The expedient of equating virtual to actual temperature is equivalent to assuming a gross buoyancy force \(gT_e^{-1} (T_{sa} - T_e)\) acting on cloud air embedded in a stagnant saturated environment.

In Eq. (1) a non-hydrostatic pressure gradient, \(-\rho_e^{-1}dp'/dz\) and an additional buoyancy
term \( gp_e^{-1} p' \) (cf. List and Lozowski 1970) have been omitted. The latter term is normally negligible in comparison with others, but the former term is included in the formulation of buoyancy for the structure model (section 5). Note that integration of Eqs. (1) and (2) with respect to height, after inserting for \( L_M \) and \( L_H \) the mixing-rate expressions in (1) and (2a), will yield numerical top-hat solutions for \( w \) and \( T' \) as functions of height.

3. **Top-Hat Model (External Turbulence)**

The open-parcel concept envisages, and defines parametrically, the transfer of heat and momentum from or to the element by eddies on the sub-elemental scale, the eddies being associated with pre-existing turbulence. It is therefore to be expected that a top-hat designed open column model will yield realistic velocities and cloud depths for comparatively narrow updraughts only, since internal turbulence associated with buoyancy fluctuations will there be less significant than in broad updraughts.

Criteria for dry convection derived by Priestley (1953) yield thresholds between oscillatory and asymptotic modes and between asymptotic and absolute-buoyancy (limitless ascent) modes of the motion of a buoyant element, the thresholds depending on mixing rate and element size but not on initial conditions. For example, in an unstably stratified environment with \( |dT_e/dz - \gamma_s| \) of the order \( 10^{-3} \) to \( 10^{-4} \) deg C m\(^{-1}\), the critical element radius for absolute buoyancy is of the order 100m. For larger elements the velocity will exceed the asymptotic value and the element will overshoot the asymptotic ceiling height. This aspect of the theory will be reflected in the numerical results from the top-hat model of columnar moist convection for an unstable environment as specified by the curves E and SA in Fig. 1.

![Graph](image-url)

**Figure 1.** Model stratification, dry adiabat and saturation pseudo-adiabats in skew\( T - \log p \) diagram.

E: unstable environment applying to all results except Fig. 5 where E\(_1\) applies.

SA: rigorously derived pseudo-adiabat.

SA*: same adiabat on Refsdahl's skew\( T - \log p \) aerological diagram (see appendix I).
A comparison with the inherent performance of entrainment models is desirable as it will give a better appreciation of the structure model. The momentum and buoyancy equations (1) and (2) are basic also to models of turbulent plume entrainment but in Eqs. (1a) and (2a) the turbulence terms are replaced by \( L_M = -\alpha R^{-1}w^2 \) and \( L_H = -\alpha R^{-1}T'w \), where \( \alpha \) is the entrainment constant. Setting for simplicity \( K_M = K_H = K \), the ratio of the relevant turbulence terms is given by

\[
e = \frac{L(\text{m. rate})}{L(\text{entrainm.})} = \frac{cK}{\alpha Rw} \tag{3}
\]

If \( e = 1 \), the performances of the models will be similar, indeed identical but for the change of \( R \) with elevation in the entrainment model. For this case, Table 1 lists values of \( K \) corresponding to various combinations of \( w \) and \( R \). The entrainment constant \( \alpha \) has the value 0·1 and the form constant \( c \) has the value 8, see section 5(c). For comparatively broad based clouds \( (R \geq 1\text{ km}) \) Squires and Turner obtained maximum updraught speeds of the

\[
\begin{array}{cccc}
R&(km) & w (m s^{-1}) \\
10 & 13 & 25 & 38 \times 10^6 \text{cm}^2 s^{-1} \\
20 & 25 & 50 & 75 \times 10^6 \text{cm}^2 s^{-1} \\
30 & 50 & 100 & 150 \times 10^6 \text{cm}^2 s^{-1} \\
\end{array}
\]

order 10 to 20m s\(^{-1}\) for ascent in a saturated environment of comparable convective instability as specified by curves E and SA in Fig. 1 and for the cloud base values \( w_0 = 1\text{·}0\text{m s}^{-1} \), \( T' = 0 \). The diffusion coefficients indicated in Table 1 for velocities \( w \) of the above magnitude are in the range \( 10^6 \) to \( 10^7\text{cm}^2 s^{-1} \) whereas, as will be seen, \( 10^5 \) to \( 10^6 \) is more appropriate to the scaling of large convective elements. The ratio \( e \) will thus be less than unity so that, other things being equal, the net turbulent transfers of momentum and heat to the ambient stagnant and cooler atmosphere are less in the mixing rate model than in the entrainment model. This will account for the unrealistically large velocities resulting from the top-hat version of the latter model for all but very narrow updraughts (section 5).

4. The horizontal diffusion coefficient

With regard to the structure version of the model, mixing by eddies scaling well below element size might well be different from mixing across the sharp-edged lateral boundary of the aggregate. Telford (1966) proposed a modified entrainment mechanism for clear-air convection, entrainment being scaled with the intensity of internal turbulence (Reynolds stresses) rather than the mean flow. Morton (1968) argued that such formulation of entrainment does not allow turbulence generated by fluctuating buoyancy. In the present context, one should then discriminate between mixing rates for external and internal turbulence involving different values of the heat and momentum transfer coefficients for the environment and the cloud. It would be, however, somewhat meaningless to assign, by trial and error, various combinations of internal and external mixing rates for the purpose of arriving at realistic velocities, cloud depths, etc.; for a proliferation of adjustable constants will obscure the interpretation of the results from a numerical model. For this reason, too, a range of acceptable values of the diffusion coefficient will be set in advance.

Scanty information on the diffusion coefficient appropriate to convective elements
calls for the following simplifying assumptions: (i) the rate of mixing with the environment equals the rate of mixing between adjacent elements; (ii) \( K_M = K_H = K \), whence \( k_M = k_H = k \); (iii) \( K \) is constant with respect to height, whence \( k \) is also constant since in columnar convection \( R \) does not change with height. With these assumptions, to any given \( k \) value a range of \( K \) values rather than a single value should be assigned. This implies a corresponding range of the element radius. A search of the literature for representative \( K \) values is briefly summarized.

Warner (1970) in the above-mentioned adaptation of mixing-rate theory to a numerical assessment of liquid water content in small to medium Cu clouds adopted a \( K \) value as low as \( 5 \times 10^6 \text{cm}^2 \text{s}^{-1} \). Pasquill (1962) analysed data from measured cross-wind spread at varying distances from a point source. Taking 5km as representative of convection scale, Pasquill’s regression of \( K \) on distance gives a value \( 8 \times 10^5 \). Taylor et al. (1971) carried out extensive measurements of turbulent fluctuations and horizontal gradients of wind and temperature within and above a smoke plume with an associated large Cu cloud which were generated by a controlled surface fire of comparable horizontal dimensions. The average value of \( K \) deduced from these data is \( 7 \times 10^5 \). Setting thus the limits of an acceptable range of \( K \) as \( 10^5 \) and \( 10^6 \), on account of Eqs. (1a) and (1b), an experimental value for \( k \) implies a range of the radius \( R \), its lower and upper limits differing by a factor of \( \sqrt{10} \). With this tolerance it will still be possible to interpret vertical velocity, temperature and cloud depth as functions of the horizontal dimension of the element or aggregate of elements.

5. The structure model

(a) Geometry and horizontal turbulent mixing

Ideally the model would consist of an axisymmetric aggregate of columns with identical cross section shape and size. Except for slightly unequal density associated with non-uniform buoyancy, the elements at any level will then have the same volume and mass. These requirements are met most closely by a symmetric array of columns with hexagonal or diamond shaped cross-section. A simpler, more workable design approaching closely the ideal one is shown in Fig. 2. A centrally placed circular shaped element (the central column) is surrounded by concentric rings (annuli). The number of ring-shaped regimes determines the degrees of freedom of differentiation. Here we chose two such regimes whence, together with the ambient

\* The low value may well account for the high liquid water content deduced by Warner.
atmosphere surrounding the aggregate of columns, there are three degrees of freedom, or \( i = 1, \ldots, 4 \) as explained in the notation.

From the requirement of equal cross-section size for all elements it follows that the intermediate regime will consist of 6 elements and the rim of 12 elements. Denoting the radius of the central core by \( R_1 \), then \( R_2 = \sqrt{7} R_1 \) and \( R_3 = \sqrt{19} R_1 \) (see Fig. 2). For comparison with top-hat design, \( R_3 \) is to be identified with the radius \( R \) of that design.

Axial symmetry implies equality of all properties among the elements within both the intermediate régimes and the rim. It will thus suffice to specify mixing between elements 1, 2, 3 and the environment. The reason for nominating the elements 2 and 3 rather than the corresponding rings, is that this avoids the necessity for introducing mass-dependent weighting factors in the calculation of mean velocity, temperature excess, etc., in these régimes.

The turbulent mixing rate terms \( L \) in Eqs. (1a) and (2a) will have the form

\[
L_i = k \Sigma \left( A_i \frac{Q_i - Q_i \pm 1}{Q_i \pm 1} \right)
\]

the symbol \( Q \) signifying here velocity \( w \) or cloud temperature \( T \). The signs indicate that the elements farther in and farther out from the centre of the aggregate refer in turn. The constant factors \( A_i \) are derived in appendix II.

(b) Buoyancy force and pressure gradient

How is 'environment' to be defined for any one of the three sample elements? Conventionally, one would define it as the ambient atmosphere in which case the buoyancy forces equal \( g(T_{i} - T_{e})T_e^{-1} \). However, the consideration of individual fluid columns within a mass of buoyantly rising air demands that buoyancy be formulated as local, i.e. density deficiency or excess must be regarded as relative to adjacent fluid columns. The pressure gradient \(- dp/dz\) must then no longer be equated to the hydrostatically balanced gradient in the ambient atmosphere, \(- dp/dz(= gp'_e)\), as is customarily done with respect to a horizontally uniform mass of buoyant fluid. Therefore, the locally defined buoyancy force will now represent the vertical body force exerted on the element and will equal the sum of the forces of buoyancy relative to the ambient hydrostatically balanced atmosphere and the unbalanced part of the vertical pressure gradient force:

\[
g \Sigma \left( A_i \frac{T_i - T_i \pm 1}{T_i \pm 1} \right) = g \frac{T_i - T_e}{T_e} - \frac{1}{\rho_e} \frac{dp_i}{dz}
\]

where on the r.h.s. the density \( \rho_i \) has been approximated by \( \rho_e \) applying as before the Boussinesq approximation. Since by definition \( p'_i \) is the local departure of air pressure from hydrostatic pressure in the ambient atmosphere, it represents a perturbation and therefore a non-hydrostatic pressure. It could be argued that the perturbation pressure gradient is simply proportional to the difference in hydrostatic pressure gradient as between any one element and the environment, but this is not so; for that difference is given by \((\rho_i^{-1} - \rho_e^{-1}) + dp'/dz\) and vanishes on account of the approximation in the second term on the r.h.s. of Eq. (5).

An explanation for the identity of weighting factors \( A_i \) in Eqs. (4) and (5) is given in appendix II.

The equations of momentum and buoyancy for the structure model take the following form exemplified here only for element 3:

* The 'sum' is actually a residual of opposing forces.
\[ w_3 \frac{dw_3}{dz} = A_{3,e} \left[ \frac{g(T_3 - T_e)}{T_e} - kw_3 \right] + A_{3,2} \left[ \frac{g(T_3 - T_e)}{T_2} - k(w_3 - w_2) \right] \]  \hspace{1cm} (6)

and

\[ w_3 \frac{dT_3}{dz} = -w_3 \gamma_{sa} - k\left[ A_{3,e}(T_3 - T_e) + A_{3,2}(T_3 - T_2) \right] \]  \hspace{1cm} (7)

(c) Computational procedure

Introducing in Eqs. (1) and (2) the turbulent mixing and buoyancy terms of the form indicated in Eq. (4) and left side of (5), and specifying the ambient temperature as function of height (see appendix I), numerical solution of \( w_i \) and \( T_i \) as functions of height were obtained; also of \( \dot{w}_i, T'_i, T_e, p_e, dT_e/dz \) and \( \gamma_{sa} \). The evaluations of the saturation-adiabatic lapse rate \( \gamma_{sa} \) and the pseudoadiabat \( T_0 - \gamma_{sa}(z - z_0) \) are given in appendix I.

The calculations were programmed using height steps for the integrals so that the time elapsed for each parcel was different:

\[ w_i = \int_{z_0}^{z} (...) w_i^{-1} \, dz = \int_{0}^{t} (...) \, dt \]

and

\[ T'_i = \int_{z_0}^{z} (...) w_i^{-1} \, dz = \int_{0}^{t} (...) \, dt \]

The integrations were carried out using a standard Runge-Kutta routine with 50m equivalent steps although smaller steps were required in the asymptotic domain of \( w_3 \), the integration being terminated when \( w_3 \to 0 \).

The results of computer runs with mixing rates varying over a very wide range, viz. from 0.0005 to 0.95s\(^{-1}\), were obtained with the cloud base conditions \( (w_i)_0 = 1\, m\, s^{-1} \) and \( (T_i)'_0 = 0 \). Various additional runs for selected values of \( k \) were made to examine the effects of larger initial momentum \( w_0 \) or of finite positive temperature excess \( T'_0 \); also of non-uniform momentum or temperature excess at cloud base. Lack of space does not permit us to present the results of these experiments.

Different mixing rates must be assigned to top-hat and structure models for one and the same horizontal dimension of the updraught column and a given \( K \) value. In the latter model the equivalent cross section radius of each element of the aggregate is equal to \( R_1 \), the radius of element 1 or central core (Fig. 2). The mixing rate pertaining to all elements is therefore \( k = cKR_1^{-2} \) where \( c = 8 \) for the central core (Priestley 1953), this value being assigned also to the other elements. On the other hand, the mixing rate for the top-hat cloud of lateral dimension equal to the structured cloud, is given by \( k = cKR_3^{-2} = cK(\sqrt{19} \cdot R_1)^{-2} = (1/19)cKR_1^{-2} \). For this reason in Figs. 3 and 4, showing results from the structure model, \( k \) values relating to the top-hat curves are to be divided by 19.

On account of the termination of computer runs at the asymptotic height \( (z_3)_{\text{max}} \) ceiling heights of the central core and the intermediate region, i.e. elements 1 and 2, were derived by extrapolating distributions of \( w_1 \) and \( w_2 \) with height above cloud base. Once the element 3 attained its ceiling height, the radius of the aggregate nominally decreased from \( R_3 \) to \( R_2 \). Relevant top-hat computer outputs were examined to adjust the extrapolated velocity-height curves. For almost the entire range of \( k \) this procedure entailed negligible errors in evaluating \( (z_3)_{\text{max}} \) but slight errors were involved in evaluating \( (z_1)_{\text{max}} \).

The non-hydrostatic or 'dynamic' pressure gradient was derived from Eq. (5) but this necessitated the solution of a set of equations similar to (6) and (7) except that the generating buoyancy was replaced by the conventional term \( g(T'_i - T_e)T_e^{-1} \).
Figure 3. Maximum updraught velocity as function of mixing rate $k$ and radius of cross section $R$ scaled for two values of the turbulent transfer coefficient $K$. Curves labelled I refer to the structure-model equations with buoyancy defined as local; curves labelled II with buoyancy relative to cloud environment. (a) small to medium radii (b) medium to large radii. Asterisks indicate computer runs.

6. RESULTS

In Figs. 3 and 4, the label CORE applies to both the central core and the intermediate regions. In accordance with the specifications in section 4 the aggregate radius $R(\equiv R_3)$ is indicated on two separate scales for $K$ values $10^5$ and $10^6$. The curves labelled I refer to the structure model equations (6) and (7) while the curves labelled II refer to the similar equations in which the buoyancy term is replaced by the conventional term implying the
omission of non-hydrostatic pressure according to Eq. (5). Curves for top-hat profile are shown for comparison, the values of $k$ to be divided by 19 as was explained in section 5.

The main feature conveyed by Figs. 3 and 4 is an increase of maximum updraught velocity and cloud depth with increasing diameter. However, as will be seen from Fig. 3(b), the maximum updraught velocity in the CORE tends to decrease for diameters exceeding a relatively large value.

Fig. 5 illustrates radial structure of velocity, temperature excess over the ambient atmosphere, and ceiling heights for three values of $k$ which correspond to narrow, medium and large-sized cross sections of the cloud. In these diagrams the transfer coefficient $K$ has been set equal to $10^6$ for two reasons. Firstly, the range of the updraught radii $R$ in Figs. 3 and 4 pertaining to the $10^6$ scale of $K$ is a priori more realistic than that pertaining to the $10^5$ scale. Secondly, measurements of updraught speed and cloud base diameter agree with the (structure) model results significantly better with the higher of the $K$ values in question, as will be shown further below (see section 8(a)). The environment differs slightly from that
Figure 5. Showing half of vertical cross section through model cloud column. The modelling equations are those for curves labelled 1 in Figs. 3 and 4. Lower arrow indicates \( \frac{1}{2} w_{\text{max}} \), middle arrow \( w_{\text{max}} \), upper arrow (where inserted) again \( \frac{1}{2} w_{\text{max}} \). Temperature excess or deficit in degC relative to cloud environment. Vertical and horizontal dimensions not drawn exactly to scale. \( K = 10^6 \text{cm}^3\text{s}^{-1} \), see text.

Applying to Figs. 3 and 4 in that the spurious shallow adiabatic layer just above cloud base in curve E of Fig. 1 was omitted so that curve E of Fig. 1 applies. The stratification being less unstable, vertical velocity and cloud depth are somewhat reduced in comparison with Figs. 3 and 4.

Figure 6. Buoyancy forces and non-hydrostatic pressure gradient as functions of height, see Eq. (5). Mixing rate \( k = 0.0019 \text{s}^{-1} \).
It will be seen from Fig. 5 that comparatively broad updraughts exhibit a dome-shaped top penetrating the tropopause. As an observed phenomenon, tropopause penetration of thunderheads has an important bearing on the injection of water substance into the stratosphere. Oscillations associated with overshooting of the thermal equilibrium level and their bearing on tropopause penetration will be dealt with in section 8(b).

Referring back to Fig. 3, the velocities pertaining to the set I curves are greatly reduced in comparison with set II: in the former case, vertical momentum is generated by the coupling of vertical body forces and lateral mixing of momentum and heat, the resultant body force being the residual of the terms on the r.h.s. of Eq. (5). This is demonstrated in Fig. 6, where the environment buoyancy force is opposed by the non-hydrostatic pressure gradient.

We cannot give a convincing assurance that the above briefly outlined results are not dependent upon the particular design of the model, e.g. the choice of the number of rings in Fig. 2. But we believe that despite the many simplifications made in this treatment, the processes controlling salient features of the internal structure of the Cb updraught are reasonably well presented by the model.

7. DISCUSSION: THE GENERATION OF INTERNAL STRUCTURE

Height integration of the pressure gradient \( dp'/dz \) with the initial condition \( p'_0 = 0 \), yields \( p' \) as function of height. An example of the resulting lateral stress is shown in Table 2, from which one may infer the unexpected result that in a comparatively broad-based buoyant updraught a relative pressure excess rather than a deficit occurs in the warm core through much of the vertical reaches of the cloud. However, the perturbation pressures \( p'_1, p'_2, p'_3 \).

<table>
<thead>
<tr>
<th>Height above cloud base (km)</th>
<th>( p'_1 - p'_2 ) (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00</td>
</tr>
<tr>
<td>1.0</td>
<td>0.13</td>
</tr>
<tr>
<td>2.0</td>
<td>0.32</td>
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<tr>
<td>4.0</td>
<td>0.52</td>
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<tr>
<td>5.0</td>
<td>0.55</td>
</tr>
<tr>
<td>6.0</td>
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<tr>
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<td>0.21</td>
</tr>
<tr>
<td>9.0</td>
<td>-0.03</td>
</tr>
<tr>
<td>9.8</td>
<td>-0.33</td>
</tr>
</tbody>
</table>

Table 2. The lateral stress (dynamic pressure difference \( p'_1 - p'_2 \)) as function of height above cloud base. The mixing rate \( k = 1.9 \times 10^{-3} \text{s}^{-1} \) corresponds to an updraught radius of 2.83 km when \( K = 10^6 \text{cm}^2\text{s}^{-1} \).

They are themselves negative from 1 to 3 mb in qualitative agreement with observation. The mentioned pressure excess in the core of the cloud is a somewhat uncertain result since it might well be inconsistent with the lateral flow direction implied by continuity of mass, particularly at levels where the magnitude of the peripheral updraught exceeds that of the inner core. Nevertheless, Smith, Morton and Leslie (1975) arrived at an analogous result from a numerical 2-dimensional field-of-motion model of dry convection over a strip fire. In such conditions the terms equivalent to those on the r.h.s. of Eq. (5) are naturally very large just above the intense buoyancy source but decrease rapidly upward. In Cb updraught the gross buoyancy source associated with the release of latent heat is distributed through a considerable depth of the cloud which accounts for the distribution in Fig. 6.
Some of the significant processes controlling both the fire convection and Cb convection are best described by quoting from the paper in question: 'In natural convection the fluid motion is generated wholly by the gross buoyancy field and consists of ascent of heated fluid (and its replacement by cooler ambient fluid). However, the buoyancy is strictly a local force which acts to modify the relative position of horizontally adjacent fluid elements at different temperatures or specific weights, and is communicated between distant elements of the fluid only through the dynamic pressure field. . . .'

In the light of the foregoing considerations, the results may be discussed in some more detail: (i) comparatively narrow clouds possess a warm jet core, the erosion of momentum and (additional) depletion of heat being dominated by mixing at the edges bordering onto the turbulent environment; (ii) with broader updraughts the core has still a larger ceiling height than the outer parts, but the central jet vanishes and the highest updraught speed occurs in the peripheral regions. With increasing radius the rim velocity continues to increase whereas the core velocity decreases slightly although the inner parts of the cloud retain the largest temperature excess over the ambient atmosphere i.e. up to the level of thermal equilibrium.

The retarded build-up of vertical momentum in the inner regions of a broad based cloud is attributed to the process by which buoyancy is conveyed to these regions. Smith et al. showed that this is associated with the momentum generated by the horizontal component of the dynamic pressure field, the equivalent of which in the present model are the discrete stresses $p_i - p_{i-1}$. Since, however, radial mass flow is not part of the present model, the buoyancy in this case is conveyed by turbulent heat transfer only instead of a combination of such transfer and an organised lateral flow.

The combined effects of buoyancy being communicated from element to element and of the turbulent transfer of momentum is shown in Fig. 7. The first effect is seen in the rapid height increase of the local buoyancy force in the outer region and its subsequent rapid decrease, in contrast to its gradual and higher extending build-up in the inner region. This explains the comparatively high momentum generated at low levels in the outer region. Again, up to some height depending on the available gross buoyancy, rim momentum is being lost by turbulent mixing, to both the environment and the inner regions of the cloud; for the resulting acceleration in the rim is less whereas in the core it is larger than the net buoyancy force, as is apparent from Fig. 7.

In nature there is a counterpart to these processes in the formation of a protected core in Cb clouds. The existence of such a core was first proposed by Richter and Malkus (1958).

There is a known tendency of very broad based convection clouds, especially those forming by orographic or frontal lifting, to flatten out rather than to grow into fully developed cumulonimbi, even in the presence of high convective instability. This pheno-

![Figure 7. Net local buoyancy and resultant acceleration. Mixing rate $k = 0.0019 s^{-1}$.](image)
menon suggests a greatly impeded communication of buoyancy from the inner to the outer regions owing to the distance involved. Photographic and visual evidence from Cu-cong. cloud supports the interpretation: there is in such cases peripheral development giving the appearance of towering while there is suppressed vertical growth in the central parts of the cloud (the middle part as seen in profile). Although a steady-state model cannot of course simulate phases of cloud growth, its results are nevertheless applicable to the prediction of maximum development providing a quasi-stationary state is reached in the vicinity of the maximum.

8. Observational evidence

(a) Maximum updraught speed

From experiments with radar tracking of towed free-lift balloons and corner reflectors in Cu-cong. and Cb clouds, Sulakvelidze (1966) deduced an average maximum updraught speed of 12m s\(^{-1}\) at an average height of 2km above cloud base. As part of the National Hail Research experiment in N.E. Colorado; Marwitz (1972) measured horizontal and vertical air velocity by radar tracking of chaff released into the updraught (weak echo) region of storm clouds. The average \(w_{\text{max}}\) from 17 measurements in Cu-cong. and Cb clouds was 14.1m s\(^{-1}\) with a s.d. of 3.6. In the weak-echo updraught regions of 15 hailstorms, \(w_{\text{max}}\) averaged 17-4m s\(^{-1}\), s.d. 4-2.

Marwitz had drawn attention to the surprisingly low magnitude (and altitude) of maximum updraught speeds measured by the Soviet team. However, when his data largely confirmed the Soviet scientist's results, he was satisfied that there was little doubt about their validity.

The Colorado updraught data are entered on Fig. 3(b) on the abscissa scale for \(K = 10^6\) at radii 2.4 and 4.0km, the first of these values being based on chaff trajectories given in Marwitz' paper, the second on seeder aircraft measurements at the base of updraughts (NHRE 1973; and unpublished data). As will be seen, the observational data fit the set I curves for \(w_{\text{max}}\) rather well. Related to the \(K = 10^5\) scale, agreement would be not so good though incomparably better than with the set II and top-hat curves.

(b) Tropopause penetration

For much of the experimental range of mixing rate the top-hat model programme outputs \(w\) and \(T'\) beyond the ceiling height \(z_{\text{w}}\), resulting in loops or oscillations with varying amplitude and damping. These are the counterpart of the oscillatory damped modes among the analytical solutions in Priestley's open parcel theory briefly discussed in section 3. The level about which an oscillation occurs is invariably above the thermal equilibrium level, but was located either below or above the tropopause of the model atmosphere shown in Fig. 1. The amplitude is the larger the smaller the mixing rate \(k\). The amplitude of loops pertaining to the structure version of the model should then be considerably smaller than those generated by the top-hat version for the same radius. As was explained in section 5(c), for a given radius \(R_3\) of the aggregate the values appropriate to the structure and top-hat models are in the ratio 19:1. Oscillations referred beneath to CORE are based on top-hat computer outputs applying \(k\) values appropriate to the ratio of the radius \(R_2\) to the element radius \(R_1\), viz. \(\sqrt{7}\).

Radar observations of thundercloud tops were examined to see whether the oscillations have a counterpart in Cb convection in the real atmosphere. The storms occurred in the Brisbane (Queensland) area during the period of soundings from which the model environment curve E in Fig. 1 was constructed. Time sequences of radar measurements
Figure 8. The data plots indicate rise and fall speeds of radar echo tops of thunderclouds not penetrating the tropopause (for definition see text). Average reflectivity black-hole radius 1.5km; matching updraught radius for velocity curves from model 1.7km. $K = 10^6$ except where otherwise indicated.

indicated quasi-periodic rises and falls of individual echo tops with periods of the order 20 minutes which was about twice as long as the oscillation period in the model outputs of vertical velocity (the period changing very little with mixing rate). A systematic study of amplitude as well as period was not feasible with the available data. Instead rise and fall speeds were determined for two groups of data: (i) one or both of any two successive echo-top heights used in the derivation of rise or fall speed, exceeded the tropopause height indicated on the antecedent sounding; (ii) tropopause penetration did not occur in either of these measurements.

Rise and fall speeds are plotted for case (ii) in Fig. 8 and for case (i) in Fig. 9. The

Figure 9. As in Fig. 8 but echo tops penetrating the tropopause. Average reflectivity black-hole radius, 4.0km; matching updraught radius 3.9km.
superimposed curves of vertical (air) velocity pertaining to the structure model are results from solutions of equations of type (6) and (7) assuming $K$ to have the value $10^6$ consistently with Fig. 5. Specifically, these results are for mixing ratios $k$ of values such that the corresponding radius $R$ matches a radar parameter which is correlated to the updraught area. This is the equivalent radius of a high-reflectivity radar echo ascribed to accumulation of water being typically located above the weak-echo updraught (Marwitz et al. 1972). Although primarily correlated with cloud depth, this high-reflectivity parameter is also a measure of the horizontal dimension of the parent Cb cloud (Berson 1971). Specifically, the here adopted parameter is the equivalent radius of the maximum reflectivity area painted as a 'black-hole' on PPI and RHIs displays of the Plessey WF 44S radar operating with 0–60 nautical mile swept gain and pulse length 1.5km. The corresponding reflectivity factor threshold is $Z = 8 \times 10^5 \text{mm}^6/\text{m}^3$.

The velocity curves shown in Figs. 8 and 9 were calculated for radii $R$ closely approximating the average values of the mentioned radar parameter. Significantly, they are 1.5km in the case of non-penetration, but 4-km in the case of penetration of the tropopause. Now it will be seen that in either case the derived oscillation loops more or less envelop the plots of the echo tops with respect to both height and rise/fall speed. For comparison top-hat model loops are also shown for the same matching radii $R$. The velocity oscillations are almost undamped and their amplitude much too large to start with. On Fig. 8 an additional loop for $K = 10^7$ is shown to demonstrate that while the ceiling height can be made to fit the observations, the amplitude cannot.

The echo top speeds had been so far tacitly regarded as a measure of vertical air motion which is only partly true because of the movement of the reflecting hydrometeors (largely consisting of ice particles) relative to the air. Nevertheless, the results are significant in that the structure model predicts the height about which oscillatory modes of motion would seem to occur at the top of storm clouds, the order of magnitude of the oscillation amplitude, and penetration of the tropopause by cloud tops.

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NUMERICAL MODEL OF CUMULONIMBUS CONVECTION


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**APPENDIX I**

**Specification of the environment**

The average of five soundings at Brisbane during a thunderstorm period 26 Nov. to 3 Dec. 1968 can be represented to a good approximation by the curve labelled E in the skew$T - \log P$ diagram, Fig. 1. It consists of three linear parts whence denoting their slope by $a_j$, the environment temperature gradient in Eq. (2) can be written as

$$
\frac{dT_e}{dz} = a_j \frac{d \log p_e}{dz} = a_j \frac{d \log P}{dz} = a_j \frac{g}{RT_e} \quad j = 1, 2, 3
$$

where $R$ here denotes the gas constant. The ambient temperature $T_e$ pertaining to curve E in Fig. 1 is then given as function of height by the equation

$$
T_{ej} = T_{e0} - \sum_{j=1}^{3} \int_{z_{j-1}}^{z_j} \frac{a_j g}{RT_e} \, dz
$$

**The saturation pseudo-adiabat**

In the subroutine for the pseudo-adiabatic lapse rate the following standard formula was used:

$$
\gamma_{sa} = \gamma_a \left( 1 + \frac{0.622 L e_s}{RT_p} \right) \left( 1 + \frac{0.622 L e_s}{c_p p} \frac{dT}{dT} \right)^{-1}
$$

where $L$ is latent heat of vaporization of water, $c_p$ specific heat of air at constant pressure and $e_s$ saturation vapour pressure. The Goff-Gratch formulation for $e_s$ in the pure phase over a plane surface of pure water (Smithsonian Meteorological Tables, 6th Ed., 1958, Eq. (1), p. 350) was applied and the variation of $e_s$ with temperature was deduced by differentiating the Goff-Gratch equation without any subsequent approximation. The resulting pseudo-adiabat commencing at $T_0 = 286K$, $p_0 = 780mb$ (and saturation mixing ratio $q_0 = 0.012$) is shown as the curve SA in Fig. 1.

In the Smithsonian Met. Tables on top of p. 372 there is an approximate expression for $de_s/dT$ while Tables 78 on pp. 318–322 list temperatures and pressures along pseudo-adiabats based on a differential equation of the pseudo-adiabatic condensation stage involving
saturation with respect to water'. As author of that equation, von Bezold is cited and annotated with the year 1888!

We have examined all known kinds of aerological diagrams and found that the pseudo-adiabats displayed on them are more or less consistent with Tables 78 in Smithsonian Met. Tables. The curve labelled SA* in Fig. 1 is the relevant pseudo-adiabat on the Refsdahl skewT - logP diagram which is also the format of Fig. 1.

The discrepancy between the curves SA and SA* is considerably larger than between the corresponding pseudo-adiabats among the various aerological diagrams, and also between these and the cited Tables 78. The most likely explanation of this state of affairs is the underlying approximation of the expression for $\frac{de}{dT}$.

The adiabat SA is 1 to 2 degC cooler than SA*. This will be the order of magnitude of the discrepancy for much of the temperature range in middle and low geographical latitudes shown on aerological diagrams.

To give an indication of the size of the discrepancy in terms of kinetic energy released by buoyantly ascending saturated mass of air, it suffices to apply the ordinary 'closed' parcel method. With the above mentioned initial values of $p_0$, $T_0$ and $d_0$ and the environment labelled E, the maximum updraught speed computed with the pseudo-adiabat SA is 38 m s$^{-1}$ whereas with the pseudo-adiabat SA* it is 50 m s$^{-1}$. This represents a difference in kinetic energy of over 1000 m$^2$s$^{-2}$, i.e. a decrease by 42% of the value based on SA* and thus routinely derived.

**APPENDIX II**

*Derivation of the factors A*

The mixing of momentum and heat is formalized so as to take place between any one of the three elements and its immediate environment (Fig. 2). The mixing occurs across concentric circles, but here it need be defined only per unit mass of a sample element. Thus element 2 and 3 interchange property across inner and outer arc-shaped boundaries while element 1 exchanges property with element 2 across the circle $2\pi R_1$ only. Denoting the common boundaries between elements by $B_{1,2}$, $B_{2,1}$, $B_{2,3}$ and $B_{3,4}$ (= $B_{3,a}$) the factors $A$ are given by the ratio of any one of these boundaries to the length of the element's periphery. For example, the outer boundary of element 2 is $B_{2,3} = (2\pi\sqrt{7})/6$; the periphery of this element is given by $(\pi/3 + 2)\sqrt{7} + (\pi/3 - 2)$ whence $A_{2,3} = (\pi\sqrt{7})/3 [(\pi/3 + 2)\sqrt{7} + (\pi/3 - 2)]^{-1} = 0.3897$.

A listing of the factors applying to Eq. (4) follows:

\[
\begin{align*}
A_{1,2} &= 1.0 \\
A_{2,1} &= 0.1473 \\
A_{2,3} &= 0.3897 \\
A_{3,2} &= 0.1953 \\
A_{3,4} &= 0.3217 \quad (= A_{3,a}).
\end{align*}
\]

The same factors apply to the buoyancy terms on the l.h.s. of Eq. (5). When buoyancy is defined as local, i.e. relative to adjacent fluid elements, then in our model the buoyancy equals $g$ multiplied by the weighted average temperature excess (deficit) of the element over its environment. The weighting factors of the radial temperature differences $T_1 - T_{1±1}$ will be obtained in the same manner as in the case of mixing between element and its environment, and are identical to those listed above.