A split semi-implicit reformulation of the Bushby-Timpson 10-level model

By D. M. BURRIDGE*
Meteorological Office, Bracknell

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SUMMARY

A split semi-implicit reformulation of the Bushby-Timpson 10-level model is described. The integration cycle is split into two main parts. Firstly, the dependent variables are advected by the wind using the explicit two-step Lax-Wendroff integration scheme with the grid staggered in time and space. Secondly, these advected values are adjusted to account for the terms that govern the motion of small amplitude inertia-gravity waves in the model, using a combination of an unconditionally stable implicit scheme and a conditionally stable explicit scheme. A novel feature of the scheme is the implicit treatment of only two of the model's ten gravity wave modes. This split semi-implicit scheme is more than four times as efficient as the original explicit scheme designed by Bushby and Timpson. The results of one semi-implicit integration to 36 hours are described and compared with an integration made with Bushby and Timpson's original explicit scheme.

1. INTRODUCTION

The widespread use of primitive equation models in place of filtered baroclinic models in routine numerical weather prediction has led to some intensive research on efficient numerical methods designed to overcome the computational disadvantages associated with the primitive equation approach. The most commonly used schemes are explicit, non-iterative time marching schemes, which are subject to severe stability criteria which restrict the time-step, $\Delta t$, that can be used. For the leap-frog scheme with simple centred difference approximations to the spatial derivatives with an un staggered grid we must have $\Delta t < 2^{-4} \Delta x / (C + V_{\text{max}})$ for a stability, whilst for the two-step Lax-Wendroff scheme, with the grid staggered in both space and time as suggested by Eltassen (1956) and used by Bushby and Timpson (1967), we require

$$\Delta t < \Delta x / (2(C^2 + V_{\text{max}}^2))^{\frac{1}{4}}$$

(1)

$C$ is the speed of the fastest moving gravity wave in the model, $V$ is the wind speed and $\Delta x$ is the horizontal grid length; since $C \approx 300 \text{ m s}^{-1}$ and $V$ rarely exceeds $100 \text{ m s}^{-1}$ these criteria are dominated by the speed of the gravity waves, which themselves form only an insignificant part of the pressure amplitude of synoptic scale weather disturbances, and with $\Delta x = 100 \text{ km}$ allow a maximum time-step of about three minutes. The stability criterion for filtered models, which have no gravity wave solutions, takes the form

$$\Delta t < \Delta x / (\sqrt{2} V_{\text{max}})$$

(2)

with a leap-frog scheme which is much less severe than (1).

With a stable algorithm, time integrations in steps of about 15 minutes, with $\Delta x = 100 \text{ km}$, could proceed many times faster and would still give sufficiently accurate predictions. Semi-implicit methods belong to this category. Semi-implicit methods for primitive equation models involve the implicit treatment of the terms in the model's equations that govern the motion of small amplitude gravity waves and stability criteria for these methods can be made to depend on the wind speed giving criteria such as (2), thus allowing a much larger time-step.

* Present address: European Centre for Medium Range Weather Forecasts, Fitzwilliam House, Bracknell.
The essential difference between explicit and implicit schemes is most easily understood by considering some time-stepping schemes for the equation \( \partial X/\partial t = F(X) \). The centred scheme \( X^{n+1} = X^n - 1 + 2\Delta t F(X^n) \) defines \( X^{n+1} \) explicitly in terms of values at earlier time levels, whereas the scheme \( X^{n+1} = X^n + \frac{1}{2}\Delta t(F(X^n) + F(X^{n+1})) \) defines \( X^{n+1} \) implicitly in terms of itself and in terms of earlier values and \( X^{n+1} \) is subsequently determined by an inversion which generally involves the inversion of a matrix either by a direct or an iterative procedure.

Most of the research on implicit and semi-implicit techniques for meteorological models has been done by atmospheric scientists in the USSR; in particular Marchuk (1965) has developed many highly efficient and practical algorithms. Outside the USSR semi-implicit integrations have been made by Robert (1968) and Kwizak and Robert (1971) for a non-linear barotropic model and recently Robert et al. (1972) have described a semi-implicit reformulation of the baroclinic Schuman–Hovermale (1968) model.

The fine mesh version of the Bushby–Timpson (1967) 10-level model has been reformulated to use a split semi-implicit scheme in which the model's two fastest moving gravity waves are treated implicitly and the remaining eight modes are treated explicitly. In this split scheme the integration cycle for each time-step is split into two main parts. Firstly, the dependent variables are 'advected' by the wind using the two-step Lax–Wendroff scheme with a grid staggered in both space and time. In the second part these 'advected' values are adjusted to account for the terms governing the motion of the model's inertia-gravity waves using a combination of stable implicit and explicit schemes. Two Helmholtz equations have to be solved, one for each of the gravity waves that has been treated implicitly, for each time-step of the integration. The fine mesh version of the 10-level model (\( \Delta x = 100 \text{km} \)) is integrated using this scheme with a 12 minute time-step compared with the 24 minute time-step used with Bushby and Timpson's explicit scheme and results of integrations with both these schemes are compared.

Since the original description of the 10-level model by Bushby and Timpson and the publication of the revised version by Benwell et al. (1971) a number of changes have been made to the model's equations and to the vertical distribution of the model's variables, so in the next section a brief description of the model is given which covers the stages of developments up to the end of 1973.

2. THE GOVERNING EQUATIONS

The horizontal co-ordinates \((x,y)\) are taken on a stereographic map projection with the origin at the north pole and the hydrostatic pressure \(p\) as the vertical co-ordinate. The vertical resolution is 100 mb, the model atmosphere being bounded by the 1000mb and 100mb surfaces. Horizontal components of velocity \(u\) and \(v\) are used to define the motion on the pressure surfaces \(p = 1000\text{mb}, 900\text{mb}, \ldots, 100\text{mb}\). The vertical velocity \(\omega(= dp/dt)\) is kept at the nine levels midway between these pressure surfaces and at 1000mb. The contour heights of the ten pressure levels define the vertical temperature structure at any given time, the thickness of a 100-mb slab between adjacent pressure levels being a measure of the mean temperature of that slab. The humidity mixing ratio, \(r\), is the moisture variable, the values of \(r\) being the mean values for the seven 100-mb slabs between 1000mb and 300mb; the model atmosphere is dry above 300mb. Fig. 1 shows the vertical arrangement of the dependent variables.

The equations of motion are

\[
\frac{\partial u^*}{\partial t} + \mu \left( u^* \frac{\partial u^*}{\partial x} + v^* \frac{\partial u^*}{\partial y} \right) + \frac{\partial u^*}{\partial p} + \frac{1}{2}(u^* + v^* \partial h/\partial x) + \frac{g}{\partial x} - ru^* = F_x. \tag{3}
\]
for each of the pressure levels; \( h \) is the height of the pressure surface, \( u^* = u/m \), \( v^* = v/m \), \( \mu = m^2 \), \( m = \sec^2(\pi/4 - \phi/2) \), \( \phi \) being the latitude) is the map magnification factor, \( f \) the Coriolis parameter and \( g \) the acceleration due to gravity. The terms \( F_x \) and \( F_y \) represent the frictional and horizontal eddy diffusion terms. The equation of continuity is used in the form

\[
\mu \left( \frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \right) + \frac{\partial \omega}{\partial p} = 0. \tag{5}
\]

The integration of this equation with respect to pressure is described below. The thermodynamic equation expressed in terms of 100-mb-layer thicknesses \( h' \), where \( h' \) is proportional to the mean temperature of the 100-mb slab of atmosphere between two adjacent pressure surfaces, takes the form

\[
\frac{\partial h'}{\partial t} + \mu \left( u^* \frac{\partial h'}{\partial x} + v^* \frac{\partial h'}{\partial y} \right) + \omega p^{k-1} \frac{\partial}{\partial p} \left( p^{\frac{1}{2}} \cdot h' \right) = F'_h + S'_h \tag{6}
\]

for each of the nine layers of the model; \( \kappa = R/c_p \) where \( R \) is the gas constant and \( c_p \) is the specific heat at constant pressure. \( F'_h \) represents the horizontal eddy diffusion of heat and the source term \( S'_h \) is the non-adiabatic heating and includes latent heat effects of condensation and evaporation, heating from the surface and heating due to sub-grid scale convection. The lower boundary condition

\[
W = \frac{dh}{dt} = \mu \left( u^* \frac{\partial}{\partial x} + v^* \frac{\partial}{\partial y} \right) H
\]

at \( Z = H \), where \( H \) is the topographic height, is approximately satisfied by applying it at \( p = 1000\text{mb} \); this gives the height tendency equation

\[
\begin{array}{cccc}
p=1000\text{mb} & h_1 \quad (u_1,v_1) & \omega_1 & h' = h_1 - h_2 \text{ 2 dry layers} \\
p=700\text{mb} & h_2 \quad (u_2,v_2) & \omega_2 & h' = h_2 - h_3 \text{ 2 dry layers} \\
p=300\text{mb} & h_3 \quad (u_3,v_3) & \omega_3 & h' = h_3 - h_4 \text{ 2 dry layers} \\
p=400\text{mb} & h_4 \quad (u_4,v_4) & \omega_4 & h' = h_4 - h_5 \text{ 2 dry layers} \\
p=500\text{mb} & h_5 \quad (u_5,v_5) & \omega_5 & h' = h_5 - h_6 \text{ 2 dry layers} \\
p=600\text{mb} & h_6 \quad (u_6,v_6) & \omega_6 & h' = h_6 - h_7 \text{ 2 dry layers} \\
p=700\text{mb} & h_7 \quad (u_7,v_7) & \omega_7 & h' = h_7 - h_8 \text{ 2 dry layers} \\
p=800\text{mb} & h_8 \quad (u_8,v_8) & \omega_8 & h' = h_8 - h_9 \text{ 2 dry layers} \\
p=900\text{mb} & h_9 \quad (u_9,v_9) & \omega_9 & h' = h_9 - h_{10} \text{ 2 dry layers} \\
p=1000\text{mb} & h_{10} \quad (u_{10},v_{10}) & \omega_{10} & \\
\end{array}
\]

Figure 1. The vertical arrangement of the 10-level model's dependent variables.
\[
\frac{\partial h_{10}}{\partial t} + \mu \left( u^* \frac{\partial}{\partial x} + v^* \frac{\partial}{\partial y} \right) (h_{10} - H) + \omega \frac{\partial h_{10}}{\partial p} = F_h
\]

where \( h_{10} \) is the height of the 1000mb surface and \( F_h \) represents an eddy diffusion term. The remaining equation is the water balance equation for the humidity mixing-ratio \( r \) which is used in the form

\[
\frac{\partial r}{\partial t} + \mu \left( u^* \frac{\partial r}{\partial x} + v^* \frac{\partial r}{\partial y} \right) + \omega \frac{\partial r}{\partial p} = F_r + S_r
\]

for the seven moist layers of the model. \( F_r \) represents the horizontal eddy diffusion of water vapour and the source term \( S_r \) includes the rate of increase of humidity mixing-ratio due to gain or loss due to evaporation of rain or condensation to prevent supersaturation, transfer from the surface and transfer due to sub-grid scale convection. For our purpose it is convenient to recast the governing equations into a 'vector' form and to this end we define the following ten-dimensional 'vectors':

\[
\begin{bmatrix}
 u_1^* \\
 u_2^* \\
 \vdots \\
 u_{10}^*
\end{bmatrix}, \quad
\begin{bmatrix}
 v_1^* \\
 v_2^* \\
 \vdots \\
 v_{10}^*
\end{bmatrix}, \quad
\begin{bmatrix}
 \omega_1 \\
 \omega_2 \\
 \vdots \\
 \omega_{10}
\end{bmatrix}, \quad
\begin{bmatrix}
 h_1 \\
 h_2 \\
 \vdots \\
 h_{10}
\end{bmatrix}
\]

and the seven dimensional 'vector'

\[
\begin{bmatrix}
 r_3 \\
 \vdots \\
 r_9
\end{bmatrix}
\]

where \( u_k^* \) and \( v_k^* \) are the horizontal components of 'velocity' in the pressure surface \( p = k \times 100mb \), \( h_k \) is the height of this surface, \( \omega_k \) is the vertical velocity at \( p = (k+\frac{1}{2}) \times 100mb \), \( k = 3, \ldots, 9 \). The vector forms of the Eqs. (3), (4), (6), (7) and (8) are

\[
\begin{align*}
\frac{\partial u}{\partial t} + u_x + g \frac{\partial h}{\partial x} - f v &= f_x \\
\frac{\partial v}{\partial t} + u_y + g \frac{\partial h}{\partial y} + f u &= f_y \\
A \frac{\partial h}{\partial t} + n_h - H_x u - H_y v - \Gamma \omega &= f_h + s_h
\end{align*}
\]

and

\[
\frac{\partial r}{\partial t} + n_r = f_r + s_r
\]

where the vectors \( n_x, n_y, n_h \) and \( n_r \) contain the remaining non-linear terms on the left-hand sides of Eqs. (3), (4), (6), (7) and (8), the vectors \( f_x, f_y, f_h, f_r, s_h \) and \( s_r \) have obvious definitions,

\[
A = \begin{pmatrix}
1 & -1 & & & & & & & & \\
& 1 & -1 & & & & & & & \\
& & & \ddots & & & & & & \\
& & & & 1 & -1 & & & & \\
& & & & & & 1 & -1 & & \\
& & & & & & & & 1 & -1 \\
& & & & & & & & & 1
\end{pmatrix}, \quad
H_x = \text{diag}(0, 0, \ldots, \partial H/\partial x), \quad H_y = \text{diag}(0, 0, \ldots, \partial H/\partial y)
\]
and $\Gamma = \text{diag}(\Gamma_1, \Gamma_2, \ldots, \Gamma_{10})$ are $10 \times 10$ matrices

with $\Gamma_k = -\left( p^{k-1} \frac{\partial}{\partial p} (p^{1-k}h') \right)_{k+\frac{1}{4}}$, $k = 1, 2, \ldots, 9$

$\Gamma_{10} = -\frac{\partial h_{10}}{\partial p} = \frac{h'_{10}}{\Delta p}$, $\Delta p = 100 \text{ mb},$

where centred differences are used to evaluate the derivative of $(p^{1-k}h')$ except for $k = 1$ and 9 where one-sided differences are used. We evaluate $\left( \frac{\partial \omega}{\partial p} \right)_k$ as

$$\left( \frac{\partial \omega}{\partial p} \right)_k = \frac{1}{\Delta p} (\omega_{k+\frac{1}{4}} - \omega_{k-\frac{1}{4}}), \quad k = 2, \ldots, 9$$

$$= \frac{1}{\Delta p} \omega_{k+\frac{1}{4}}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
matrix, corresponding to ICAO values of the $\Gamma_k (k = 1, 2, \ldots, 10)$ and that $\mu = 1$. Writing $u_2$ for the velocity vector $(u, v)$ we can rewrite Eqs. (16) to (18), with $G = G_0$, in the form

$$\frac{\partial u_2}{\partial t} + g\nabla_2 h + f_2 = 0$$

and

$$\frac{\partial h}{\partial t} + \frac{1}{g} G_0 \nabla_2 \cdot u_2 + t = 0$$

(19)

where $\nabla_2 \equiv \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$, $f_2 = (fv, -fu)$, and $t = -(T_xu + T_yv)$.

We now introduce the model matrix $E_0$, where

$$E_0^{-1} G_0 E_0 = C_0^2$$

where $C_0^2 = \text{diag} (C_{0,1}^2, C_{0,2}^2, \ldots, C_{0,10}^2)$,

and $C_{0,k}^2 (k = 1, \ldots, 10)$ are the ten distinct positive eigenvalues of $G_0$. If we order the $C_{0,k}$ such that $C_{0,k} < C_{0,k+1}$ then $C_{0,1}$ is the speed of the external gravity wave and the $C_{0,k} (k + 1)$ are the speeds of the $(k - 1)$th internal gravity wave. The ten values of $C_{0,k}$ are shown in Table 1.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$C_{0,k} \text{ (m s}^{-1})$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>298</td>
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<tr>
<td>2</td>
<td>99</td>
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<tr>
<td>3</td>
<td>47</td>
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<tr>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
</tr>
</tbody>
</table>

We now make the linear transformations

$$a_2 = E_0^{-1} u_2$$

$$y = E_0^{-1} h$$

(20)

to a new velocity vector $a_2$ and height $y$ and substitute for $u_2$ and $h$ in (19) to give

$$\frac{\partial a_2}{\partial t} + g\nabla_2 y = -E_0^{-1} f_2$$

$$\frac{\partial y}{\partial t} + \frac{1}{g} C_0^2 \nabla_2 \cdot a_2 = -E_0^{-1} t$$

(21)

If the right-hand sides of Eqs. (21) were zero, then these equations would be uncoupled, since $C_0^2$ is a diagonal matrix, and would describe the motion of infinitesimal amplitude gravity waves in our atmosphere with an ICAO vertical structure. Explicit time-stepping schemes such as the leap frog or the Lax-Wendroff schemes, with simple centred differences
for the spatial derivatives, for the equations in (21) are stable provided the time-step, $\Delta t$, between time levels, satisfies the Courant–Friedrichs–Lewy condition

$$\Delta t < \Delta x/(\sqrt{2}C_{0,k}), \quad k = 1,2,\ldots, 10.$$ 

In the fine mesh version of the 10-level model, with a 100km horizontal grid length, this restriction allows a maximum time-step of about three minutes because $C_{0,1} \approx 300\text{m s}^{-1}$. Unconditionally stable implicit schemes can be used for each gravity wave mode necessitating the solution of ten Helmholtz equations, or some other partial differential equation at each time step, but because $C_{0,3} \approx 47\text{m s}^{-1}$ it is possible to treat two of the modes implicitly ($k = 1,2$) and the remaining eight explicitly and still take a time-step of about 15 minutes. The scheme of Robert et al. (1972) amounts to an implicit treatment of Eqs. (16), (17) and (18) which entails the solution of a three-dimensional partial differential equation at each time-step or equivalently ten coupled Helmholtz equations.

\[(a)\] A finite difference scheme for Eqs. (21)

A suitable explicit scheme for (21) is

$$\begin{align*}
\psi_{k}^{n+1} &= \psi_{k}^{n} - \frac{\Delta t}{g} C_{0,k}^2 V_{2} \cdot a_{x,k}^{n} - \Delta t E_{0}^{-1} t_{n} \\
a_{x,k}^{n+1} &= a_{x,k}^{n} - \Delta t g V_{2} \psi_{k}^{n+1} - \Delta t E_{0}^{-1} f_{2}^{n} \\
 f_{2}^{n} &= f\left(\psi_{k}^{n} - \frac{\Delta t}{2} \left(g \frac{\partial h}{\partial y} + f u_{k}^{n}\right), -u_{k}^{n} + \frac{\Delta t}{2} \left(g \frac{\partial h}{\partial x} - f v_{k}^{n}\right)\right)
\end{align*}$$

(22)

For the two leading modes ($k = 1,2$), the external and first internal gravity waves, the implicit scheme

$$\begin{align*}
y_{k}^{n+1} &= y_{k}^{n} - \frac{\Delta t}{g} C_{0,k}^2 V_{2} \cdot a_{x,k}^{n+1} \left(a_{x,k}^{n+1} + a_{x,k+1}^{n+1}\right) - \Delta t (E_{0}^{-1} t_{n})_{k} \\
a_{x,k}^{n+1} &= a_{x,k}^{n+1} - \Delta t g V_{2} \left(y_{k}^{n+1} + y_{k+1}^{n+1}\right) - \Delta t (E_{0}^{-1} f_{2}^{n})_{k}
\end{align*}$$

(23)

is unconditionally stable for zero $f_{2}$ and $t$. If we now introduce $\psi_{k}^{n+1} = y_{k}^{n+1} + y_{k}^{n}$, for $k = 1,2$, then eliminating $a_{x,k}^{n+1}$ from (23) gives

$$\begin{align*}
\psi_{k}^{n+1} &= y_{k}^{n} - \frac{\Delta t}{g} C_{0,k}^2 V_{2} \cdot a_{x,k}^{n+1} - \Delta t (E_{0}^{-1} t_{n})_{k} + \left(\frac{\Delta t}{2}\right)^2 C_{0,k}^2 V_{2} \cdot V_{2} \psi_{k}^{n+1} \\
&\quad + \frac{1}{2} \frac{\Delta t^2}{g} C_{0,k}^2 V_{2} \cdot (E_{0}^{-1} f_{2}^{n})_{k}
\end{align*}$$

which we rearrange as

$$-\left(\frac{\Delta t}{2}\right)^2 C_{0,k}^2 V_{2}^{2} \psi_{k}^{n+1} + \psi_{k}^{n+1} = \psi_{k}^{n+1} + y_{k}^{n} + \frac{(\Delta t)^2}{2} C_{0,k}^2 V_{2} \cdot (E_{0}^{-1} f_{2}^{n})_{k},$$

(24)

$$y_{k}^{n+1} = \psi_{k}^{n+1}$$

a Helmholtz equation for $\psi_{k}^{n+1}$, $k = 1,2$. We now define the corrections

$$C_{k}^{n+1} = \psi_{k}^{n+1} - \psi_{k}^{n+1} - y_{k}^{n},$$

so that

$$y_{k}^{n+1} = \psi_{k}^{n+1} + C_{k}^{n+1}, \quad k = 1,2$$

(25)

and hence $y_{k}^{n+1}$ is given by the forward time-step of (22) plus the correction $C_{k}^{n+1}$. Combining (22) and (23) to give an implicit treatment of the leading modes results in the scheme
\[ y^{n+1} = y^n - \frac{\Delta t}{g} C_0^2 \nabla_2 \cdot a_2^n - \Delta t E^{-1} \cdot t^n + C_1^{n+1} \hat{e}_1 + C_2^{n+1} \hat{e}_2 \]  \tag{26}

\[ h^{n+1} = h^n - \frac{\Delta t}{g} G_0 \nabla_2 \cdot u_2^n - \Delta t t^n + C_1^{n+1} \hat{y}_1 + C_2^{n+1} \hat{y}_2 \]  \tag{27}

that is
\[ h^{n+1} = \hat{h}^n + C_1^{n+1} \hat{y}_1 + C_2^{n+1} \hat{y}_2 \]  \tag{28}

where
\[ \hat{e}_1 = (1,0,\ldots,0)^T, \quad \hat{e}_2 = (0,1,0,\ldots,0)^T \]

and the \( \hat{y}_k = E_0 \hat{e}_k, k = 1,2, \) are the leading eigenfunctions of \( G_0, \) corresponding to \( C_{0,1} \) and \( C_{0,2}. \) Similarly the basic explicit integration of \( a_2^{n+1} \) in (22) is modified to give

\[ u_2^{n+1} = u_2^n - \Delta t g \nabla_2 \hat{h}^{n+1} - \Delta t t_2^n - \frac{1}{2} \Delta t g \nabla_2 [(y^{n+1} - 2 \hat{y}_1^{n+1}) \hat{y}_1 + \hat{y}_2] \]  \tag{29}

The spatial differences for the explicit/implicit scheme of Eqs. (28) and (29) are approximated using simple centred differences, with the grid shown in Fig. 2(a), where the velocities and heights are staggered in space with respect to each other. At grid points where variables are needed but are not kept they are determined by linear or bi-linear interpolations of neighbouring values. The Helmholtz equations of (24) are solved by an alternating implicit method with \( \hat{y}_1^{n+1} + y_1^n \) providing an accurate first guess for \( \psi_1^{n+1}. \)

It now remains to account for the variation of \( G \) from the ICAO matrix \( G_0. \) The straightforward addition of the explicitly evaluated term \( -\Delta t/g)(G^n - G_0) \nabla_2 \cdot u_2^n \) to the right-hand side of (28) results in an unstable scheme for the implicitly treated modes if \( C_k^n > C_{0,k}, k = 1,2, \) where \( (C_k^n)^2 \) are the eigenvalues of \( G \) and this instability is certainly realized in practice and is easily demonstrated theoretically for the special case \( G^n = (1 + \alpha)G_0 \) for positive \( \alpha. \) One solution to this problem is to choose \( G_0 \) so that \( C_{0,k} > C_k^n, k = 1,2, \) for all matrices \( G^n. \) Insight into a second and better solution is provided by the explicit scheme (22). There the new values of the heights \( y^{n+1} \) are used to compute \( \Psi_2^n \) and this suggests the following scheme:

\[ y_{k}^{n+1} = y_k^n - \frac{\Delta t}{2g} C_{0,k}^2 \nabla_2 \cdot (a_{2,k}^n + a_{2,k}^{n+1}) - \Delta t E_0^{-1} t^n \]  \tag{30}

\[ a_{2,k}^{n+1} = a_{2,k}^n - \frac{\Delta t}{2} g \nabla_2 (y_k^n + (y_k^{n+1} - y_k^{n+1})) - \Delta t g y_k^{n+1} - \Delta t (E_0^{-1} \hat{y}_2^n)_k \]  \tag{31}

where
\[ y_k^{n+1} = -\Delta t \left( E_0^{-1} \left( \frac{G^n - G_0}{g} \right) \nabla_2 \cdot u_2^n \right)_k \]  \tag{32}

for \( k = 1,2. \)

In order to incorporate this modification into the integration scheme of Eqs. (28) and (29) we need only redefine \( \psi_2^{n+1} \) and \( \hat{h}^{n+1} \) so that
\[ \psi_2^{n+1} = y_2^{n+1} + y_2^n + y_2^{n+1} \]  \tag{33}

\[ \hat{h}^{n+1} = h^n - \frac{\Delta t}{2} G \nabla_2 \cdot u_2^n - \Delta t t^n \]  \tag{34}
REFORMULATION OF A 10-LEVEL MODEL

$$\bar{y}^{n+1} = \bar{E}^{-1}_0 \bar{y}^{n+1}$$

with

$$\bar{y}^{n+1} = -\Delta t \left( \bar{E}_0^{-1} \left( \frac{(G^a - G^b)}{g} \nabla^2 \mathbf{u}^a \right)_k \right)$$

$$\bar{y}^{n+1}_k = \bar{y}^{n+1} + \bar{y}^{n+1}$$

The finite difference scheme of (28), (29) with the Helmholtz equations of (24) and definitions (33)–(35) is stable provided

$$\Delta t < \min \left\{ \frac{\Delta x}{m\sqrt{2C_{0,k}}} \right\} = \frac{\Delta x}{m\sqrt{2C_{0,3}}}$$

and

$$\Delta t < \frac{\Delta x}{m\sqrt{2(C_k - C_{0,k})^2}} \quad k = 1, 2, \text{ for } C_k > C_{0,k}$$

where we have now included the effect of a variable map factor $m = \sqrt{\mu}$.

4. THE SEMI-IMPLICIT ALGORITHM

The model’s Eqs. (9), (10), (11), and (12) together with (13) are integrated in two stages by splitting the time rate of change of dependent variables into a number of parts, a technique pioneered mainly by Russian mathematicians and meteorologists particularly Marchuk (see Marchuk et al. 1968). Let $\mathbf{x}_n$ be a vector whose components are all the grid point values of the dependent variables after $n$ complete integration cycles where the model’s equations are represented by

$$\frac{\partial \mathbf{x}}{\partial t} + \mathbf{f} = \mathbf{s}$$

We define a splitting of $\mathbf{f}$ into $\mathbf{n} + 1$ where $\mathbf{n}$ represents the non-linear advective terms $\mathbf{n}_a$, etc., and the frictional and diffusion terms and $\mathbf{I}$ the terms which appear in Eqs. (16), (17) and (18), apart from the source/sink terms of Eqs. (11) and (12) which form $\mathbf{s}$. The two main parts of the cycle are advection and adjustment.

(a) The advection stage

Here we integrate forward in time through a single time-step, $\Delta t$, the equation $\partial \mathbf{x}/\partial t + \mathbf{u} = 0$ using the two-step Lax–Wendroff scheme for the advective terms with the grids shown in Figs. 2(a) and (b) and a forward time-step for the friction and diffusion terms with the grid in Fig. 2(a); simple centred differences are used for the spatial derivatives. The details of this process are described fully in the original paper by Bushby and Timpson (1967) and in Benwell et al. (1971), though in their grids the positions of $u$ and $v$ are interchanged. We represent this part of the integration cycle by the matrix operation $\mathbf{x}_{n+1} = (\mathbf{I} + \mathbf{N})\mathbf{x}_n$ where $\mathbf{N}$ is the matrix representing the ‘advective’ changes and $\mathbf{I}$ is the identity matrix.

(b) The adjustment stage

This involves the integration of $\partial \mathbf{x}/\partial t + 1 = 0$ through a single time-step, $\Delta t$, with initial conditions $\mathbf{x}_{n+1}$ using the explicit/implicit scheme described in section 3 above. The state of the model atmosphere after this stage is given by $\mathbf{x}_{n+1} = (\mathbf{I} + \mathbf{L})\mathbf{x}_{n+1}$ where $\mathbf{L}$ is the operator representing the adjustment step and we can write

$$\mathbf{x}_{n+1} = (\mathbf{I} + \mathbf{L})(\mathbf{I} + \mathbf{N})\mathbf{x}_n$$

(38)
Figure 2. (a) The horizontal distribution of variables at the beginning and end of each integration cycle. (b) The horizontal distribution of variables at the end of the intermediate stage of the advection cycle.

The linear stability of the integration scheme (38) is governed by the stability properties of the individual operators. The stability criteria for the adjustment stage have been given in section 3; the advective stage is stable provided

\[ \Delta t < \Delta x/(m |V|_{\text{max}}) \]  

(39)

where \( |V|_{\text{max}} \) is the maximum wind speed; we have neglected the modifying effect of diffusion on the stability criteria for this stage. A reasonable value of \( |V|_{\text{max}} \) is 100 m s\(^{-1}\) and so (39) provides the most stringent stability criterion and compares favourably with (1).

The contribution from the source term \( s \) is now introduced. This term accounts for the effects of condensation and evaporation, heat and moisture transfers from the surface and changes due to sub-grid-scale convection. Details of the procedures used and the parameterizations for these processes are described in Gadd and Keers (1970) and Benwell et al. (1971), the latter also gives the formulations for horizontal diffusion and surface friction. Since these publications an ice phase has also been included in the cloud physics which provides an objective scheme for forecasting snow. Experience with this scheme has shown that it is advisable to modify it slightly to include the heating due to surface transfers in the adjustment step, before the Helmholtz equations are solved, in order to avoid excessive small-scale roughnesses, particularly over sea areas. This is achieved by the addition of the appropriate source vector to Eq. (34).

5. Results

The split semi-implicit scheme described in the preceding sections has been used successfully to produce forecasts for the North Atlantic fine mesh rectangular area shown in Fig. 3. The horizontal grid length is 100 km in the stereographic co-ordinate system, corresponding to a true grid length of 100 km at 60°N, and there are 64 × 48 = 3072 height points in this area. The boundary conditions are time dependent, the time rate of change of the boundary heights and the outer most ring of normal velocities is obtained by spatial and temporal interpolation from coarse mesh explicit integrations on an almost hemispheric octagonal area, also shown in Fig. 3, where the grid length is 300 km at 60°N
in the stereographic system. The model used for these coarse mesh integrations is basically the explicit version of the 10-level model described by Benwell et al. (1971). The time rate of change of the tangential velocities at the boundary are extrapolated from the interior of the rectangle. Control of the boundary instabilities that generally arise because of this over specification of the boundary conditions is maintained by the use of a horizontal linear diffusion coefficient of $10^6 \text{m}^2 \text{s}^{-1}$ within three grid lengths of the lateral boundaries. This procedure was adopted by Benwell and Bushby (1970) for the explicit Lax–Wendroff scheme. The time-step used is 12 minutes and this allows for a maximum wind of $100 \text{m s}^{-1}$ in the linear stability criteria at the southernmost part of the grid, whereas the time-step used with the original Bushby–Timpson scheme is 2.5 minutes. The average computation time needed for a semi-implicit 36-hour forecast for the rectangular area is 12 minutes compared with a computation time of 50 minutes for the original explicit scheme, a factor of four increase in efficiency. These computations were performed on the Meteorological Office IBM 360/195 computer.

A comparison between the semi-implicit integrations and the explicit integrations is illustrated by the results of 24-hour and 36-hour forecasts produced from initial data at 12 GMT 17 November 1973. Surface isobars and fronts as analysed by the Central Forecasting Office are shown in Fig. 4 for this data time. Both integrations started from objectively analysed height and humidity fields, the procedures used for these analyses have been described by Atkins (1970, 1974). The initial non-divergent part of the wind field derived from a solution of the non-linear balance equation, the vertical velocity from the solution of an ‘omega’ equation and the divergent component of the wind is obtained from the vertical velocity through the continuity equation; see Benwell et al. (1971) for a description of these procedures. The 24-hour and 36-hour forecasts of surface pressure and rainfall from the two schemes are shown in Figs. 5, 6, 7 and 8. The surface pressure patterns are almost identical, with both schemes forecasting a central value of 965 mb for the depression north of Scotland at 00 GMT 19 November. Similarly, the extent of the rain areas (stippled) and the rainfall intensity forecast by both schemes is to all intents and purposes also
identical. The evolution of the weather pattern as analysed by the Central Forecasting Office is illustrated by the subjective analyses shown in Figs. 9 and 10. The rainfall areas (stippled) were analysed using synoptic reports and satellite pictures. During the 36 hours covered by the forecast the anticyclone over the English Channel moved eastwards into Europe intensifying slightly, and the deepening low over mid-Atlantic moved quickly into the North Sea to near south Norway, where its central value was analysed to be 960mb and a west to north-west airstream covered the U.K. Also, high pressure built to

Figure 4. Actual synoptic situation at 12GMT 17 November 1973.

Figure 5. Semi-implicit 24 hour forecast, 12GMT 18 November.

[Legend: Moderate rain, 0.5 to 4mm/h  Slight rain, 0.5mm/h  Purely convective rain, 0.5mm/h.]
south of Iceland and the depression near the western boundary at 12 GMT 17 November moved slowly towards the southern tip of Greenland filling slightly and evolving into a complex system of lows with the development of a warm front and small frontal waves on the cold front in this system. This evolution of the pressure pattern was quite accurately forecast by both integration schemes, with the depth of the low in the North Sea, at 00 GMT on the 19th, being underestimated by about 5 mb and the intensity of the high cell that moved into Europe was also underestimated. The main defect in these forecasts lies in the handling of the complex cyclonic system near the western boundary with the main centre

Figure 6. Explicit 24 hour forecast, 12 GMT 18 November (key as in Fig. 5).

Figure 7. Semi-implicit 36 hour forecast, 00 GMT 19 November (key as in Fig. 5.)
being held back at the boundary, though both schemes have extended a trough to the west of Greenland. The development of the warm front and the waves on the cold front in this system has been forecast, as can be more readily inferred through the rainfall distribution. Some of the errors near the boundaries may be attributed to the inadequate boundary scheme which needs a high diffusion zone to preserve stability and indeed the boundary scheme may have contributed towards the loss in intensity in the anticyclone over Europe. The frontal rainfall from the developing depression that moved into the North Sea was well

Figure 8. Explicit 36 hour forecast, 00amt 19 November (key as in Fig. 5).

Figure 9. Actual synoptic situation, 12amt 18 November.
Rain areas analysed from synoptic reports and satellite pictures.
forecast including the rain from the cold front moving across Scotland. As the warm front moved into Norway the precipitation associated with it was mostly snow and both models are forecasting precipitation with a high probability of snow. Behind the cold front in this system there are showers in the north-westerly air stream, Fig. 10, and some of these are being forecast as dynamic precipitation just to the north of Scotland and as convective precipitation further to the west, again with a high probability of snow.

Figure 10. Actual synoptic situation, 00GMT 19 November (key as in Fig. 9).

6. DISCUSSION

All the comparisons between the split semi-implicit integrations with a 12-minute time-step and the explicit integrations with a 2.5-minute time-step for the rectangular area shown in Fig. 3 have shown that both schemes give almost exactly the same result on all occasions. Since March 1973 this scheme has been used routinely to produce forecasts for the North Atlantic fine mesh rectangular area and the results have been assessed by the operational forecasting and the forecasting research branches of the Meteorological Office. The fine mesh model with this scheme was subsequently incorporated into the operational system at Bracknell in December 1973.

The extension of this split semi-implicit integration scheme to the coarse mesh octagonal area is straightforward but the scheme cannot be used with the largest time-step allowed by the linear stability criteria because of the significant time truncation arising from the splitting technique. Time truncation with this split scheme appears to be significant when the time-step is more than 15 minutes. This has led to a reformulation of the integration scheme to use a non-split leap-frog time-stepping scheme which does not suffer from this severe time truncation, and experimentation with this scheme is at present being conducted.

A semi-implicit scheme which enables forecasts, comparable with those obtained by explicit methods, to be produced much more quickly is certainly beneficial in an operational environment but an enormous benefit is also accrued on the research side enabling much more investigation and experimentation on other aspects of the model to proceed more quickly and economically. In a direct way the economy of semi-implicit methods may be
used to increase the complexity and accuracy of the parameterizations of physical processes for the model or to increase the model’s spatial resolution either by reducing the grid length with the simple spatial differencing schemes used in this paper or by using more sophisticated and accurate spatial representations of the dependent variables.

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