Ice particle multiplication in cumulus clouds

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SUMMARY

Ice particle multiplication in shallow supercooled clouds is described by a stochastic process, which depends on collisions between ice particles and supercooled water droplets and drops. The multiplication is a consequence of the riming and drop splintering processes, or possibly just one of them. In the riming process, a large ice particle collects water droplets which eject splinters on freezing. In the drop splintering process, a supercooled drop which collides with a splinter, freezes and may eject splinters. Account is taken of the stronger evidence of significant drop splintering for small supercooled drops by considering those drops with diameters less than 250μm as a separate group. The probability of collision between an ice splinter and a water drop is estimated by considering their fall velocities and assuming a collection efficiency of unity. As a consequence of the small and large water drop division, the ice splinters are also divided; small ice splinters collide with all water drops, but large splinters having fall velocities comparable with those of small water drops collide only with large water drops. The growth of all ice particles by diffusion of water vapour or accretion of water droplets is taken into account, splinters not captured by water drops grow to rimer status after an appropriate time. Rimers are removed from the system at the time they fall through the 0°C isotherm. A rimer may also be formed as a result of the freezing of a large supercooled water drop, following its collision with an ice splinter. As the early production of splinters by such a rimer depends on its initial mass, which equals that of the drop, a distribution of large water drop masses is considered.

Generating functions are formulated for the probabilities of particular numbers of the various categories of ice particle existing at a general time t, as a consequence of one initial ice particle. These functions lead to a set of renewal equations for the estimated mean numbers of ice particles at time t. The Laplace transforms of the renewal equations are studied and it is shown that a simple large time analysis provides a valid description of the growth mechanism during most of the cloud lifetime. A small time behaviour marching technique completes the growth description. The large time analysis predicts that $A_t \exp(-p_0 t)$ ice particles exist at time t as a consequence of one initial small ice splinter. Values of the growth parameter, $p_0$, are given for many combinations of the physical parameters. An integration of this large time result provides a description of ice particle multiplication initiated by the nucleating particles introduced into a cloud by a constant updraught. A thermal model is also considered and results in growth times near to those obtained for the constant updraught model. These growth times are considerably less than the growth times predicted by earlier workers.

The field observations of several workers are all found to be consistent with this analysis. In many situations the most likely explanation of the multiplication involves two processes. These are: splinter production by riming; and the capture of ice splinters by drops, the drops becoming riming agents when they freeze.

1. INTRODUCTION

There now exists a substantial body of evidence showing that the concentration of ice particles in some fairly shallow supercooled clouds can be a factor, $f$, of up to $10^4$ times the measured concentration of ice nuclei effective at the cloud top temperature. Particularly detailed field investigations have been reported recently by Mossop, Ono and Wishart (1970) and Mossop, Cottis and Bartlett (1972). In these cases, where the clouds studied never had summit temperatures $T_s$ below $-13°C$, and a typical value was $-10°C$, the ice crystal concentrations sometimes exceeded those of the ice nuclei by values of $f$ between $10^2$ and $10^4$. The observations suggested that this multiplication occurred in about 40 minutes to one hour. In these clouds the measured values of the concentration of small ice particles were typically in the range 10 to 100 per litre. Such clouds were generally found to contain large supercooled drops and rimed ice particles, each usually in concentrations of 0.1 to 1 per litre. The diameter of the largest rimed particles was 2mm and of the largest
drops in excess of 1.5 mm. These measurements of particle size and concentration were made close to the cloud top.

The observation that rimed particles were always present in appreciable quantities within these clouds suggests that splinter production during riming may be responsible for this multiplication phenomenon. Mossop, Brownscombe and Collins (1974) performed laboratory experiments indicating that $M_p$, the number of ice particles ejected per unit mass of rime deposited, was around $1 \text{mg}^{-1}$. Parallel experiments by Bader et al. (1974), in which submicron ice particles could be detected, provided values of $M_p$ for situations which may occur within clouds, of up to $5 \text{mg}^{-1}$. A simple four-stage model of cloud glaciation by riming presented by Mossop et al. (1974) yielded the conclusion that these values of $M_p$ are much too low to explain the observed maximum levels of multiplication. Mason (1973) made calculations based on the assumption that cumulus towers rose to a greater altitude than that of the cloud top. Even with the greater concentration of ice nuclei activated at these lower temperatures, a value of $M_p \approx 100 \text{mg}^{-1}$ was required in order to produce $f = 10^4$ in one hour.

Explanations of this multiplication phenomenon in terms of splintering by riming are facilitated by laboratory experiments conducted by Hallett and Mossop (1974) and Mossop and Hallett (1974), in which a cylindrical rod was moved at known velocity through a cloud of supercooled droplets. They found that copious splinter production occurred within a temperature range $-3$ to $-8^\circ\text{C}$ for velocities in excess of about $0.9 \text{m s}^{-1}$; values of several hundred $\text{mg}^{-1}$ for $M_p$ were reported and they established that the presence of droplets of diameter greater than $24 \mu\text{m}$ was crucial to the occurrence of significant splintering.

The observations of Koenig (1963, 1968) and previously mentioned authors show that significant numbers of raindrops are present in clouds which exhibit rapid multiplication. The collisions of these drops with ice splinters affect the multiplication rate of ice particles in two distinct ways.

Firstly the drops may eject splinters on freezing. The present authors (1974) considered this mechanism in the absence of riming and concluded that $f = 10^4$ will be achieved within one hour if six splinters are ejected on average during the freezing of drops in the diameter range 50 to 200 $\mu\text{m}$. An analysis of the available evidence suggested that such an average splinter number is feasible. More recently Mason (1975a) has commented on some inconclusive aspects of the experimental evidence; and Pruppacher and Schlamp (1975) have concluded that drop splintering, although significant, is not primarily responsible for the observed multiplication rates.

The second consequence of a drop collision with an ice splinter is the immediate formation of an ice particle which is either large enough to commence riming at once or at least will grow to rimer size in a significantly shorter time than the capturing splinter. Thus the presence of water drops can significantly enhance ice particle multiplication by riming.

In the present paper both effects of the presence of water drops are included in a stochastic treatment of ice particle multiplication which includes the processes of splinter production by both riming and drop splintering. This combined treatment produces far stronger multiplication rates than previous treatments of either single process. In the case of splinter production by riming this is partly due to the non-stochastic approach of earlier authors, which considers the birth of all splinters to occur at the discrete death times of the parent rimers.

2. THE PHYSICAL MODEL

In this section we outline the chain of processes starting with one small ice particle,
by which multiplication occurs within a model cloud of dimensions and microphysical properties based on the observations of Mossop et al. (1972).

The cloud has a supercooled depth $Z = 1.2$ km, a summit temperature $T_s = -10^\circ$C, and a cloud droplet water concentration $C = 1.0$ g m$^{-3}$. An important feature is the assumed presence of a significant concentration of supercooled raindrops. Their diameter generally ranges from $60 \mu$m to 1 mm. The lower diameter of $60 \mu$m marks the boundary between the cloud droplets and raindrops. The droplets have negligible fall velocities and may be swept out by the much larger falling rimed particles. By contrast the raindrops, particularly the larger ones, have significant fall velocities in comparison with the ice splinters which may have been formed in the riming process. The probability of collision between raindrops and a splinter is considered in section 3. When such a collision occurs it is assumed that the drop freezes immediately, thereby becoming an ice particle of mass equal to that of the drop; the splinter remains embedded within the frozen drop. When a drop freezes secondary ice particles may be ejected. Splinters formed by riming and drop freezing are assumed to have identical properties. The evidence referred to in section 1 suggests that drops of diameter, $d$, less than about $250 \mu$m will on average eject more splinters than larger drops. Accordingly we have divided the drops into two categories:

![Diagram](image)

Figure 1. Possible life histories of an ice particle, and associated processes of secondary splinter production. The four stages of growth are: 1, small splinter; 2, large splinter; 3, growth rimer; 4, capture rimer. 3 and 4 are alternatives. The processes of transition from one stage to another are: A, diffusional growth; B, capture of small drops; C, capture by large drop; D, growth by diffusion and accretion; E, riming. E continues until the death of the rimer with a mass $W_f$. The processes giving rise to the production of secondary ice splinters are: a, riming; b, freezing of a large drop; c, freezing of a small drop.
‘small drops’ with $60\mu m < d < 250\mu m$ being characterized by a larger average number of ejected splinters on freezing than ‘large drops’ having $d > 250\mu m$. A distribution of sizes of large drops is introduced in order to take account of the observations of Mossop et al. (1972). In most calculations these drops are confined within the diameter range 0-5 to 1mm. The concentrations of water particles of all types are constant in space and time. The implications of these and other assumptions are discussed in section 6.

Fig. 1 illustrates the possible life histories of an ice splinter and the assumed processes of secondary ice particle production. The splinter may have been formed by the activation of an ice nucleus or by the riming or drop-freezing processes. There are four possible stages in the development of this ice particle.

**Stage 1.** The particle is small and grows principally by vapour diffusion. This type 1 particle is termed a ‘small splinter’. If it avoids capture by raindrops it grows for a time $T_1$ before achieving a mass $W_1$ and entering the second stage. In all calculations of ice particle growth made in this paper considerable use is made of the detailed computations of Koenig (1972). $W_1$ is chosen to be 2$\mu g$, which is the typical mass of a small raindrop; inspection of Koenig’s growth curves for a mean temperature of $-5^\circ C$ and a cloud-water concentration of $1g\text{m}^{-3}$ shows that $T_1 \approx 8\text{min}$. If, however, the ice splinter collides with a supercooled raindrop before the time $T_1$ has expired, the raindrop will freeze and capture the splinter. If the collision is with a small raindrop, some splinters may be ejected as the drop freezes and the resultant particle enters stage 2 immediately. If the collision is with a large raindrop, some splinters may be ejected during freezing and the resultant frozen drop enters stage 4 directly.

**Stage 2.** The particle has an initial mass $W_1$, having been formed either by continuous growth of a small splinter, or by the freezing of a small raindrop. This type 2 particle is termed a ‘large splinter’. It may grow by diffusion and accretion for a time $T_2$ before it achieves a mass $W_g(0-1\text{mg})$ and enters stage 3. Koenig’s work shows that $T_2 \approx 4-8\text{min}$. The amount of rime deposited on the large splinter during its lifetime is so small that the possibility of associated splinter production can be neglected. Since its fall velocity will be comparable with that of a small raindrop we assume that collisions between such particles will not occur. However, collision between the large splinter and a large raindrop may occur at any time in the growth period $T_2$. This collision is considered as large raindrops have much higher fall velocities than the large splinters. In this event the large raindrop will freeze, possibly ejecting some splinters, and the resulting frozen drop enters stage 4.

The last two stages of growth are alternatives and are populated by rimers.

**Stage 3.** The particles are formed by the growth of a large splinter. This type 3 particle is called a ‘growth rimer’ and has an initial mass $W_g = 0-1\text{mg}$. Type 3 particles remain within the supercooled region of the cloud, growing exclusively by accretion for a time $T_3$, until they fall through the $0^\circ C$ isotherm with a mass $W_f$ and are removed from the system. For any given calculation it is assumed that all growth rimers, and also all stage 4 particles, achieve the same final mass $W_f$. The validity of this assumption is discussed in section 5. Values of $W_f$ ranging from 1-0 to 15-7mg have been employed in the calculations. Koenig’s growth curves have been used to estimate the rate of growth of these particles. The number of splinters ejected by a rimer is governed by $M_p$. Since $W_f \gg W_g$ the number ejected is approximately $M_p W_f$.

**Stage 4.** This stage is populated by ‘capture rimers’. It is entered by a small or large splinter being captured by a large drop; its initial mass $w$ is that of the drop. Since a range of values of $w$ is considered there is a range of values of growth time $T_4(w)$ before the final mass $W_f$ is achieved. The growth and splintering behaviour of these type 4 particles is assumed to be identical to that of the growth rimers.
We see that the presence of raindrops in the cloud may assist the multiplication in two ways. When a drop collides with an ice splinter and freezes it may eject secondary splinters. This process is particularly effective when the drops are small. Also, the capture of a splinter by a drop reduces the time for which the ice particle must exist before becoming a rimer capable of producing its own splinter progeny. This effect is particularly important when the drops are large.

3. Mathematical development and capture probabilities

In this section a description of the stochastic growth of the population of ice particles resulting from one initial ice particle is presented.

Small splinters may be ejected during the freezing of any of the three types of water particle. Let \( v \) be the number of splinters ejected at the freezing of a particular water drop or droplet of mass \( w \). The splinter number is a random variable and the probability of any particular value of \( v \) occurring at a freezing event is given, for a particular value of \( w \), by the probability generating function

\[
h(w, s_1) = \sum_{j=0}^{\infty} Pr(v = j)s_1^j
\]

with \( h(w, 1) = 1 \). The suffix of the variable \( s_1 \) denotes the production of type 1 ice particles. For any choice of \( w \), the mean number of splinters ejected at the freezing of this size of water drop, or droplet, is obtained by differentiation of (3.1). The mass ranges of droplets, small drops and large drops are taken to be \( w < w_1 \), \( w_1 < w < w_2 \) and \( w_2 < w < w_3 \) respectively. The mean number of small splinters ejected at the freezing of water drops of mass \( w \) is

\[
r(w) = (\partial h/\partial s_1)(w, 1) \quad w_1 < w < w_3
\]

For droplets, which are captured by rimers, the corresponding splinter number is taken to be independent of the droplet mass and is given by

\[
r_s = (\partial h/\partial s_1)(w, 1) \quad w < w_1
\]

The probabilities of the occurrence of splintering events produced by each of the four types of ice particle are now discussed. Let the lifetime \( \tau_1 \) of a type 1 ice particle have a probability distribution function \( G_1(t) = Pr(\tau_1 \leq t) \), then \( dG_1(t) \) is the probability that a small splinter born at \( t = 0 \) is captured by a small or large water drop in the time interval \( t \) to \( t + dt \). This probability of capture exists only for a time \( T_1 \); if the splinter is not captured in the time interval 0, \( T_1 \) it becomes a type 2 particle at \( t = T_1 \), giving \( G_1(t) = 1 \) for \( t > T_1 \). As the water drop that captures the splinter becomes a type 2 or type 4 ice particle depending on the mass \( w \) of the water drop, we need to isolate the contributions to the elementary death probability \( dG_1(t) \) arising from different water drop sizes. Let \( S(w)dw \) \( dG_1(t) \) be the probability that a small splinter born at \( t = 0 \) is captured in time \( t, t + dt \) by a water drop with a mass in the range \( w \) to \( w + dw \). Integration over the complete water drop spectrum requires

\[
\int_{w_1}^{w_3} S(w) \, dw = 1.
\]

For future use we separate the contributions to this integral arising from small and large water drops,

\[
S_1 = \int_{w_1}^{w_3} S(w) \, dw, \quad S_2 = \int_{w_2}^{w_3} S(w) \, dw.
\]
Similarly let \( G_2(t) \) be the lifetime probability distribution of a large splinter. Such a type 2 ice particle can only be captured by a large drop and if captured a type 4 rimer is produced with an initial mass equal to that of the drop. If the splinter avoids capture for a time \( T_2 \) it is assumed to have grown to a mass \( W_g \) and becomes a type 3 rimer, giving \( G_3(t) = 1 \) for \( t > T_2 \). If the large splinter is captured by a large water drop, the behaviour of the resulting type 4 rimer depends on the mass \( w \) of the capturing drop. The contribution to \( dG_2(t) \) arising from large water drops in the mass range \( w \) to \( w + dw \) is \( \{ S(w)/S_2 \} dw \) \( dG_2(t) \), the \( S_2 \) normalization factor arising as small water drops do not contribute to \( dG_2(t) \).

Unlike a splinter which dies at its first collision with a water drop, the third type of ice particle expects to have many collisions with cloud droplets before its death due to passing through the 0°C isotherm. The lifetime of the rimer is denoted by \( T_3 \). The probability that a growth rimer of age \( t \) captures a cloud droplet in the time \( t \) to \( t + dt \) is denoted by \( dG_3(t) \). Let \( G_3(t) = \int_0^t dG_3(u) \) and note that \( dG_3(t) = 0 \) for \( t > T_3 \).

The fourth type of ice particle is treated similarly, except that the mass of the rimer and hence its effectiveness in capturing droplets at time \( t \) depends on its initial mass. The initial mass is that of the capturing drop, \( w \), and we write \( dG_4(w,t) \) as the corresponding capture probability. Rimers of the two different types, having the same mass, are given equal capturing probabilities \( dG_3, dG_4 \). The lifetime of the second type of rimer is written \( T_4(w) \), giving \( dG_4(w,t) = 0 \) for \( t > T_4(w) \). The splinter number for droplets, \( r_3 \), is the same whichever type of rimer effects the capture.

The generating functions for the numbers of ice particles existing at time \( t \) that result from the birth of one ice particle at \( t = 0 \) are now considered. Let the coefficient of \( s_1^i s_2^j s_3^k s_4^l \) in \( F_i(s_1, s_2, s_3, s_4, t) \) denote the probability that \( i, j, k \) and \( l \) ice particles exist at time \( t \) of types 1, 2, 3, 4 respectively, given that one small splinter, or type 1 particle, is born at \( t = 0 \). The \( s_1, s_2, s_3, s_4 \) dependence of the generating functions is not explicitly displayed in the subsequent analysis. Similarly let \( F_2(t) \) be the generating function for a large splinter and \( F_3(t), F_4(w, t) \) relate to rimers with initial masses \( W_g, w \) respectively. In formulating equations for the generating functions, we assume that progeny of any one ice particle are independent of the possible existence of other ice particles. In these circumstances the generating function for an assemblage of ice particles existing at time \( t = 0 \) is the product of the generating functions of the individual ice particles.

The probabilities of the possible life histories of a type 1 particle born at \( t = 0 \) are given by

\[
F_1(t) = s_1 [1 - G_1(t)] + H(t - T_1)(1 - G_1(T_1))F_2(t - T_1) + \int_{w_1}^{w_2} \int_{u_1}^{t} h\{w, F_1(t-u)\}F_2(t-u)S(w)\, dw \, dG_1(u) + \int_{w_2}^{w_3} \int_{u_1}^{t} h\{w, F_1(t-u)\}F_4(w,t-u)S(w)\, dw \, dG_1(u),
\]  

(3.6)

where \( H(t) = 1 \) for \( t \geq 0 \) and \( H(t) = 0 \) for \( t < 0 \). The four terms of the right-hand side of this equation arise respectively from the following possibilities:

(i) the splinter may still exist at time \( t \),
(ii) the splinter may avoid capture and be converted to a type 2 particle at time \( T_1 \),
(iii) at any time \( u \) in \([0, t]\) the initial splinter may encounter a water drop in the small drop spectrum and produce one type 2 particle and a number of type 1 ice particles given by (3.1),
(iv) an encounter with a large drop of mass \( w \) producing a type 4 particle and some type 1 ice particles.
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Apart from the absence of encounters with small drops, as in (iii) above, the large splinter has similar possible life histories. The type 2 particle generating function satisfies

\[ F_2(t) = s_2 \{ 1 - G_2(t) \} + H(t - T_2) \{ 1 - G_2(T_2) \} F_2(t - T_2) + \]

\[ + \int_{w_2}^{\infty} \int_{u_2}^{t} h(w, F_1(t - u)) F_2(w, t - u) \{ S(w)/S_2 \} \, dw \, dG_2(u) , \]

where this time the second term on the right-hand side arises from growth after time \( T_2 \) into a rimer of mass \( W_2 \).

In determining the generating function for a growth rimer the probability \( dG_3(t) \) of a droplet capture in \( t, t + dt \) is assumed to be independent of its previous capturing events. These events then constitute a variable time Poisson process and the probability of there being \( l \) events in \([0, t]\) is \( \frac{1}{l!} G_3(t) \exp \{ -G_3(t) \} \). In \([0, t]\) let a particular growth rimer have capturing events at \( u = u_j \) and produce \( v_j \) splinters for \( j = 1, 2, \ldots, J \). The generating function for the rimer satisfies

\[ F_3(t) = \{ s_3 H(T_3 - t) + H(t - T_3) \} \exp \prod_{j=1}^{J} \left( f_j \right) \]

where \( \exp \) denotes expected value. The first bracket describes the death of the rimer at \( T_3 \). To evaluate the expected value we first consider the possible different values of \( v_j \) at one particular splintering event occurring at time \( u_j \). Eq. (3.1) gives the generating function \( h(F_i(t - u_j)) \) for the splinters produced at this event, the mass of the droplets not needing to be displayed. By considering the probabilities of the event occurring at any \( u_j \) in \([0, t]\), the expected value of this generating function is \( \int_{u_2=0}^{t} h(F_i(t - u)) \, dG_3(u)/G_3(t) \). For \( l \) such independent splintering events occurring in \([0, t]\) this generating function is raised to the \( l \)th power. Finally we sum over all values of the number of collisions \( l \), giving

\[ \exp \prod_{j=1}^{J} \left( f_j \right) \int_{u_2=0}^{t} h(F_i(t - u)) \, dG_3(u) \]

and

\[ F_3(t) = \{ s_3 H(T_3 - t) + H(t - T_3) \} \exp \int_{u_2=0}^{t} \left[ h(F_i(t - u)) - 1 \right] \, dG_3(u) . \]

The same analysis applies for the fourth type of ice particle, though its initial mass \( w \) needs to be displayed in appropriate places

\[ F_4(w, t) = \{ s_4 H(T_4(w) - t) + H(t - T_4(w)) \} \exp \int_{u_2=0}^{t} \left[ h(F_i(t - u)) - 1 \right] \, dG_4(w, u) . \]

Equations for the mean numbers of ice particles existing at time \( t \) that arise from one ice particle of a particular type at \( t = 0 \) may now be derived. We introduce the sixteen means

\[ M_i(t) = \partial F_i(t)/\partial s_i , \]

where after the \( s_j \) differentiation all \( s_i \) variables are given the value unity. The first suffix in \( M_{ij} \) denotes the type of originating ice particle and when \( j \) takes the values 1, 2, 3, 4 the mean number of resulting ice particles of each type is obtained. The means \( M_i(w, t) \) depend on the initial mass of the rimer. Differentiation of (3.6), (3.7), (3.10) and (3.11) with respect to \( s_j \), noting that each \( F_i(t) \) function has the value unity when the \( s_i \) are each unity, gives
\[ M_{1j}(t) = \delta_{1j}(1 - G_1(t)) + H(t - T_1)(1 - G_1(T_1))M_{2j}(t - T_1) + \]
\[ + \int_{w_1}^{w_3} \int_{u=0}^{t} \{r(w)M_{1j}(t-u) + M_{2j}(t-u)H(w-w_1)\}dwdG_1(u), \]
\[ M_{2j}(t) = \delta_{2j}(1 - G_2(t)) + H(t - T_2)(1 - G_2(T_2))M_{3j}(t - T_2) + \]
\[ + \int_{w_2}^{w_3} \int_{u=0}^{t} \{r(w)M_{1j}(t-u) + M_{4j}(w,t-u)\} \{S(w)/S_2\}dwdG_2(u), \]
\[ M_{3j}(t) = \delta_{3j} H(T_3 - t) + \int_{u=0}^{t} r_3 M_{1j}(t-u) dG_3(u), \]
\[ M_{4j}(w, t) = \delta_{4j} H\{T_4(w)-t\} + \int_{u=0}^{t} r_3 M_{1j}(t-u) dG_4(w, u), \]

for \( j = 1, 2, 3, 4. \)

These renewal equations give the mean numbers of different types of ice particle at time \( t \) arising from any one type of initial particle. Each equation possesses a simple physical interpretation, which may be illustrated by the second of the equations with \( j = 1 \). The number of small splinters arising from an initial large splinter is the estimated number of splinters arising from the progeny of (i) a growth rimer should the splinter not be captured, (ii) the splinters possibly formed during the freezing of the capturing drop, (iii) the rimer formed by the freezing of the drop.

We now derive expressions for the probabilities of collision between ice splinters and water drops, which occur in (3.13). Our derivation of the probability distribution function \( G_i(t) \) for small splinters rests on the assumption that the fall velocity of a small splinter remains small compared with the fall velocity of all raindrops. The probability of a splinter being swept up by a falling drop in any time interval is then just the proportion of volume swept out by the water drops in this time. Let the proportional volume sweep out rate be \( \lambda \). For a small splinter born at \( t = 0 \), the probability that the splinter is alive at time \( t \) is \( \{1 - G_1(t)\} \) and the probability that the splinter will be captured in \( t, t + dt \) is
\[ dG_1(t) = \{1 - G_1(t)\} \lambda dt. \]  
\[ (3.14) \]

We further assume that \( \lambda \) is independent of the time and position in the cloud, giving for all small splinters
\[ G_1(t) = 1 - e^{-\lambda t}H(T_1 - t), \]  
\[ (3.15) \]
as \( G_1(0) = 0 \) and the probability of capture ceases after the growth lifetime \( T_1 \).

For large splinters we similarly neglect their fall velocity in comparison with the large water drops which sweep them up. The probability distribution function \( G_2(t) \) is then determined by the proportional volume sweep out rate for large drops. Let the proportional volume sweep out rates of small and large water drops be \( \lambda_1 \) (\( = S_1 \lambda \)) and \( \lambda_2 \) (\( = S_2 \lambda \)) respectively, with \( \lambda_1 + \lambda_2 = \lambda \) and \( S_1, S_2 \) defined in (3.5). There follows
\[ G_2(t) = 1 - e^{-\lambda t}H(T_2 - t), \]  
\[ (3.16) \]
\( T_2 \) being the growth lifetime of a large splinter. The freezing of small drops due to encountering type 2 ice particles is not considered as they have comparable fall velocities.

Formulae for the \( \lambda \)'s may be derived in terms of \( N(d), V(d), \) the concentration and fall velocity of drops of diameter \( d \). For a collision efficiency of unity
\[
\lambda = \frac{\pi}{4} \sum_d d^2 V(d) N(d),
\]

(3.17)

the summation being over the small drop diameters for \( \lambda_1 \) and the large drop diameters for \( \lambda_2 \). We assume that the diameter range covered by the small raindrops is 60 to 250\,\mu m, and following Chisnell and Latham (1974) estimate 28 minutes as a typical value of \( \lambda_1^{-1} \).

The large drop sweep out rate is more difficult to estimate. The foil impactor used by Mossop et al. (1972) encountered hydrometeors infrequently – the horizontal spacing between events typically being several metres – so that a truly representative sample of large raindrops could not be obtained. Nevertheless values of \( \lambda_2 \) can be determined by summing the individual contributions of large drops of all sizes listed for the various flights made. This analysis leads to a typical value of \( \lambda_2^{-1} \) of 21 minutes. For a large splinter to rimer growth time of 4-8 minutes, Eq. (3.16) gives a mean large splinter lifetime of 4-3 minutes. Combining the small and large drop results we obtain a typical value of \( \lambda^{-1} \) of 12 minutes. Consequently small splinters, which will be swept out by small and large drops have a mean lifetime of 5-8 minutes, using a growth time to large splinter size of 8 minutes in (3.15).

We next consider the production of splinters by either type of rimer. Experimental evidence relates to \( M_r \), the number of splinters ejected per unit mass of accreted rime. Let the estimated increase in the mass \( W \) of a growth rimer in time \( dt \) be \( dW \), then the estimated number of splinters produced in this time is

\[
M_r dW = r_3 dG_3(t).
\]

(3.18)

We assume \( M_r \) to be the same for all rimers and to be independent of rimer size. For a rimer moving with speed \( V \) through supercooled droplets in concentration \( C \), with a collection efficiency of unity, mass is collected at a rate \( \pi r^2 V C \), \( r \) being the rimer radius. Introducing the simplifying assumption \( V = kr \), where \( k \) is a constant, we obtain a mass collection rate of \( 3W/kC/4\rho \), where \( \rho (= 0.49\, \text{g cm}^{-3}) \) is the density of ice. The work of List and Schemenauer (1971) suggests that \( k/\rho \) is nearly constant over a wide range of ice particle sizes. The integrated rimer mass law is therefore

\[
W = W_0 e^{\beta t}, \quad \beta = 3kC/4\rho,
\]

(3.19)

\( W_0 \) being the initial mass of the rimer. Taking \( k/\rho = 8300\, \text{cm}^3\, \text{g}^{-1}\, \text{s}^{-1} \) and \( C = 1\, \text{g m}^{-3} \), the value of \( \beta \) is 0.3735 min\(^{-1}\). It is reassuring to note that over the range of interest the growth rates given by (3.19) agree well with those predicted by the comprehensive treatment of Koenig (1972). The estimated number of splinters produced by a growth rimer born at \( t = 0 \) in the time interval \( t \) to \( t + dt \) is then

\[
r_3 dG_3(t) = M_r W_0 \beta e^{\beta t} dt.
\]

(3.20)

The same considerations apply to rimers formed by drop capture and lead to

\[
W = w e^{\beta t}, \quad r_3 dG_4(w, t) = M_r w \beta e^{\beta t} dt,
\]

(3.21)

where \( w \) is the initial mass of the rimer and \( \beta \) is unaltered. A spectrum of large water drops is considered as the early effectiveness of the rimer depends on its initial mass which is that of the capturing drop.

A rimer continues to produce splinters in this manner until it passes through the 0°C isotherm with a mass \( W_f \). The lifetimes of the type 3 and 4 rimers are given in terms of \( W_f \) as

\[
T_3 = \frac{1}{\beta} \ln(W_f/W_0), \quad T_4(w) = \frac{1}{\beta} \ln(W_f/w).
\]

(3.22)
These results for \( dG \) may now be substituted in the renewal equations (3.13) and solved by Laplace transforms. The details are presented in the appendix and the main conclusions are stated at the beginning of the next section.

4. Results

The mathematical analysis of the renewal equations for the mean numbers of the various types of ice particle is presented in the appendix. A study of the Laplace transforms of these equations reveals that a simple one-term large time result provides a valid description of the growth law once the total number of ice particles arising from one initial ice particle has reached about 100. In this analysis, the estimated number of particles of type \( j \) arising from one initial small splinter is given as

\[
M_{1j}(t) = A_{1j} e^{p_0 t}
\]  

and summation over the four \( j \)-values gives the total number of ice particles present at time \( t \) as

\[
M_1(t) = A_1 e^{p_0 t}.
\]

The exponent \( p_0 \), rather the coefficient \( A_1 \), provides the key measure of the growth law and most results in this section will relate to \( p_0 \), though some results for \( A_{1j} \) are presented in order to describe the structure of the ice particle population. It will be argued in section 5, that values of \( p_0 \) greater than about 0.2 min\(^{-1}\) give a growth law consistent with the highest observed multiplication rates.

In the formulation of the renewal equations a distribution of large water drops is considered. In the appendix three distributions are considered, each having the same value of proportional volume sweep-out rate \( \lambda \). The first distribution describes one of the observations of Mossop et al. (1972), the second is a top-hat distribution in which contributions to \( \lambda \) are proportional to the range of drop diameters and the third is an exponential distribution. It is concluded that the choice of distribution does not significantly affect the growth law. The results in this section all relate to the top-hat distribution.

Calculations of the growth parameter \( p_0 \) and the coefficients \( A_{1j} \) were made for a range of values of the various input parameters. For the ranges of parameters considered \( M_p \) and \( \lambda_2 \) have a key influence on \( p_0 \). \( M_p \) was given the values 1, 3, 10, 30, 60 and 140 mg\(^{-1}\). The values 140 and 60 mg\(^{-1}\) are based on the experiments of Hallett and Mossop (1974) and Mossop and Hallett (1974). An average of the values of \( M_p \) found by these workers over the temperature range covered by our model cloud gives \( M_p \approx 140 \) mg\(^{-1}\) and an average of these same values weighted to take account of the different number of splinters produced by the rimers of this model at different temperatures within the cloud gives \( M_p \approx 60 \) mg\(^{-1}\). The range of values of \( \lambda_2 \) utilized were based on observations, cited in the introduction, of the numbers and sizes of large drops in clouds which exhibit the multiplication phenomenon. Four values of \( \lambda_2 \) have been used, ranging from \( \lambda_2 = 0 \) in the absence of large drops to \( \lambda_2 = 11 \) \text{min}^{-1}.

For each choice of \( M_p \) and \( \lambda_2 \) the growth parameter \( p_0 \) was calculated with the rimer exit mass \( W_f \) having the values 3, 6, 8, 10, and 15, and a few calculations were performed for lower values of this parameter.

On the basis of the evidence mentioned in the introduction, the small drop splintering parameter \( r_1 \) was given even integral values between 0 and 10, while \( r_2 \) – the corresponding parameter for large drops – was assumed to be either 0 to 1.

The other parameters which influence \( p_0 \), namely \( T_1, T_2, \lambda_1, W_f, w_2, w_9 \), were found to have secondary importance and accordingly were held constant for the great
majority of calculations. Their values were \( T_1 = 8.0 \text{ min.}, T_2 = 4.8 \text{ min.}, \frac{1}{\lambda_1} = 28 \text{ min}, \)
\( W_g = 0.1 \text{ mg}, \ w_2 = 6.5 \times 10^{-2} \text{ mg} \) and \( w_3 = 0.52 \text{ mg} \), the reasons behind these choices having already been given. Combination of this value of \( \lambda_1 \) with the four chosen values of \( \lambda_2 \), gives values of the total inverse volume sweep out rate \( \lambda^{-1} \) of 28, 20, 12 and 8 minutes.

**TABLE 1. Values of the growth parameter \( p_0 (\text{ min}^{-1}) \) for complete ranges of the primary parameters \( M_p, \lambda_2 \) and \( W_f \), in the absence of drop splintering, \( r_1 = 0, \ r_2 = 0 \). Values of \( p_0 \) in the absence of drops, \( \lambda_1 = 0, \ \lambda_2 = 0 \) are also included**

<table>
<thead>
<tr>
<th>( W_f ) (mg)</th>
<th>( \lambda_2^{-1} ) (min)</th>
<th>( \lambda_1^{-1} ) (min)</th>
<th>( M_p (\text{ mg}^{-1}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>11.2</td>
<td>28</td>
<td>0.20</td>
<td>0.49</td>
</tr>
<tr>
<td>21</td>
<td>28</td>
<td>0.17</td>
<td>0.27</td>
</tr>
<tr>
<td>15.7</td>
<td>70</td>
<td>0.14</td>
<td>0.21</td>
</tr>
<tr>
<td>( \infty )</td>
<td>28</td>
<td>0.13</td>
<td>0.18</td>
</tr>
<tr>
<td>( \infty )</td>
<td>( \infty )</td>
<td>0.12</td>
<td>0.17</td>
</tr>
<tr>
<td>8.6</td>
<td>70</td>
<td>0.12</td>
<td>0.19</td>
</tr>
<tr>
<td>( \infty )</td>
<td>28</td>
<td>0.10</td>
<td>0.16</td>
</tr>
<tr>
<td>( \infty )</td>
<td>( \infty )</td>
<td>0.10</td>
<td>0.14</td>
</tr>
<tr>
<td>3.0</td>
<td>( \infty )</td>
<td>0.059</td>
<td>0.12</td>
</tr>
<tr>
<td>( \infty )</td>
<td>( \infty )</td>
<td>0.055</td>
<td>0.11</td>
</tr>
</tbody>
</table>

We first consider a model in which no drop splintering occurs, \( r_1 = 0, r_2 = 0 \). Table 1 lists values of \( p_0 \) for various values of the three parameters \( M_p, \lambda_2 \) and \( W_f \). We note that although \( W_f \) has some importance for small \( M_p, W_f \) changes have a very small effect when \( M_p \) is large, especially when \( \lambda_2 \) is also large. By contrast it is noted that \( \lambda_2 \) and \( M_p \) cause significant changes in \( p_0 \) throughout the whole of their ranges, with variations due to \( M_p \) being the more pronounced. Thus \( M_p \) emerges as the dominant parameter, with the presence of large water drops significantly enhancing the multiplication process even if the drops do not splinter on freezing. The role of the large water drops in this situation is to reduce the time interval between the birth of some splinters and their transition to rimers, and that of the small drops is to hasten the transition of some small splinters to large splinter status. The importance of the presence of small drops in the model is seen by comparing the last two rows of the results for each \( W_f \), section of Table 1. Removal of the small drops is achieved by putting \( \lambda_1 = 0 \) and is seen to be a significant effect, though of lesser importance than the removal of the large drops.

The composition of the population of ice particles originating from one small splinter is illustrated in Table 2. The values of the \( A_{ij} \) coefficients, and their sum \( A_{11} \), is given for two values of each of the parameters \( M_p, \lambda_2 \) and \( W_f \). We observe that the small splinters provide about 70\% of the population when \( M_p \) is 3 mg\(^{-1}\) and about 90\% when \( M_p \) is 60 mg\(^{-1}\). The proportion of large splinters is reduced by about 2.5 when \( M_p \) is raised from 3 to 60 mg\(^{-1}\). Variations in \( \lambda_2 \) change the balance between the two types of rimer. When \( \lambda_2^{-1} \) is 21 min there are more rimers formed by drop capture (\( A_{14} \)) than by growth from a splinter (\( A_{13} \)). As \( \lambda_2 \) decreases \( A_{13} \) overtakes \( A_{14} \). The changes in \( W_f \) do not significantly affect the ratios of the different types of ice particle.

We next consider the effects on \( p_0 \) of the remaining parameters \( T_1, T_2, \lambda_1, W_g, w_2 \) and \( w_3 \) in this non-splintering drop model. All these tests have been carried out at each of the
TABLE 2. VALUES OF THE COEFFICIENTS $A_{11}$, $A_{12}$, $A_{13}$, $A_{14}$ AND THEIR SUM $A_1$ FOR TWO VALUES OF EACH OF $M_p$, $\lambda_2$ and $W_f$ with $r_1 = 0$, $r_2 = 0$, $\lambda_1^{-1} = 28$ min. IN THE LARGE TIME ANALYSIS, THESE COEFFICIENTS ARE PROPORTIONAL TO THE NUMBERS OF SMALL SPLINTERs, LARGE SPLINTERs, GROWTH RIMERS, DROP-FREEZING RIMERS AND THE TOTAL NUMBER OF ICE PARTICLES RESPECTIVELY, THAT ARISE FROM ONE INITIAL SMALL SPLINTER

<table>
<thead>
<tr>
<th>$W_f$ (mg)</th>
<th>$\lambda_1^{-1}$ (min)</th>
<th>$M_p$ (mg$^{-1}$)</th>
<th>$A_{11}$</th>
<th>$A_{12}$</th>
<th>$A_{13}$</th>
<th>$A_{14}$</th>
<th>$A_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>15.7</td>
<td>3</td>
<td>0.27</td>
<td>0.038</td>
<td>0.012</td>
<td>0.051</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60</td>
<td>0.32</td>
<td>0.014</td>
<td>3.3 x 10$^{-4}$</td>
<td>0.021</td>
<td>0.36</td>
</tr>
<tr>
<td>15.7</td>
<td>70</td>
<td>3</td>
<td>0.23</td>
<td>0.050</td>
<td>0.026</td>
<td>0.017</td>
<td>0.32</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60</td>
<td>0.28</td>
<td>0.020</td>
<td>1.7 x 10$^{-3}$</td>
<td>8.2 x 10$^{-3}$</td>
<td>0.31</td>
</tr>
<tr>
<td>70</td>
<td>21</td>
<td>3</td>
<td>0.30</td>
<td>0.047</td>
<td>0.018</td>
<td>0.060</td>
<td>0.42</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60</td>
<td>0.34</td>
<td>0.015</td>
<td>3.5 x 10$^{-4}$</td>
<td>0.022</td>
<td>0.37</td>
</tr>
<tr>
<td>8.6</td>
<td>21</td>
<td>3</td>
<td>0.25</td>
<td>0.060</td>
<td>0.036</td>
<td>0.020</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60</td>
<td>0.30</td>
<td>0.022</td>
<td>2.0 x 10$^{-3}$</td>
<td>8.9 x 10$^{-3}$</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Two extreme $M_p$ values of 1 and 140 mg$^{-1}$, with $\lambda_2^{-1}$ and $W_f$ kept fixed at typical values of 21 min and 15.7 mg respectively. The subsidiary parameters are varied in turn, except for $T_1$ and $T_2$ which are treated together. During the test of any one of these parameters, the remaining parameters retain their typical values $T_1 = 8$ min, $T_2 = 4.8$ min, $\lambda_1^{-1} = 28$ min, $W_f = 0.1$ mg, $w_2 = \pi/48$ mg and $w_3 = \pi/6$ mg.

The boundary between small and large splinters has been tested by changing $T_1$ from 8 to 9.6 and 6.4 min keeping $T_1 + T_2$ fixed at 12.8 min. The parameter $\lambda_1^{-1}$ has been changed to 42 and 21 min and $W_f$ varied by factors of 3 and 4. The lower mass limit of the large drops $w_2$ was varied by changing the diameter from $\frac{1}{4}$ to $\frac{3}{4}$ and $\frac{5}{6}$ mm. Of these parameters only the $\lambda_1$ changes affected values of $p_0$ in the second decimal place and they may all be regarded as minor parameters. By contrast, the discussion in the appendix relating to the preponderance of the largest permitted drops in different distributions suggests that the upper large drop mass, $w_3$, may be an important parameter. Variations in the largest drop diameter from $\frac{1}{4}$ to $\frac{3}{4}$ and $\frac{5}{6}$ mm have been considered, resulting in $w_3$ variations from 0.52 to 0.22 and 1.77 mg respectively. Not surprisingly these large changes in maximum

TABLE 3. VALUES OF THE GROWTH PARAMETER $p_0$ (MIN$^{-1}$) ILLUSTRATING ITS DEPENDENCE ON THE DROP SPLINTERING NUMBERS $r_1$, $r_2$ IN CONJUNCTION WITH VARIATIONS IN $M_p$, $\lambda_2$ AND $W_f$, WITH $\lambda_1^{-1} = 28$ MIN

<table>
<thead>
<tr>
<th>$M_p$ (mg$^{-1}$)</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$\lambda_2^{-1}$ (min)</th>
<th>$\lambda_1^{-1}$ (min)</th>
<th>$w_f$ = 15.7 mg</th>
<th>$w_f$ = 8.6 mg</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0.21</td>
<td>0.27</td>
<td>0.32</td>
<td>0.19</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>0.23</td>
<td>0.29</td>
<td>0.34</td>
<td>0.21</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0.21</td>
<td>0.29</td>
<td>0.35</td>
<td>0.19</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1</td>
<td>0.26</td>
<td>0.34</td>
<td>0.40</td>
<td>0.24</td>
</tr>
<tr>
<td>60</td>
<td>0</td>
<td>0</td>
<td>0.52</td>
<td>0.76</td>
<td>0.94</td>
<td>0.51</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>0.54</td>
<td>0.78</td>
<td>0.97</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
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<td>1</td>
<td>0.53</td>
<td>0.77</td>
<td>0.98</td>
<td>0.51</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1</td>
<td>0.57</td>
<td>0.83</td>
<td>1.03</td>
<td>0.56</td>
</tr>
</tbody>
</table>
ICE PARTICLE MULTIPLICATION

drop mass produce significant changes in \( p_0 \). For \( M_p = 1 \) the growth parameter \( p_0 \) changes from 0.17 to 0.16 and 0.20 and for \( M_p = 140 \), \( p_0 \) changes from 1.05 to 0.80 and 1.69 as the largest diameter changes from 1 to \( \frac{1}{4} \) and \( \frac{1}{2} \) mm respectively. Although these diameter changes are perhaps extreme, the need for further experimental information relating to the top end of the large drop spectrum is clearly indicated.

The remaining numerical results relate to models which include drop splintering. Table 3 gives some \( p_0 \) values for small drop splinter numbers \( r_1 = 0, 2, 4 \) and large drop splinter numbers \( r_2 = 0, 1 \) for two values of \( M_p \) and \( W_f \) and the three non-zero values of \( \lambda_2 \). Markedly different behaviour occurs for the two chosen \( M_p \) values. For \( M_p = 3 \) \( mg^{-1} \) the splinter values \( r_1 = 4, \ r_2 = 1 \) provide an increase in \( p_0 \) comparable with that produced at zero \( r_1 \) and \( r_2 \) by the increases of about 0.04 \( min^{-1} \) in \( \lambda_2 \) listed in Table 3. However, at the larger \( M_p \) value of 60 \( mg^{-1} \), the increases in \( \lambda_2 \) produce much larger changes in \( p_0 \) than are achieved by the splinter numbers \( r_1, r_2 \). At \( M_p = 60 \) \( mg^{-1} \) even \( r_1 = 10, \ r_2 = 1 \) does not produce an effect as big as that of the \( \lambda_2 \) changes. Thus drop splintering emerges as an important effect at modest \( M_p \) values.

TABLE 4. VALUES OF THE GROWTH PARAMETER \( p_0 (min^{-1}) \) FOR ZERO \( M_p \), i.e., A DROP SPLINTERING ONLY MECHANISM. FOR THE NON-ZERO VALUES OF \( \lambda_2 \), RESULTS ARE GIVEN FOR VARIOUS \( r_1 \) AND \( r_2 = 0 \) OR 1; FOR \( \lambda_2 = 0 \) THE \( r_2 \) DEPENDENCE IS REMOVED. THE RESULTS ARE INDEPENDENT OF \( W_f \) AND \( \lambda_1^{-1} \) REMAINS FIXED AT 28 MINUTES. THE FINAL COLUMN OF \( p_0 = (r_1 - 1)\lambda_1 \) VALUES DESCRIBES THE SMALL SPLINTER UNBounded LIFETIME MODEL, \( \lambda_2 = 0, T_1 = \infty \); IN THE EARLIER COLUMNS \( T_1 = 8 min \)

<table>
<thead>
<tr>
<th>( r_1 )</th>
<th>( \lambda_2^{-1} (min) )</th>
<th>( 21 )</th>
<th>70</th>
<th>( \lambda_2^{-1} (min) )</th>
<th>( 21 )</th>
<th>70</th>
<th>( \lambda_2 = 0 )</th>
<th>( r_1 - 1 ) ( \lambda_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>8.9 ( \times 10^{-4} )</td>
<td>0.038</td>
<td>0.058</td>
<td>0.023</td>
<td>0.036</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>0.024</td>
<td>0.066</td>
<td>0.099</td>
<td>0.16</td>
<td>0.13</td>
<td>0.11</td>
<td>0.18</td>
</tr>
<tr>
<td>6</td>
<td>0.12</td>
<td>0.20</td>
<td>0.24</td>
<td>0.23</td>
<td>0.22</td>
<td>0.21</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.21</td>
<td>0.25</td>
<td>0.28</td>
<td>0.32</td>
<td>0.31</td>
<td>0.30</td>
<td>0.30</td>
<td>0.32</td>
</tr>
</tbody>
</table>

Finally in Table 4 results for a drop splintering only mechanism, \( M_p = 0 \), are presented. If large drops produce no splinters on freezing, \( r_2 = 0 \), they act only to deplete the concentration of splinters and \( p_0 \) decreases as \( \lambda_2 \) increases. However, if we assume that each large drop produces one splinter on freezing, \( r_2 = 1 \), then \( p_0 \) increases with \( \lambda_2 \); in this case the large drops just rejuvenate the splinters. As a rough guide, it is seen that \( r_1 \) needs to be about eight in order to produce \( p_0 \) values comparable with those corresponding to \( M_p = 3 \) \( mg^{-1} \) in the earlier tables. The results in Table 4 for the case \( \lambda_2 = 0 \), which are independent of \( r_2 \) and \( W_f \), may be compared with those of Chisnell and Latham (1974). In this earlier work, the growth of the splinters was neglected and they were assumed to persist unless captured, i.e. \( T_1 = \infty \). This earlier analysis gave \( p_0 = (r_1 - 1)\lambda_1 \) and the comparison is given in the last two columns of the table. The additional realism which results from a consideration of the growth of splinters is seen to result in a slight decrease in \( p_0 \) at large values of \( r_1 \). As \( r_1 \) decreases the effect of finite \( T_1 \) becomes more pronounced.

The growth condition for the general model discussed in this paper, will now be derived. The exponential form of the leading term in the large time analysis indicates that the growth condition is that the denominator of the Laplace transforms considered in the appendix has a positive zero. When \( p \) is large, the denominator is near 1, so that a positive zero exists if the denominator is negative at \( p = 0 \). This condition is
\[ 1 < \lambda_1 r_1(1 - e^{-\lambda T})/\lambda + E e^{-\lambda T} M_p W_0 (e^{\theta T} - 1) + \]
\[ + \lambda_2 (1 - e^{-\lambda T})/\lambda + E(1 - e^{-\lambda T}) [r_2 + M_p (W_f - \frac{1}{2} (w_2 + w_3))]. \]  
(4.3)

where \( E = e^{-\lambda T} + \lambda_1 (1 - e^{-\lambda T})/\lambda \) is the probability that a small splinter becomes a large splinter, either by growth or as a result of capture by a small drop. The coefficient of \( r_1 \) is the probability that a small splinter will be captured by a small drop and the coefficient of \( r_2 \) is the probability that an initial small splinter, or a possible resulting large splinter, is captured by a large water drop. To \( r_2 \) is added the estimated number of small splinters produced by the resulting type 4 rimer, assuming a top-hat distribution of large water drops. Finally \( M_p W_0 (e^{\theta T} - 1) \) is the estimated number of splinters produced by a type 3 rimer, and its coefficient is the probability that a small splinter will become a growth rimer. Hence it is seen that the growth condition has an obvious physical interpretation; the estimated number of small splinters arising from one splinter, allowing for the various possible life histories of the splinter, must be greater than one. For \( M_p = 0 \), the inequality shows that the missing entries in Table 4 do not satisfy the growth condition.

5. Multiplication times for various physical models

The calculations so far presented give the estimated number of ice particles resulting from one initial splinter. However, the experimental data relates to the multiplication factor \( f \), the ratio of the concentrations of ice particles to ice forming nuclei effective at the cloud top temperature \( T_e \). To derive an expression for \( f \) we now embed the results of earlier sections in a rudimentary model of the airflow within the cloud. We shall see that the form assumed for the airflow does not have a major effect upon the multiplication rate, a constant updraught and a thermal model giving similar results.

For most calculations we assume that an updraught of constant speed \( U \) passes through the cloud except in a thin layer at the summit, where the airflow becomes horizontal and the vertical speed falls to zero. For the cloud properties assumed, the values of \( W_f \), 15.7, 8.6 and 3.0 mg can be shown to correspond to \( U = 2.0, 1.3 \) and 0.5 m s\(^{-1}\) respectively. In this model we assume that a concentration, \( n_p \), of ice nuclei effective at \( T_e \) is carried to the cloud top by the updraught \( U \). The ice nuclei are all assumed to form ice particles on arrival at the cloud top, so that the rate of formation of small splinters per unit cross-sectional area of the cloud is \( Un_i \). If the nucleation commences at \( t = 0 \), the estimated number of resulting ice particles per unit area at time \( t \) is given by (4.2) as

\[ N \approx Un_i A_1 e^{\nu t}/p_0. \]  
(5.1)

These particles are distributed over the height \( Z \) between the \( T_e \) and 0°C isotherm. Hence the overall cloud multiplication factor is

\[ f = \frac{UA_1 e^{\nu t}}{Zp_0}. \]  
(5.2)

The largest multiplication factor measured by Mossop et al. (1972) is about \( 10^4 \) and the time taken for \( f \) to reach this value is

\[ \tau = \frac{1}{p_0} \ln \left\{ \frac{10^4 Zp_0}{UA_1} \right\}. \]  
(5.3)

This formula for \( \tau \) provides a convenient test for the efficacy of various possible physical growth mechanisms. Four mechanisms of ice particle multiplication will now be discussed. In each case \( \tau \) is estimated for appropriate values of the physical parameters.
The first three mechanisms involve splinter production by riming and for each of these cases the experimental results of Bader et al. (1974) are represented by $M_p = 3\text{mg}^{-1}$ and those of Hallett and Mossop (1974) by $M_p = 60\text{mg}^{-1}$. Some of these results are shown in Table 5.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\lambda_1^{-1}$ (min)</th>
<th>$\lambda_2^{-1}$ (min)</th>
<th>$U$ (m/s)</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$M_p (\text{mg}^{-1})$</th>
<th>$p_0\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>60</td>
</tr>
<tr>
<td>A</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>103</td>
<td>48.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2</td>
<td></td>
<td></td>
<td>66.5</td>
<td>39.8</td>
</tr>
<tr>
<td>B</td>
<td>28</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>62.0</td>
<td>35.5</td>
</tr>
<tr>
<td></td>
<td>70</td>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
<td>52.7</td>
<td>23.1</td>
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<td></td>
<td>41.4</td>
<td>16.2</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td>0</td>
<td></td>
<td>35.3</td>
<td>13.1</td>
</tr>
<tr>
<td>C</td>
<td>28</td>
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<td></td>
<td></td>
<td></td>
<td>2</td>
<td>0</td>
</tr>
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<td>8</td>
<td>1</td>
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<td>2</td>
<td>0</td>
<td>38.3</td>
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<td></td>
<td></td>
<td>8</td>
<td>1</td>
</tr>
</tbody>
</table>

TABLE 5. VALUES OF $\tau$, THE NUMBER OF MINUTES REQUIRED TO ACHIEVE A MULTIPLICATION FACTOR OF $10^4$, AND $p_0\tau$ FOR VARIOUS VALUES OF $\lambda_1$, $\lambda_2$, $U$, $r_1$, AND $r_2$. MECHANISMS A, B, C ARE RIMING ONLY, RIMING WITH RAIN DROPS PRESENT AND A COMBINED MODEL WITH RIMING AND DROP SPLINTERING RESPECTIVELY. THE VALUES OF $W_f$ CORRESPONDING TO $U = 0.5$ AND $2\text{m s}^{-1}$ ARE $3\times 10^3$ AND $15\times 10^2\text{mg}$ RESPECTIVELY

Model A utilizes a riming only mechanism. No supercooled raindrops are present in the cloud, so that $\lambda_1 = 0$, $\lambda_2 = 0$. Table 5 gives results for $U = 0.5$ and $2\text{m s}^{-1}$. For both of these values of $U$ it is seen that whereas the lower value of $M_p = 3\text{mg}^{-1}$ does not provide an adequate growth mechanism, the higher value of $M_p = 60\text{mg}^{-1}$ easily does so.

Model B contains a rimer-raindrop mechanism. Supercooled raindrops are present in the cloud, but they produce no splinters on freezing, $r_1 = 0$, $r_2 = 0$. Results are shown for the standard value $\lambda_1^{-1} = 28\text{min}$ covering the cited range of $\lambda_2$ values, with an updraught of $2\text{m s}^{-1}$. A comparison with the results for model A, reveals that the presence of the raindrops has a marked effect; the time taken to achieve a multiplication factor of $10^4$ is significantly reduced for both values of $M_p$. Thus even when raindrops do not directly produce splinters, they have the important role of shortening some splinter lifetimes and in consequence producing some rimers earlier. The influence of $U$, or the related parameter $W_f$, on $\tau$ is much less pronounced.

In model C a combined mechanism produces splinters by both riming and drop freezing. Model C differs from model B by having non-zero splinter numbers $r_1$ and $r_2$. At large $M_p$, it is seen in Table 5 that these splinter numbers have little effect, but when $M_p = 3\text{mg}^{-1}$ $\tau$ is reduced significantly for large $r_1$, especially when $\lambda$ is small. In this model, as well as in model B, variations in $U$ have little influence on $\tau$.

Table 5 also presents values of $p_0\tau$. It is seen that $p_0\tau$ has little variation in the situations considered: all values are within about $10\%$ of 12. Consequently $\tau$ may be estimated in other situations from the $p_0$ values presented in Tables 1 and 3. If we assume that 50
TABLE 6. VALUES OF $r$ FOR A DROP SPLINTERING ONLY MODEL. THESE $r$ VALUES CORRESPOND TO THE $p_0$ VALUES GIVEN IN THE LAST THREE ROWS OF TABLE 4. THE SMALL AND LARGE SPLINTER LIFETIMES USED ARE $T_1 = 8\text{min}$ AND $T_2 = 48\text{min}$ RESPECTIVELY, EXCEPT IN THE LAST COLUMN WHICH AGAIN RELATES TO THE SMALL SPLINTER UNBOUNDED LIFETIME MODEL. VALUES OF $\lambda^{-1} = 28\text{min}$ AND $U = 1.3\text{m s}^{-1}$, CORRESPONDING TO $W_f = 8.6\text{mg}$, ARE USED

<table>
<thead>
<tr>
<th>$r_2 = 0$</th>
<th>$r_2 = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1$</td>
<td>$\lambda^{-1}(\text{min})$</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>6</td>
<td>276</td>
</tr>
<tr>
<td>8</td>
<td>73.7</td>
</tr>
<tr>
<td>10</td>
<td>46.9</td>
</tr>
</tbody>
</table>

minutes are available to produce $f = 10^4$ we conclude that $p_0$ must be greater than about $0.24\text{min}^{-1}$.

Model D contains the drop splintering mechanism. Splinter production by riming does not occur in this model, $M_p = 0$. Values of $r$ for $r_1 = 6, 8, 10$ are presented in Table 6 and it is seen that $r_1$ must be as large as eight to produce values of $r$ around 50 minutes. Comparison of the final two columns in Table 6 reveals that, in the absence of large drops, the finite lifetime of the small splinters, $T_1 = 8\text{min}$, produces values of $r$ significantly longer than those predicted by Chisnell and Latham (1974).

Although all four models are capable of explaining the observed multiplication factor of $10^4$ within 50 minutes, models B and C which involve large supercooled drops result in considerably stronger growth laws and consequently do not require such large values of the key physical parameters to explain the observed multiplication rates. As Mossop et al. (1970, 1972) report that large supercooled drops are present whenever significant multiplication takes place, we conclude that model B or C is the most likely description of many observed growths. Further support for this proposition is supplied by the observation of Koenig (1963) that the multiplication factor reached $10^3$ in only 5 minutes in one particular shallow supercooled cloud over Missouri. Such rapid multiplication can only be achieved in models B and C, with the dominant parameters $M_p$ and $\lambda_2$ being high; Koenig noted that high concentrations of drops were present in the cloud. It should be emphasized that this observation must be treated with some caution until further evidence of such a prodigious multiplication rate has been obtained.

Mossop et al. (1968) conducted a particularly detailed study, off the southern coast of Australia, of an inversion-limited, long-lived maritime cumulus cloud whose top was at $-4\degree\text{C}$. The supercooled depth of the cloud, $Z$, was 1km and its mean water content was about 0.4g m$^{-3}$. The reported multiplication factor was about $10^3$ and the time available to produce it was between 1 and 1$\frac{1}{2}$ hours. Appreciable concentrations of small raindrops were reported, corresponding to an estimated $1/\lambda \sim 28$ minutes and significant numbers of frozen drops were found. No evidence was presented for the existence of large raindrops or of ice particles of mean dimension in excess of about 400$\mu$m. These observed particle sizes suggest an average updraught velocity not greater than $1\text{m s}^{-1}$. In these circumstances multiplication by riming only, as in model A, is unlikely to occur, as at no stage of its lifetime within the cloud will a rimer satisfy simultaneously the temperature and fall velocity conditions which must be met in order for the large values of $M_p$ measured by Hallett and Mossop to be applicable. However, model B or C, with $M_p = 3\text{mg}^{-1}$, or model D with $r_1 = 6$, are each capable of explaining the reported multiplication.

Hobbs (1974) has reported values of $f$ between $10^3$ and $10^4$ in a layer cloud of thickness
600m and temperature range $-5$ to $-2^\circ\text{C}$, located above Snoqualmie Pass in the state of Washington. No drops of diameter in excess of 100 $\mu\text{m}$ were observed, but substantial quantities of rimed ice crystals existed in the cloud. It thus appears that model A would be most appropriate to this situation. We estimate from the evidence provided that $W_f \approx 1\text{mg}$ and a weighted mean value of $M_p \approx 30\text{mg}^{-1}$. The corresponding value of $p_0$ is $0.20\text{min}^{-1}$ and multiplication factors $f$ of $10^3$ and $10^4$ are achieved in $44.6$ and $56.3\text{min}$ respectively if an updraught of $1\text{m s}^{-1}$ is assumed. The times are increased by $3.5\text{min}$ if the updraught is reduced to $0.5\text{m s}^{-1}$. These predictions appear to be consistent with the evidence presented by Hobbs.

We conclude that the analysis presented in this paper is capable of explaining all the ice particle multiplication observations that have been considered, although the choice of model A, B, C or D may not be the same in all situations.

The foregoing discussion is concerned with values of $\tau$ resulting from a 'constant updraught' model of the airflow. We now demonstrate that a 'thermal' model gives similar results.

Mason (1975b) gives the concentration of ice forming nuclei at a depth $z$ below the cloud top as $n_i \exp(-\alpha z)$, where $\alpha = 2.9 \times 10^{-5}\text{cm}^{-1}$. The total number of ice nuclei activated during the ascent of a thermal is approximately $n_i/\alpha$ per unit area, as $\exp(-\alpha Z)$ is small if the cloud depth $Z$ is a kilometre or more. Each of these activated nuclei results in $A_1 \exp(p_0 t)$ ice particles at time $t$ and the multiplication factor is $f = A_1(\alpha Z)^{-1} \exp(p_0 t)$. Thus the multiplication time is

$$\tau = \frac{1}{p_0} \ln \left( \frac{10^4 \alpha Z}{A_1} \right).$$

Following Mason we assume that each ice particle formed by nucleation will achieve a radius of $0.1\text{mm}$ after 10 minutes and then begin to fall through the quiescent cloud. If $Z = 1.2\text{km}$, $\rho = 0.3\text{g cm}^{-3}$ and the collection efficiency is unity we estimate that the exit mass $W_f$ of rimers formed at the top of the cloud is $1.7\text{mg}$. We adopt this value of $W_f$ for all rimers, as we believe that following the ascent of the thermal there will be a considerably reduced updraught, possibly arising from the same energy source as the thermal. This reduced updraught will carry splinters towards the cloud top. We take the same value of $T_1 + T_2 = 12.8\text{min}$ as before and estimate from the data of Koenig that $T_3 = 5.2\text{min}$, the corresponding value of $\beta$ being $0.54\text{min}^{-1}$. The estimated weighted average value of $M_p$ is $60\text{mg}^{-1}$. Assuming $\lambda_1 = 0 = \lambda_2$, there follows $p_0 = 0.28\text{min}^{-1}$, $A_1 = 0.22$ and from Eq. (5.4) $\tau = 42.8\text{min}$. This agrees well with the value of $\tau = 48.2\text{min}$ calculated for identical parameter values on the constant updraught model. If we permit small drops to be present within the cloud but not to splinter on freezing, $r_1 = 0$, and take $1/\lambda_1 = 28\text{min}$ the growth parameter is increased from $0.28$ to $0.33\text{min}^{-1}$.

The reason why the constant updraught and thermal models yield values of $\tau$ which are in close agreement can most readily be seen by comparing equations (5.3) and (5.4). In each case the value of $\tau$ is approximately $p_0^{-1} \ln 10^4$ as both $Zp_0/U A_1$ and $\alpha Z/A_1$ are about 1. Thus $\tau$ depends only on $p_0$ and $p_0$ is governed principally by the microphysical parameters $M_p$, $\lambda_2$ and $r_1$. The nature of the airflow affects $p_0$ only through the less sensitive parameter $W_f$ and we conclude that the precise form of airflow does not exercise a major influence on the multiplication.

6. DISCUSSION

We conclude this paper by discussing the validity of some of our assumptions and suggesting an experiment which may help to define the importance of supercooled raindrops in ice particle multiplication.
The only mechanism of loss of ice particles from the cloud in this model is the removal of rimers on reaching a mass \( W_f \). Particles may be also lost as a consequence of entrainment of undersaturated air at the cloud boundaries and by the lateral motion at the cloud top produced by the updraught. An estimate of the effect on the growth law of the loss of a fraction, \( \delta \), of the ice particles by these two processes may be crudely obtained by multiplying the splinter production parameters \( M_p \), \( r_1 \) and \( r_2 \) by \((1 - \delta)\). For slow growth conditions this factor can have a significant effect on \( \tau \), but in satisfactory growth conditions only modest increases in \( \tau \) result. For instance, Table 1 shows that even when \( M_p \) is halved, values of \( p_s \) greater than \( 0.2 \text{min}^{-1} \) are only slightly reduced and consequently \( \tau \) is only marginally increased.

An assumption in our model is that the cloud properties do not change during the multiplication process. In particular, we are obliged to question the assumption that the liquid water properties of the cloud are not significantly changed before the multiplication factor has reached \( 10^4 \).

It may easily be demonstrated that the depletion of cloud droplet water by accretion onto rimers is negligible. Even for the rimer concentration corresponding to \( f = 10^4 \) the rate of removal of droplet water by accretion is less than the average rate of production of cloud water by condensation over the multiplication time \( \tau \).

Depletion of the small raindrops by collision with small ice splinters and consequent freezing has been considered in detail by Chisnell and Latham (1974), who showed that even when \( f = 10^4 \) the numbers were not significantly diminished.

Finally, we consider the depletion of large raindrops. The values of \( \lambda_2 \) used in this paper are based on the measurements of large drops by Mossop et al. (1970, 1972) which were made after significant multiplication of ice particles had taken place. A model based on these values of \( \lambda_2 \) may therefore be safely used to explain the observed multiplication factors. In clouds exhibiting significant multiplication the concentration of large ice particles found by Mossop at the same time as the measurements of large drops was typically 1 per litre. For the cases considered in Table 2 this concentration of rimers is achieved when \( f \) has reached about \( 10^4 \). We conclude that the \( \lambda_2 \) values used in this paper may be applied throughout the growth time \( \tau \). Although the depletion of large raindrops by the formation of capture rimers may well eventually necessitate a decrease in \( \lambda_2 \), this change is not needed before \( f \) has reached \( 10^4 \).

It is difficult to define field experiments that establish which of the models of ice particle multiplication is appropriate. The much-reported observation that rimers and large drops co-exist within clouds that exhibit multiplication can be adduced as evidence in favour of models B or C. However, as Mason (1975a) points out, the drops may play no role in the multiplication, being simply a consequence of the age of the cloud. Table 2 shows that the ratio \( A_{13}/A_{14} \) of growth rimers to rimers produced by drop-freezing is very sensitive to \( \lambda_2 \), the sweep out rate of large drops. The importance of raindrops in the multiplication can therefore be assessed by measurement of \( A_{13}/A_{14} \). Although records already obtained by Mossop et al. (1968, 1970, 1972) demonstrate that both types of rimer exist within clouds exhibiting multiplication, the rimers can be usually distinguished only in the early stages of their growth if the formvar sampler is used. However, if these large ice particles could be collected in flight, subsequent analysis could reveal their origins.

ACKNOWLEDGMENT

We are grateful to Mr. J. Siemieniuch for writing the programme used to obtain most of the results in section 4. This work was in part carried out within the research programme supported by NERC under grant GR3/2425.
The production of sub-micron ice fragments by water droplets freezing in free fall or on accretion upon an ice surface, *Quart. J. R. Met. Soc.*, 100, pp. 420–426.


**APPENDIX**

In this appendix the renewal equations derived in section 3 are studied. Results from this study are presented in section 4.

The renewal equations (3.13) for the means are solved by use of Laplace transforms. Let

\[ m(p) = \int_0^\infty e^{-pt} M(t) \, dt \quad . \quad . \quad . \quad (A.1) \]

with the appropriate \( M \) suffixes and \( w \)-dependence of \( M_{4j} \) being transferred to the \( m \) functions. The terms involving a \( u \)-integration in (3.13) are simplified by reversing the order of the \( t \) and \( u \) integrations. For instance

\[ \int_0^\infty \int_0^t M_{4j}(t-u) \, dG_4(u) e^{-pu} \, dt = \left( \int_0^\infty M_{4j}(t) e^{-pt} \, dt \right) \left( \int_0^\infty e^{-pu} \, dG_4(u) \right) \quad . \quad . \quad . \quad (A.2) \]

indicating the need for the finite \( G \)-transforms

\[ g_i(p) = \int_{u=0}^{T_i} e^{-pu} \, dG_i(u) \quad . \quad . \quad . \quad (A.3) \]
i = 1, 2, 3, 4 where \( g_{4}, T_{4}, G_{4} \) depend on \( w \). The upper limits of integration arise as \( G_{i}(t) \) is constant for \( t > T_{i} \). For the \( r(w) \) integrals occurring in (3.13), the mean small and large drop splinter numbers

\[
\begin{align*}
 r_{1} &= \int_{w_{1}}^{w_{2}} r(w)\{S(w)/S_{1}\} \, dw, \\
 r_{2} &= \int_{w_{2}}^{w_{3}} r(w)\{S(w)/S_{2}\} \, dw
\end{align*}
\]  

(A.4)

are introduced. Finally the large drop integrals involving \( M_{4}(w, t-u) \) in the first two of equations (3.13) require the Laplace transform means

\[
\overline{m_{4}}(p) = \int_{w_{2}}^{w_{3}} m_{4}(w, p)\{S(w)/S_{2}\} \, dw.
\]  

(A.5)

These definitions enable the Laplace transforms of (3.13) to be written as

\[
\begin{align*}
 m_{1j}(p) &= \delta_{1j} \int_{t=0}^{\infty} \{1 - G_{1}(t)\}e^{-pt} \, dt + \{1 - G_{1}(T_{1})\}e^{-pT_{1}}m_{2j}(p) + \\
 & \quad + g_{1}(p)\{r_{1}S_{1} + r_{2}S_{2}\}m_{1j}(p) + S_{1}m_{2j}(p) + S_{2}\overline{m_{4}}(p), \\
 m_{2j}(p) &= \delta_{2j} \int_{t=0}^{\infty} \{1 - G_{2}(t)\}e^{-pt} \, dt + \{1 - G_{2}(T_{2})\}e^{-pT_{2}}m_{3j}(p) + \\
 & \quad + g_{2}(p)\{r_{2}m_{1j}(p) + \overline{m_{4}}(p)\}, \\
 m_{3j}(p) &= \delta_{3j}\{1 - e^{-pT_{3}}\}/p + r_{3}g_{3}(p)m_{1j}(p), \\
 m_{4j}(w, p) &= \delta_{4j}\{1 - e^{-pT_{4}(w)}\}/p + r_{4}g_{4}(w, p)m_{1j}(p).
\end{align*}
\]  

(A.6)

The \( \overline{m_{4}}(p) \) occurring in the first two equations may be expressed in terms of a mean time \( T_{4} \) and a mean mass \( \overline{w} \) by integration of the fourth equation,

\[
\overline{m_{4}}(p) = \delta_{4j}\{1 - e^{-pT_{4}}\}/p + r_{3}g_{4}(\overline{w}, p)m_{1j}(p),
\]  

(A.7)

where the large drop means \( T_{4}, \overline{w} \) are given by

\[
\begin{align*}
 e^{-pT_{4}} &= \int_{w_{2}}^{w_{3}} e^{-pT_{4}(w)}\{S(w)/S_{2}\} \, dw, \\
 g_{4}(\overline{w}, p) &= \int_{w_{2}}^{w_{3}} g_{4}(w, p)\{S(w)/S_{2}\} \, dw.
\end{align*}
\]  

(A.8)

It should be noted that these means have a \( p \)-dependence and do not therefore have a direct physical interpretation. Equations (A.6) may now be solved for \( m_{1j}(p) \). Use of the formulae for \( G_{i} \) derived in section 3 enables the solution to be written as

\[
\begin{align*}
 m_{11} &= g_{1}(\lambda D), \\
 m_{12} &= Eg_{2}(\lambda_{2} D), \\
 m_{13} &= Ee^{-(p+\lambda_{3})T_{3}(1-e^{-pT_{3}})}/(pD), \\
 m_{14} &= (1-e^{-pT_{4}})(\lambda_{2}g_{1} + \lambda Eg_{2})/(\lambda p D),
\end{align*}
\]

where

\[
\begin{align*}
 D &= 1 - (r_{1}\lambda_{1} + r_{2}\lambda_{2})g_{1}/\lambda - Er_{2}g_{2} - (\lambda_{2}g_{1} + \lambda Eg_{2})r_{3}g_{4}(\overline{w}, p)/\lambda - \\
 & \quad - r_{3}g_{3}(p)Ee^{-(p+\lambda_{3})T_{3}}, \\
 E &= e^{-(p+\lambda T_{1})} + \lambda_{1}g_{1}/\lambda_{1}, \quad \lambda = \lambda_{1} + \lambda_{2},
\end{align*}
\]  

(A.9)
ICE PARTICLE MULTIPLICATION

\[ g_1(p) = \lambda\{1 - e^{-(p + \lambda)}T_1\}/(p + \lambda), \]
\[ g_2(p) = \lambda_2\{1 - e^{-(p + \lambda_2)}T_2\}/(p + \lambda_2), \]
\[ r_3 g_3(p) = M_p W_f \beta \{e^{(\beta - p)T_3} - 1\}/(\beta - p), \]
\[ r_3 g_4(w, p) = M_p w\beta \{e^{(\beta - p)T_4(w)} - 1\}/(\beta - p). \]

These equations involve the large drop means \( T_4 \) and \( \bar{w} \). For a top-hat distribution in which large drop masses are distributed over the range \( w_2 < w < w_3 \) so as to provide uniform contributions to the sweep out rate \( \lambda_2 \), \( S(w) \) has the constant value \( S_2/(w_3 - w_2) \) and the means are given by

\[
e^{-pT_4} = \frac{\beta W_f}{(w_3 - w_2)(p + \beta)} \left[ e^{-(p + \beta)T_4(w)} \right]_{w_2}^{w_3},
\]

(A.10)

\[
r_3 g_4(\bar{w}, p) = \frac{M_p \beta W_f}{(\beta - p)} \{e^{-pT_4} - \frac{1}{2}(w_2 + w_3)/W_f\}.
\]

Other choices of large drop distributions will be considered after the inversion of the Laplace transforms has been considered.

The presence of the exponential functions occurring in \( D \) in (A.9) precludes the possibility of a simple interpretation of the transforms. Two methods of approximating

![Figure 2](image.png)

Figure 2. The estimated number of small splinters arising from one growth rimer born at \( t = 0 \). For both \( M_\mu = 1 \) and \( M_p = 140 \text{mg}^{-1} \), the exact marching technique (M) merges with the simpler large time analysis (L) well within the available time for multiplication. The sharp peak in the marching solution for \( M_p = 1 \text{mg}^{-1} \) occurs at the death of the initial rimer at 13.5 minutes.
to the inverse transform are considered. Firstly, the large time analysis is given, which involves investigating the behaviour of the transforms at those singularities with a positive real part. These singularities are given by the zeros of the \( D \) function. Secondly a marching out technique is described which involves an expansion of each \( m_{ij}(p) \) in a power series of the exponential functions that occur. Each term in the resulting series has a known inversion and represents a term switched on at a particular time. For instance an \( \exp\{-(p+\lambda)T_1\} \) term is one switched on at \( t = T_1 \). An exact solution is thus built up; the complexity of the algebra limits the time interval, starting from \( t = 0 \), for which the solution has been determined. This exact solution has been obtained over a sufficient time interval to bring it into excellent agreement with the large time analysis, as shown in Fig. 2. The two solutions merge at a time well before the number of ice particles has reached 100. We conclude that the simpler large time analysis gives a valid description of the growth of the mean number of ice particles.

The demonstration of the utility of the simple large time analysis is performed for a particular large drop distribution and two choices of the physical parameters. The simple top-hat distribution of large water drops already referred to is used. The particular parameter values chosen use the two extreme values of \( M_p \) and are

\[
M_p = 1 \text{ and } 140 \text{mg}^{-1}, \quad r_1 = 0, \quad r_2 = 0,
\]

\[
W_g = 0.1, \quad W_f = 15.7, \quad w_2 = \pi/48, \quad w_3 = \pi/6 \text{mg}, \quad (A.11)
\]

\[
\beta^{-1} = 2.7, \quad \lambda_{r_1}^{-1} = 28, \quad \lambda_{r_2}^{-1} = 21, \quad T_1 = 8, \quad T_2 = 4.8 \text{min.}
\]

The analysis of this appendix deals with the number of ice particles resulting from one initial ice particle of any particular type. We have chosen to illustrate the merging of the marching out and large time analysis using an initial growth rimer switched on at \( t = 0 \). The same denominator \( D \) occurs in all \( m_{ij} \) and the conclusion that the simple large time analysis may be used is valid for any choice of initial ice particle.

For the large time analysis the zeros of \( D(p) \) in (A.9) having a positive real part are required. In the case \( M_p = 1 \text{mg}^{-1} \) they have been found to be

\[
p_0 = 0.17 \text{min}^{-1}, \quad p_1 \pm ip_2 = (0.054 \pm 0.053) \text{min}^{-1}. \quad (A.12)
\]

Corresponding to these singularities in \( m_{3j} \), the mean number of small ice splinters arising from one growth rimer at \( t = 0 \) has the large time behaviour

\[
M_{31}(t) = 0.7e^{p_0t} + 0.87e^{p_1t} \cos(p_2t - 1.21), \quad (A.13)
\]

which is displayed in Fig. 2 for \( t > 5 \text{min.} \) Similar expressions for the numbers of large splinters and rimers have been found and the total of all ice particles arising from one growth rimer is given as

\[
M_{3}(t) = 1.14e^{p_0t} + 0.71e^{p_1t} \cos(p_2t - 1.02). \quad (A.14)
\]

When \( M_p \) is increased sufficiently the oscillatory behaviour arising from the complex zeros is absent and for \( M_p = 140 \text{mg}^{-1} \) the corresponding results for small splinters and all particles are

\[
M_{31}(t) = 2.9e^{p_0t}, \quad M_{3}(t) = 3.1e^{p_0t}. \quad (A.15)
\]

with \( p_0 = 1.05 \text{min}^{-1} \). This \( M_{31}(t) \) is also displayed in Fig. 2.

To illustrate the marching out technique we derive just two terms in the \( M_{31}(t) \) series. The transform \( m_{31}(p) \) may be expanded in power series of the exponentials \( e^{-pT_1}, e^{-pT_2}, e^{-pT_3}, e^{-pT_4(w_3)} \) and \( e^{-pT_4(w_3)} \). Each term in the resulting series is a rational function of \( p \)
times one such exponential term; the $e^{-pT_1}$ term has an inversion which is switched on at $t = T_1$. By far the most important terms are those switched on at $t = 0$ and $t = T_3$. The first of these represents the system in the absence of all effects associated with the cut-off times $T_1, T_2, \ldots$. Although this first term gives the exact behaviour for $t < T_2$, it soon afterwards considerably overestimates the ice particle population. The other important term, switched on at $t = T_3$, takes account of the switching off of the initial rimer as it passes through the $0^\circ$C isotherm.

For $M_p = 1 \text{mg}^{-1}$ the marching out solution has been evaluated up to the time $t = 20 \text{min}$ and at this time these two dominant terms contribute 173 and $-141$ small splinters respectively. Only three other terms make contributions larger than one splinter at this time. The form of $m_{31}$ which produces just these two dominant terms is

$$m_{31}(p) = \frac{M_p W_g \beta (1-e^{(\beta - p)T_3})(p + \lambda_2)}{(p + \lambda)(p - p_3)(p - p_4)}, \quad (A.16)$$

where $p_3, p_4$ are the roots of

$$(p + \lambda_2)(p - \beta) - \frac{1}{2}\lambda_2 M_p \beta (w_2 + w_3) = 0. \quad (A.17)$$

For $M_p = 1 \text{mg}^{-1}$ the roots of (A.17) are $p_3 = 0.39$, $p_4 = -0.06 \text{min}^{-1}$ and the term switched on at $t = 0$ follows from (A.16) as

$$f(t) = 0.077e^{p_3 t} + 0.043e^{p_4 t} - 0.12e^{-2t}, \quad (A.18)$$

and the inversion of (A.16) is

$$M_{31}(t) = f(t) - e^{p_3 T_1} f(t - T_3) H(t - T_3), \quad (A.19)$$

where $T_3 = 13.5 \text{min}$. The full marching out solution is displayed in Fig. 2 up to the time $t = 20 \text{min}$ and is found to be in excellent agreement with the large time analysis at this time. It is of interest to note that even the oscillatory behaviour induced by the death of the initial rimer at $t = 13.5 \text{min}$ in the marching out solution is to some extent represented by the second term in the large time analysis.

For $M_p = 140 \text{mg}^{-1}$ it has already been noted that the second oscillatory term in the large time analysis is absent, there being no complex $p$-zeros with positive real part. The first term in the marching out solution, which is exact for $t < 4.8 \text{min}$, is

$$M_{31}(t) = 2.87e^{p_3 t} - 3.13e^{p_4 t} + 0.26e^{-2t}, \quad (A.20)$$

with $p_3 = 1.05, \ p_4 = -0.72 \text{min}^{-1}$. At $t = 4 \text{min}$ this gives 188 small ice splinters and is in complete agreement with the single term large time solution (A.15) at this time, as shown in Fig. 2.

Having demonstrated the validity of the large time analysis, we conclude the appendix by exploring the sensitivity of the growth law to the assumed distribution of large water drops. For this test we compare the $p_0$ values calculated for an observed distribution, a 'top-hat' distribution and an exponential distribution.

Mossop et al. (1972) present records from their drop sampler in terms of the concentrations of raindrops located at diameters $d = 0.25, 0.5, 0.75, 1.0, 1.25, 1.5$ and $>1.5 \text{mm}$, although it was only occasionally that drops were found in the top three bands. We chose for our test the flight 1a of 3 June 1970, and use the reported 'highest total concentrations'. This particular sample was selected because the size distributions show no obvious excessive random statistical fluctuations and because the associated value of $\lambda_2 (14.3 \text{min})^{-1}$ is close to the maximum considered in our model. We are therefore able to make a sensitive test of the importance of the distribution on the multiplication rate. In this chosen penetration
only the lowest four size bands are occupied, the concentrations being 0·4, 0·2, 0·25, and 0·22 per litre for \( d = 0·25, 0·5, 0·75 \) and 1·0mm respectively.

To incorporate the Mossop data in this model, the \( r_3 g_4(\bar{w}, p) \) defined in (A.8) is interpreted as

\[
r_3 g_4(\bar{w}, p) = \sum_{i=1}^{4} \frac{\Delta S(w_i)}{S_2} \frac{w_i \beta}{(\beta - p)} \{e^{(\beta - p)T_d(w_i)} - 1\},
\]

(A.21)

the four values of \( w_i \) being the drop masses at the four listed diameters and \( \Delta S(w_i) \) is the contribution to \( S_2 \) from the drops of mass \( w_i \). Small water drops, which would not have been detected by Mossop's apparatus, have been included and their proportional volume sweep out rate has been given the value \( \lambda_1 = (28\text{min})^{-1} \) used in other calculations in this paper. No drop splintering was considered, \( r_1 = 0 = r_2 \), and a value of \( M_p = 3\text{mg}^{-1} \)

was chosen. The parameters \( T_1, T_2, W_f, \beta \) were given their typical values discussed in earlier sections, \( W_f \) was given the value 15·7mg and the experimental value of \( \lambda_2^{-1} = 14·3\text{min} \) was used. For these values the exponent \( p_0 \) occurring in the large time solution has been found to be \( p_0 = 0·32\text{min}^{-1} \).

In the top-hat distribution, the large drops are assumed to lie between the diameter values of 0·25 and 1mm so as to give uniform contributions to \( \lambda_2 \). For the same choice of physical parameters, the result is \( p_0 = 0·30\text{min}^{-1} \). The exponential distribution is expressed by \( N(x) = Ae^{-bx} \), where \( x \) is diameter and \( A, b \) are constants. The contribution to the proportional volume sweep out rate from drops in the diameter range \( x, x + dx \) is

\[
d\lambda_2 = \frac{1}{4\pi kx^3} Ae^{-bx} \, dx,
\]

(A.22)

where \( \frac{1}{4}kx \) is drop speed. A choice of \( b = 5\text{mm}^{-1} \) gives a distribution in which half of the contribution to the sweep out rate comes from drops in the given diameter range of 0·25 to 1mm. The distribution is used only in this range and \( A \) is chosen to give the required value of \( \lambda_2^{-1} = 14·3\text{min} \). An integral expression for \( r_3 g_4(\bar{w}, p) \) follows and for the same choice of physical parameters the value of \( p_0 \) is 0·28min \(^{-1} \).

The proximity of the three \( p_0 \) values suggests that the form of the large water drop distribution is not critical. The ordering of the three \( p_0 \) values is instructive. In the Mossop data the largest drops make the largest contribution to \( \lambda_2 \), in the top-hat distribution the contributions are uniform and in the exponential distribution the largest drops make the smallest contribution to \( \lambda_2 \). It appears that the largest water drops enhance the growth mechanism more than other drops, though not to a critical extent.