Boundary layer flow over gentle curvilinear topography with a sudden change in surface roughness*

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SUMMARY

A new approach using a curvilinear coordinate system that enables incorporation of a change of roughness with a zero-plane displacement or elevation change in a boundary-layer model is presented. The effect of topography together with the rapid development of an internal boundary layer produce vertical wind speeds that reach a maximum value of about 25% of the horizontal component. It is found that perturbations due to a smooth-to-rough transition together with an increase of elevation are stronger than those generated from a rough-to-smooth transition with a decrease of elevation.

1. INTRODUCTION

The problem of the effect on the atmospheric turbulent boundary layer of a sudden change in surface roughness (Fig. 1) has been of current interest. Studies using mixing-length closure (Taylor 1969) or mean-energy equation closure (Peterson 1969) or higher-order closure schemes (Rao et al. 1973) have all assumed an abrupt change in surface roughness but omitted the effect of the surface geometry associated with such a roughness change. Unfortunately, terrain surfaces are seldom strictly horizontal over the entire fetch. In fact, a sudden change in surface roughness is most likely to be accompanied by an elevation change. However, conventional formulation of the problem based on a Cartesian coordinate system can become extremely complicated if a description of the surface geometry is to be included in the mathematical model.

The present study is an attempt to make a first step towards incorporating roughness change with the geometry of the surface structure (Fig. 2).

Figure 1. Schematic representation of flow over a change of surface roughness.

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2. **Mathematical model and governing equations**

In the smooth-to-rough case with increase in elevation, the on-coming wind is expected to decelerate as it approaches the point of surface change. The physics are very similar to the case of flow encountered with a forward facing step or a concave ramp. On the other hand, a rough-to-smooth change with a decrease in elevation resembles flow encountered with a rearward-facing step or a convex corner. In either case the pressure gradient terms can become important. In order to minimize such effects of induced pressure gradients, only those topographies with small curvatures are considered in the present study. Accordingly, the pressure gradient term in the $x$-momentum equation is deliberately omitted to simplify the analysis and so as to be able to evaluate the added effects due to the geometry of topography alone.

Furthermore, the mixing-length model of Taylor was chosen as the closure condition. The choice of Taylor's model is mainly for its simplicity so that the merit of the present approach can be clearly demonstrated.

The governing equations of Taylor's model in the Cartesian coordinate system are the continuity equation $\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0$ and the $x$-momentum equation $U \frac{\partial U}{\partial x} + W \frac{\partial U}{\partial z} = \frac{\partial \tau}{\partial z}$, with $\tau$ defined by $\tau^t = \ell \frac{\partial U}{\partial z}$ and $\ell$, the mixing length, given by

![Diagram](image)

**Figure 2.** Proposed model to incorporate zero-plane displacement with roughness change.
\( l = k(z + z_1) \). These equations, together with initial conditions \( U = (U_0/k) \ln \{(z + z_0)/z_0\} \) and \( W = 0 \) at \( x = 0 \), where \( U_0 \) and \( z_0 \) are upstream friction velocity and roughness length respectively, and boundary conditions \( U = W = 0 \) at \( z = 0 \) and \( x > 0 \), and \( \tau = U_0^2 \) as \( z \to \infty \), complete Taylor's model.

In order to incorporate topography into the model while preserving the mathematical formulation of the problem in its original simple fashion, the present study adopted a curvilinear coordinate system (Fig. 3) to formulate the governing equations. To simplify further the analysis, the geometry of a gentle topography is idealized (Fig. 2) in a form such that, in effect, it can be represented by the \( x \)-axis of the curvilinear coordinate system. Since the topography governs the curvature of the coordinate system, it is automatically built into the mathematical model.

The curvilinear coordinate system does have one major limitation, however, that it can only be applied to topographies whose normals do not coalesce within the domain of interest. This is another reason for choosing topographies with small curvatures in the present study.

![Curvilinear coordinate system](image)

**Figure 3.** Curvilinear coordinate system used for the proposed model.

Derivations of the governing equations in curvilinear coordinate system are rather routine. For curves \( x = \text{constant} \) (Fig. 3) we take normals to the wall and for curves \( z = \text{constant} \) we take curves parallel to the wall, each of which intersects the normals at a constant distance from the wall. Assume \( x \) is the distance measured along the wall from the origin \( O \), while \( z \) is the normal distance from the wall. If we let \( R(x) \) be the radius of curvature, the elements of length along the parallel curves and along the normals are \( ds = dx + (z + dz)dx/R \) and \( dz \). In the present study we confine ourselves to dealing with topographies of small curvatures. Then \( ds = dx(1 + z/R + dz/R) = dx(1 + sz) \) as \( dz/R \ll 1 \) can be neglected since \( dx/R \ll 1 \) and \( dz = 0(dx) \). Based on this assumption, the Jacobian of \( x', z' \) with respect to \( x, z \) is

\[
J(x', z'; x, z) = \frac{\partial(x', z')}{\partial(x, z)} = (1 + sz)
\]

Accordingly, the governing equations in a curvilinear coordinate system are:

\[
U \frac{\partial U}{\partial x} + W(1 + \rho z) \frac{\partial U}{\partial z} + \rho UW = \frac{\partial ((1 + \rho z) \tau)}{\partial z} + \rho \tau \quad \text{(1)}
\]

and

\[
\frac{\partial U}{\partial x} + \frac{\partial ((1 + \rho z) W)}{\partial z} = 0 \quad \text{(2)}
\]

where \( \rho \) is the curvature of the underlying surface. (1) and (2), together with the initial and
boundary conditions, again constitute a closed system. Note that $U$ and $W$ are now velocity components referred to the curvilinear coordinate system.

If we non-dimensionalize the governing equations with respect to $U_0$ and $z_1$, and introduce the transformation $\zeta = \ln(z/z_1) + 1$, the governing equations take the following form: Momentum equation

\[
U \frac{\partial \hat{U}}{\partial \hat{x}} + e^{-\zeta} W \frac{\partial}{\partial \zeta} [1 + \rho(e^\zeta - 1)] \frac{\partial U}{\partial \zeta} \frac{\partial^2 \hat{U}}{\partial \zeta^2} + 2k^2 \rho \left( \frac{\partial U}{\partial \zeta} \right)^2
\]

Continuity equation

\[
\frac{\partial \hat{U}}{\partial \hat{x}} + e^{-\zeta} \frac{\partial}{\partial \zeta} [(1 - \rho + \rho e^\zeta) \hat{W}] = 0
\]

where non-dimensional quantities are $\hat{U} = U/U_0$, $\hat{W} = W/U_0$, $\hat{x} = x/z_1$ and $\hat{\zeta} = \rho z_1$. $z_1$ denotes the downstream roughness length.

For the purpose of demonstrating the merit of the present approach, a close comparison with that of Taylor's model is desirable. Therefore the present study employed the same finite-difference scheme as that of Taylor during numerical calculations.

![Smooth-Rough wind profiles](image)

$M = \ln \frac{c_0 z_1}{\tau} - 6, 9/\zeta = 5$

- [present]
- [Taylor]

$\frac{c z_1}{\tau} = 10^5$

Figure 4. Horizontal wind profiles for $M = -5$. 

3. Results and Discussions

In order to be able to compare with Taylor’s results more meaningfully, wind components presented in this study have been converted back to their counterparts in the Cartesian coordinate system. Assume that $\theta_T$ is the angle of the local terrain slope, while $U$, $W$ and $U'$, $W'$ are wind components referred to the curvilinear and Cartesian coordinate systems respectively. We have $U' = U \cos \theta_T - W \sin \theta_T$ and $W' = U \sin \theta_T + W \cos \theta_T$. Other parameters used here are defined as follows: $M = \ln(z_0/z_1)$, $d$ is the zero-plane displacement or elevation change and $\delta$ is the thickness of the internal boundary layer. All results were plotted against the normal distance from the wall. Since we are only dealing with topographies of small curvature (maximum $\theta_T = 11.6^\circ$), discrepancies between $z$ and $z'$ are negligibly small (about 2%). At a fixed height the corresponding distances between $x = x' = 0$ and $x = \text{constant}$ and that between $x = x' = 0$ and $x' = \text{constant}$ are more noticeable. However they have no effect on the profile comparisons presented in this study as all comparisons with that of Taylor were done beyond the curvilinear portion of topography, where the curvilinear and the Cartesian coordinate systems coincide.

Taylor’s results do not go below a height of 20 roughness lengths. The present study
provides detailed information on the development of horizontal velocity profiles, including those near the ground surface and in the vicinity of the roughness change (Figs. 4 and 5). It is to be noted that the mixing length theory becomes less valid near the ground surface. However results presented in this study seem to be consistent with the physics of the flow and also consistent with its subsequent developments. In general, agreement with Taylor's results is good at locations further downstream.

It is to be noted that in Fig. 4 at one roughness length downstream the wind profile possesses an inflection point. The inflection point quickly disappeared within a few roughness lengths further downstream and consequently has very little physical significance; such a phenomenon nevertheless presents a more realistic transition of the wind profile associated with the change-of-roughness problem (Peterson 1969; Shir 1972).

Since Taylor's results only accounted for the effect of roughness change, the added effect due to topography is demonstrated by the differences between results of this study and those of Taylor.

The discrepancies in Reynolds stress profiles between the present study and that of Taylor (Fig. 6) diminish gradually as the flow progresses further downstream. A larger

![Figure 6. Reynolds stress profiles for $M = +4$ at $x/z_1 = 10^4$ and $10^5$.](image)

Figure 6. Reynolds stress profiles for $M = +4$ at $x/z_1 = 10^4$ and $10^5$. 
Figure 7. Vertical wind profiles for $M = -5$.

Figure 8. Vertical wind profiles for $M = +4$. 
Figure 9. Relative magnitude of vertical to horizontal winds.

Figure 10. Horizontal wind profiles for $M = -2$. 
discrepancy exists at $x/z_1 = 10^4$ than at $x/z_1 = 10^5$. This may imply qualitatively that the effect of the surface geometry is a relatively local one compared with the effect of change of roughness. It can also be interpreted as that the perturbation generated from the change of surface geometry decays faster than that due to a roughness change. Further downstream only the effect of roughness is felt. Consequently results of this study converge to that of Taylor.

Fig. 7 shows the development of the profiles of vertical wind, $W'$, for $M = -5$ and an elevation change of $+5z_1$, at various stations. The magnitude of $W'$ is very closely correlated with the slope of the terrain surface and is characterized by a rather full profile in the sloping terrain region. Values of $W'$ increase with terrain slope until a maximum is reached at $x/z_1 = 25$ (curve 2 of Fig. 7), and then decrease with increasing distance downstream. This behaviour suggests that the surface geometry played a dominant role in generating vertical winds. Location $x/z_1 = 50$ (curve 3) marks the end of the sloping terrain, but $W'$ remains appreciably large. This indicates that a downstream interaction is needed to complete the entire turning process of fluid particles and for the flow field to reach a new state of equilibrium. Far downstream, at $x/z_1 = 10^5$ (curve 4), the vertical winds are shown to be

Figure 11. Horizontal wind profiles for $M = +2$. 
Figure 12. Surface Reynolds stress distributions.

diminishing. The profile appears to be returning to its equilibrium value of nearly zero vertical wind.

Fig. 8 shows the same profiles for the rough-to-smooth case of $M = +4$ and an elevation change of $-200z_1$. Once again the surface geometry shows its dominating effect on the magnitude of vertical winds. The interesting phenomenon to be noted here is the local maximum appearing in curve 1. This rather unusual behaviour of the vertical wind profile indicates that streamlines closer to the ground tend to follow the general curvature of the idealized terrain geometry, whereas streamlines further away from the surface do not. In other words, the flow field above the maximum point of the $W$-profile is not fully aware of the perturbation generated by the transition in surface structure. This can also be interpreted to mean that the impact of such a transition on the flow field is weaker than the smooth-to-rough case shown in Fig. 7.

A comparison of the relative magnitude of $W'$ versus $U'$ for the case of $M = -5$ is given in Fig. 9. The maximum value of $W'/U' = 0.239$ is recorded at $x/z_1 = 25$. Incidentally, $W'/U'$ values shown in Fig. 9 also represent the flow angles of streamlines at their corresponding locations. Here we have assumed that $\theta_T$ is the angle of the local terrain slope, $\theta$ is the flow angle and subscript $\delta$ refers to the outer edge of the internal boundary layer. The larger flow angles than that of the local terrain slope are attributed to the rapid developments of the internal boundary layer.

Other results are shown in Figs. 10, 11 and 12. In general, they all exhibit similar characteristics as those reported by previous studies. Horizontal wind profiles for cases $M = -2$ and $M = +2$ are shown in Figs. 10 and 11 and are similar to those of Figs. 4 and 5 respectively. Fig. 12 is the surface Reynolds-stress distributions for $M = -2$ and $M = -5$, over the entire fetch. The surface Reynolds-stress experiences a sharp jump in the vicinity of the roughness change. It then adjusts itself downward rather rapidly at first and then asymptotically approaches its new equilibrium value further downstream.
4. Conclusion

The main feature of the present approach is the introduction of the curvilinear coordinate system. This coordinate system enabled inclusion of a zero-plane displacement or elevation change in the change-of-roughness problem without further complicating the mathematical analysis. Most importantly, the present approach has the potential to be extended for studying wind fields over a two-dimensional mountain ridge, a river valley or a forest belt. Such extensions can be accomplished by simply introducing two successive changes of elevation or zero-plane displacement in an opposite manner. It would be a routine matter to add the temperature and humidity equations to model a forest belt problem. Further extensions to three-dimensional topography using the curvilinear coordinate system approach are also feasible.

Key findings of the present study relate to profiles of the vertical winds. Results suggest that flow over a sudden change of surface roughness coupled with a zero-plane displacement or elevation change generates vertical winds, the magnitude of which may be larger than would be expected from intuition. Clearly these vertical winds cannot be ignored as they have a controlling impact on vertical transport by turbulence and diffusion processes, especially on pollution dispersion near the ground.

As for future work towards a more realistic model, the following points should be considered. Induced pressure gradients cannot be ignored if larger curvatures are to be considered; the governing equations should include a tangential pressure gradient term in the $x$-momentum equation, and the $z$-momentum equation ought to be added with the inclusion of a transverse pressure gradient term to counterbalance the centrifugal force generated from the turning process of fluid particles during the transition of elevation change. Thus it can be anticipated that more accurate solutions, particularly on information of the development of vertical wind profiles, will result.

REFERENCES


