Inertial and frictional effects on rotating and stratified flow over topography

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SUMMARY

The separate effects of fluid inertia and friction at the lower boundary on the flow of a rotating, stably stratified, incompressible fluid over a three-dimensional shallow isolated obstacle are investigated.

Analytic solutions of the perturbation equations are obtained as a power series expansion in the Rossby number, and some examples are presented. Possible meteorological applications to the flow over the Alps are discussed.

1. INTRODUCTION

The problem of the interaction of rotating stratified fluids with isolated topographical obstacles has become more and more important to dynamical meteorologists and oceanographers in the last few years.

The authors became involved in the subject dealing with the problem of flow over and around the Alps - a typical synoptic-scale obstacle for large-scale atmospheric flow; their aim is to investigate the role played by fluid inertia and surface friction in such a flow.

In the case of the Rossby number (Ro) much less than 1, this problem has been regarded mainly as the 'Taylor column' problem, an expression first used by Hide (1961), who suggested the possibility of geophysical applications of the well-known Proudman-Taylor theorem.

The steady, homogeneous case was first investigated, and then the effects of viscosity or of inertia were retained, as small non-geostrophic contributions, in order to remove the degeneracy implicit in the purely geostrophic equations. (See, for example, Jacobs 1964; Ingersoll 1969; Vaziri and Boyer 1971.)

The vorticity distribution in the frictional homogeneous solutions is worth attention; it shows two cells, a negative one above the obstacle and a positive, smaller one in the lee (right-hand side looking downstream). In the inviscid case only the negative cell over the obstacle is evident, explained mainly in terms of vortex tubes contracting with no modification due to the Ekman pumping.

The stratified problem, more realistic in regard to geophysical applications, has been recently investigated, but mainly in the inertial case.

Experimental works by Davies (1972), and theoretical studies by Hide (1971), Hogg (1973a and b) and Huppert (1975) have clarified the role of stratification, though not in a completely satisfactory way. The solutions of both Hogg and Huppert for the stratified case, resemble, as far as the horizontal pattern near the obstacle is concerned, the Ingersoll (1969) solution in the homogeneous case, showing that the static stability does not modify qualitatively, at least in the inviscid case, the shape of the flow around the obstacle, but introduces an alternative vertical scale \( fa/N \), where \( N \) is the Brunt-Vaisala frequency, \( a \) a typical horizontal scale, and \( f \) the Coriolis parameter. The few solutions including \( f \)-plane effects are due to McCartney (1975), to whom we refer the reader for a more complete
summary of previous results. More recently the time-dependent approach has received attention. Numerical and analytical work by Huppert and Bryan (1975) shows vorticity patterns similar to those characteristic of the steady, viscous, homogeneous solution of Vaziri and Boyer.

2. THE APPROACH TO THE PROBLEM

The problem of an incompressible, internally inviscid, barotropic, rotating, stratified, steady flow over a bell-shaped obstacle of infinitesimal height is considered. Linearized equations are expanded in powers of $\textit{Ro}$ to isolate ageostrophic effects.

As the purpose is to simulate a flow on the scale of a three-dimensional obstacle like the Alps, the $\beta$-effect is neglected since the space-scale is not large enough, and gravity waves have been filtered out. An $f$-plane approximation with $\textit{Ro} < 1$ is therefore considered, with particular attention to finite Rossby number (inertial) and, separately, Ekman pumping.

The usual quasi-geostrophic approximation, in which only the terms of zero and first order in $\textit{Ro}$ are retained, cannot be completely satisfactory in the case of an obstacle such as the Alps for which $\textit{Ro}$ is of order unity in typical meteorological conditions.

With the object of inspecting the change in the solutions as the advective terms begin to be important ($\textit{Ro}$ small but not far from 1) we retain terms up to the third order in the expansion in powers of $\textit{Ro}$. The effect of bottom friction is also separately considered in the case of very small Ekman numbers.

Our approach has been through boundary conditions linearized about flow across the obstacle, and this excludes $\textit{a priori}$ a solution of the ‘Taylor column’ type, i.e. closed streamlines, that are found to direct flow around rather than over the obstacle. However, some inference on these aspects can be deduced as limit cases of the solutions (see section 5). We have treated the case of an incompressible fluid for the sake of simplicity and because the results will apply to atmospheric flow so long as the static stability is identified with $(1/\theta)(\partial \theta/\partial z)$ rather than $-(1/\rho)(\partial \rho/\partial z)$ where $\theta$ and $\rho$ are, respectively, potential temperature and density of the fluid, and where appropriate we recognize that statements are to be interpreted as applying to $\rho^\times \times$ variable rather than to the variable itself.

In our problem we suppose the flow to be unbounded above, which is permissible if the internal vertical scale $fa/N$ is much smaller than the distance between lower and upper boundaries.

A particular smooth three dimensional bell-shaped mountain has been chosen in order to have analytic solutions. The obstacle Rossby number is defined in terms of the scale distance $a$, the decay coefficient of the exponentially decaying Fourier transform of this mountain. This obstacle is therefore representative of mountains with gentle slopes everywhere.

We make the following assumptions:

(a) incompressible, internally inviscid, Boussinesq fluid
(b) pressure in hydrostatic balance
(c) linearized problem (perturbation approach)
(d) stationary flow unbounded above with unsheared basic state
(e) constant rotation ($f$-plane) and static stability
(f) $\textit{Ro} < 1$; $\textit{E} = 0$ or (alternatively) $\textit{E}^\times \sim \textit{Ro}^2$.

The only physical mechanisms left, which act on the scale required by the problem with which we are dealing, are rotation (at finite Rossby numbers), buoyancy, and surface friction (Ekman pumping).
FLOW OVER TOPOGRAPHY

3. BASIC EQUATIONS AND SCALING PROCEDURE

The basic equations are conservation of momentum, mass and entropy, which for the incompressible model becomes conservation of density:

\[ \frac{DV}{Dt} + f \hat{K} \times V + (1/\rho) \nabla p + g \hat{K} = 0 \]  \hspace{1cm} (1)
\[ \nabla \cdot V = 0 \]  \hspace{1cm} (2)
\[ \frac{D\rho}{Dt} = 0 \]  \hspace{1cm} (3)

We consider the stratified, unsheared, hydrostatic and geostrophic basic state in a Cartesian frame of reference \((x', y', z')\) given by:

\[ \nabla \equiv (\ddot{u}, 0, 0) \quad (\ddot{u} = \text{constant}) \]  \hspace{1cm} (4)
\[ \ddot{\rho} = \ddot{\rho}(y', z') \quad \text{and} \quad \ddot{\rho} = \ddot{\rho}(y', z') \]  \hspace{1cm} (5)
\[ B = -(1/\ddot{\rho})(\partial \ddot{\rho}/\partial z') = \text{constant} \]  \hspace{1cm} (6)

where \[ \partial \ddot{\rho}/\partial z' = -g \ddot{\rho} \quad \text{and} \quad \ddot{u} = -(1/\ddot{\rho})(\partial \ddot{\rho}/\partial y') \]  \hspace{1cm} (7)

Linearizing the set (1)–(3) about the basic state (4)–(7), using the Boussinesq approximation and assuming a steady state and hydrostatic perturbation, one obtains the following set:

\[ \begin{aligned}
\ddot{u} \frac{\partial u'}{\partial x'} - fu' + (1/\ddot{\rho})(\partial \ddot{p}'/\partial x') &= 0 \\
\ddot{u} \frac{\partial v'}{\partial x'} + fu' + (1/\ddot{\rho})(\partial \ddot{p}'/\partial y') &= 0 \\
\partial \ddot{p}'/\partial z' + g\rho' &= 0 \\
\partial u'/\partial x' + \partial v'/\partial y' + \partial w'/\partial z' &= 0 \\
\ddot{u} \frac{\partial \ddot{p}'}{\partial x'} - B\ddot{p}w' &= 0
\end{aligned} \]  \hspace{1cm} (8–12)

We choose for the obstacle the form of an axisymmetric 3-D bell-shaped function whose equation is

\[ h(x', y') = H a^3/(x'^2 + y'^2 + a^2)^{1/4}, \quad a, H > 0 \]  \hspace{1cm} (13)

where \(a\) is the horizontal scale and \(H\) is the mountain height.

We now scale the set (8)–(12) with respect to the basic state:

\[ \begin{aligned}
x &= x'/a \\
y &= y'/a \\
z &= z'/H_0 \\
u &= \ddot{u}' \\
v &= \ddot{v}' \\
w &= \ddot{w}' / \{R_0 \ddot{u}H_0(a)^{-1}\} \\
p &= \rho' / \{f \ddot{u}a (gH_0)^{-1}\}
\end{aligned} \]  \hspace{1cm} (14–16)

where \(H_0\) is a dynamical vertical scale. The obstacle profile function is also scaled: \(h = h'/H\).

It is easy to recognize that the scaling procedure (14)–(16) is essentially the classic one for large-scale motions, see e.g. Charney (1970). We are looking, in fact, at scales of the same order of magnitude.

We define \(R_0 = \ddot{u}/(fa)\). This implies that the horizontal scale of motion is determined by the scale of the mountain. The set becomes therefore:

\[ \begin{aligned}
R_0 \partial u/\partial x - v + \partial p/\partial x &= 0 \\
R_0 \partial v/\partial x + u + \partial p/\partial y &= 0 \\
\partial p/\partial z + \rho &= 0 \\
\partial u/\partial x + \partial v/\partial y + R_0 \partial w/\partial z &= 0
\end{aligned} \]  \hspace{1cm} (17–20)
\[ \partial p/\partial x - w H_0^2 N^2/(f^2 a^2) = 0 \quad \text{(21)} \]

where \( N^2 = gB \).

We see immediately that \( H_0 \) is internally determined by the scaling process and it comes from the thermodynamic equation \( \text{(21)} \) that \( H_0^2 = a^2 f^2 / N^2 \). We can, therefore, eliminate \( \rho \) and \( w \) in terms of \( p \) from the set that can now be written as:

\[
\begin{vmatrix}
  1 & \partial/\partial x & \partial/\partial y & \partial/\partial z & u \\
  \partial/\partial x & \partial/\partial x & \partial/\partial y & \partial/\partial z & v \\
  \partial/\partial y & \partial/\partial y & \partial/\partial x & \partial/\partial z & p \\
  \partial/\partial z & \partial/\partial z & \partial/\partial x & \partial/\partial y & p \\
\end{vmatrix}
\]

remembering that \( \rho = -\partial p/\partial z \) and \( w = -\partial^2 p/\partial x \partial z \) . . . (23)

System (22) is linear, homogeneous and with constant coefficients. If we define

\[
\zeta = \partial u/\partial x - \partial u/\partial y \quad \text{vertical component of the relative vorticity vector} \quad \text{(24)}
\]

\[
\delta = \partial u/\partial x + \partial v/\partial y \quad \text{horizontal divergence} \quad \text{(25)}
\]

it is easy to see that we can write

\[
\delta = Ro \partial^3 p/\partial x \partial z^2 \quad \text{and} \quad \zeta = -\partial^2 p/\partial z^2 \quad \text{(26)}
\]

respectively the divergence and vorticity equations (the latter after an \( x \)-integration) for our simplified system. System (22) means that, if \( \phi(x, y, z) \) is any of \( u, v, w, \rho, p, \zeta \) and \( \delta \), we have

\[
\left[ \nabla^2 + Ro^2 \partial^4 / \partial x^2 \partial z^2 \right] \phi(x, y, z) = 0 \quad \text{(27)}
\]

4. Boundary Conditions and Expansion in Powers of \( Ro \)

Because the fluid is unbounded, the upper and lateral boundary conditions (b.c.) are

\[
\lim \phi(x, y, z) = 0 \text{ as } x \to -\infty, \text{ for any value of } y \text{ and } z \quad \text{(28)}
\]

\[
\lim \phi(x, y, z) = \text{finite as } x \to +\infty, \text{ for any value of } y \text{ and } z \quad \text{(29)}
\]

The lower boundary condition is applied at \( z = 0 \) instead of at \( z = h(x, y) \) (obstacle of infinitesimal amplitude) and includes the surface friction parameterization in the classical form first proposed by Charney and Eliassen (1949). In other words, the vertical velocity at \( z = 0 \) is the sum of the orographic term plus the friction-suction term. In its non-dimensional form the complete lower b.c. is

\[
w = (\gamma/Ro)(1 + u) (\partial h/\partial x) + (\gamma/Ro) v (\partial h/\partial y) + \zeta (\frac{1}{E})/Ro \quad \text{at } z = 0 \quad \text{(30)}
\]

where \( \gamma = H/H_0 \) is a vertical aspect ratio and \( E = v(fH_0^2) \) is the Ekman number. Since the non-dimensional slope of the mountain is of order unity, it follows that perturbation pressure, and hence any other perturbation quantity, is of order \( \gamma/Ro \). This yields immediately, for consistency with the perturbation approach, \( \gamma \ll Ro \). Remembering (23) and (24) we can derive from (30) our lower b.c.:

\[
\partial^2 p/\partial x \partial z = - (\gamma/Ro) \partial h/\partial x + (\frac{1}{E})/Ro \partial^2 p/\partial z^2 \quad \text{at } z = 0 \quad \text{(31)}
\]
The order of magnitude of any perturbation quantity is now determined; we can therefore perform the expansion in powers of $Ro$, writing

$$\phi(x, y, z) = \frac{\gamma}{Ro} \sum_{i=0}^{\infty} Ro^i \phi^{(i)}(x, y, z) \quad \ldots \quad (32)$$

The Rossby number is assumed to be less than unity. Introducing (32) into Eqs. (22)–(27) and requiring separate balance of each order, we obtain:

$$u^{(i)} = -\frac{\partial p^{(i)}}{\partial y} - \frac{\partial v^{(i-1)}}{\partial x} \quad \ldots \quad (33)$$

$$v^{(i)} = \frac{\partial p^{(i)}}{\partial x} + \frac{\partial u^{(i-1)}}{\partial x} \quad \ldots \quad (34)$$

$$w^{(i)} = -\frac{\partial^2 p^{(i)}}{\partial x \partial z} \quad \ldots \quad (35)$$

$$\rho^{(i)} = -\frac{\partial p^{(i)}}{\partial z} \quad \ldots \quad (36)$$

$$\zeta^{(i)} = -\frac{\partial^2 p^{(i)}}{\partial x^2} \quad \ldots \quad (37)$$

$$\delta^{(i)} = \frac{\partial^3 p^{(i-1)}}{\partial x \partial z^2} \quad \ldots \quad (38)$$

and

$$\nabla^2 \phi^{(i)} + \partial^4 \phi^{(i-2)} / \partial x^2 \partial z^2 = 0 \quad \ldots \quad (39)$$

where $i = 0, 1, 2, \ldots$ and it is everywhere and from now on understood that if an index takes a negative value, the term containing it vanishes.

It is worth pointing out that an immediate physical link between subsequent orders is easily given by Eqs. (37) and (38) in terms of vorticity and divergence. They yield $\delta^{(i)} = -\partial \zeta^{(i-1)} / \partial x$, which means that vorticity at a given order is linked to divergence at the subsequent order.

We note that Eq. (39) is a Poisson equation for $i \geq 2$ and reduces to a Laplace equation for $i = 0$ and $i = 1$, that is in the so-called quasi-geostrophic approximation.

In the examples we are going to present we will truncate the power series at a certain value of $i$. Let this value be $i_{\text{max}} = n$. We now want to ensure that in the expansion in powers of $Ro$ we are not retaining terms negligible with respect to the terms neglected in the first linearization procedure that led us from the set (1)–(3) to the set (8)–(12). If we consider, for example, the $x$-component of the momentum equation (1) we see that, in order to have the largest of the non-linear terms smaller than the linear terms,

$$O[Ro u \partial u / \partial x] \ll O[\rho] \quad \ldots \quad (40)$$

but, using Eq. (32), Eq. (40) becomes

$$O[\max(\gamma Ro^{i+j} \partial u^{(j)} / \partial x)] \ll O[\min(Ro^{i+j})] \quad \ldots \quad (41)$$

If we decide to retain terms until $Ro^i$ ($i_{\text{max}} = n$), the smallest linear term is $O[Ro^n \gamma]$. The greatest non-linear term is the term we get from l.h.s. of Eq. (41), setting $i = j = 0$, and is $O[\gamma]$. We therefore require

$$\gamma \ll Ro^i \ll 1 \quad \ldots \quad (42)$$

for consistency between the perturbation expansion and the expansion in powers of $Ro$.

If we now assume

$$\frac{1}{2} E^+ = O[Ro^i] = C Ro^i \quad \ldots \quad (43)$$

where $C$ is a constant of order unity, and we use Eq. (32), the lower b.c. (31) becomes:

$$\sum_{i=0}^{\infty} Ro^i \frac{\partial^2 p^{(i)}}{\partial x \partial z} = -\frac{\partial h}{\partial x} - C \sum_{i=0}^{\infty} Ro^{i+1} \frac{\partial^2 p^{(i)}}{\partial z^2} \quad \text{at } z = 0 \quad \ldots \quad (44)$$
We now derive from (44) the balances at the various orders, and use them as b.c. for each set of corresponding order.

An assumption must be made to determine the value of \( l \) defined by (43), that means the relative importance of the Ekman number with respect to the Rossby number. We will consider, in the examples of solutions, only cases in which \( l \geq 2 \rightarrow \frac{1}{2}E^4 \leq Ro^2 \), implying that the inertial effects always dominate over boundary-friction effects which are not allowed to influence zero-order solutions. We make this assumption in order to obtain analytic solutions. If we assume \( E^4 \) to be of order \( Ro \), as it is often true for both large-scale and laboratory flows, we are unable to obtain closed analytical solutions to the problem.

With these assumptions the lower b.c. for the zero order is, from (44),
\[
\partial p^{(0)} / \partial z = -h(x, y), \text{ at } z = 0 . \quad \quad \quad (45)
\]
at which point the zero-order solution is completely determined.

We are now able to deduce, from Eq. (44), the lower b.c. for \( i > 0 \):
\[
\partial^2 p^{(i)} / \partial x \partial z = C \partial^2 p^{(i-l+1)} / \partial z^2 \quad \text{at } z = 0; \quad i > 0; \quad l \geq 2. \quad \quad \quad (46)
\]

It is worth pointing out that the b.c. (46) in the particular case in which \( E \to 0 \) and hence \( l \to +\infty \) becomes \( \partial^2 p^{(i)} / \partial x \partial z = 0 \), at \( z = 0; \quad i > 0 \). This leads to identically zero perturbation pressure (and hence vertical velocity and vorticity) at all odd orders \( (i = 1, 3, 5 \ldots) \) and identically zero divergence at all even orders \( (i = 0, 2, 4 \ldots) \). We now calculate the analytic solutions for three different cases.

5. THE QUASI-GEOSTROPHIC INVISCID CASE

If \( Ro \ll 1 \) but \( E \to 0 \), then \( n = 1; \quad l \gg 2 \). This is the quasi-geostrophic frictionless solution which has already appeared in the literature, in which both viscous and higher-order inertial effects have been neglected. Ingersoll (1969), for instance, shows a similar solution for an infinitesimal mountain in the homogeneous, upper-bounded case.

The zero-order solution is obtained by solving the Laplace equation \( \nabla^2 p^{(0)} = 0 \), derived from (39), with b.c. (28), (29) and (45); the other variables are obtained immediately using Eqs. (33)–(38). This is easily solved by inspection just remembering that, since \( r^{-1} = \{x^2 + y^2 + (z + 1)^2\}^{-\frac{1}{2}} \) is a solution of the Laplace equation, so any function \( f_{i,m,n} \) defined by
\[
f_{i,m,n} = \partial^i m + n(r^{-1}) \partial x^i \partial y^m \partial z^n \quad \quad \quad (47)
\]
is also a solution (apart from the point \( x = 0, y = 0, z = -1 \) that lies below the surface so does not concern us).

If we take
\[
p^{(0)}(0, x, z) = f_{0,0,0} = r^{-1} = \{x^2 + y^2 + (z + 1)^2\}^{-\frac{1}{2}} \quad \quad \quad (48)
\]
it is easy to check that b.c. are satisfied. The first order solution is immediately found to be trivial, since \( p^{(1)} = 0 \) satisfies Laplace equation and b.c. (46) in the case \( E \to 0 \ (l \gg 2) \). All the other variables come from Eqs. (33)–(38).

The analytic solutions derived here and in the following two sections and the relevant quantities derived from them are listed in the appendix.

Figs. 1(a), (b), (c) and (d) show the solution with \( Ro = 0.1 \). In Fig. 1(a) the horizontal velocity field is displayed by arrows, the vertical one by isolines. All the quantities are non-dimensional and rotation is anticlockwise. Fig. 1(b) shows lines of \( \eta \), lateral displacement (full lines) and \( \xi \), longitudinal displacement (dotted lines). \( \eta \) and \( \xi \), often called 'displacement coordinates', are defined by \( D\eta / Dt = v \) and \( D\xi / Dt = u \). Lines of \( \xi \) represent successive positions of a line of tracer released at \( x = -\infty \) on a straight line perpendicular to the main
Figure 1. Quasi-geostrophic case. All quantities are non-dimensional. The dashed circle of radius unity represents the height contour of the obstacle at $\sqrt{2}$ of the top height, $Ro = 0.1$; $\gamma = 0.1$; $E = 0$.
(a) Arrows represent horizontal velocity, isolines vertical velocity: thick lines upwards, thin lines downwards. Isolines drawn at intervals of 0.1. Plane $z = 0$.
(b) Full lines are lines of lateral displacement, hence streamlines in the linear approximation; dotted lines are lines of longitudinal displacement, hence successive positions of a straight line of tracer released at $x = -\infty$, Plane $z = 0$.
(c) Isolines of pressure at intervals of 0.5 at $z = 0$. The dotted line marks the pressure value of the basic state at $\gamma = 0$. Pressure decreases towards the top of the figure.
(d) Vertical component of relative vorticity: isolines drawn at intervals of 0.2. Plane $z = 0$. Thick lines cyclonic, thin lines anticyclonic, dotted line zero vorticity. In this particular example the positive vorticity region far from the mountain is weaker than 0.2 so no thick isolines appear.

flow; lines of $\eta$ represent the projection of the trajectories on a horizontal plane. In our steady, linearized case the calculation of $\xi$ and $\eta$ becomes simple remembering that, in non-dimensional notation, we have $D/Dt \approx \partial/\partial x$.

In the examples shown, $\gamma$ is $O(Ro)$ in spite of condition (42). This is done in order to emphasize in the figures the contribution of perturbation with respect to the basic state; it is, furthermore, generally admitted that the linear approach describes the behaviour of such
physical systems more closely than could be expected from purely mathematical considerations. Configurations of \( p \) (here approximately a stream function, Fig. 1(c)) show good qualitative agreement with the results of Edelmann (1972a; numerical experiment), Hogg (1973a) and also with other previously mentioned results in the homogeneous case, when the disturbances are small. The \( \zeta \) pattern (Fig. 1(b)) can be compared with the dye lines shown in the experiments by Hide, Ibbetson and Lighthill (1968) and Davies (1972). The maximum velocity on the left and the minimum on the right of the obstacle together with an anticyclonic curvature over it are common features in all the examples. A circular cell of negative relative vorticity (Fig. 1(d)) is concentrated above the obstacle, surrounded by a region of weakly positive vorticity vanishing at infinity.

Figure 2. Sketch depicting the deformation of vortex tubes in a stratified rotating flow over an obstacle. The solid line represents the projection on a vertical plane of a trajectory. Large arrow shows direction of mean flow.

This vorticity pattern can be interpreted looking at Fig. 2, in terms of vortex compression or stretching. In our solutions the trajectories aloft are more smooth than the mountain contour. In Fig. 2 this effect has been emphasized for greater clarity, showing a vortex tube, starting from A (approximately zero relative vorticity) being stretched going towards B (positive relative vorticity) and compressed in C (negative relative vorticity). The steepness of the first slope of the mountain (position B in Fig. 2) is important in this respect: steeper obstacles are expected to produce stronger positive vorticity there. This, in turn, causes the streamlines to deviate more sharply around the obstacle in agreement with the conclusions in Huppert (1975) for the stratified case, possibly leading to a Taylor-column flow if \( Ro \) remains much smaller than 1.

Whereas in the previous literature the problem of Taylor-column flow (homogeneous or stratified) has been emphasized, our perturbation approach allows us only to formulate a general criterion for the occurrence of stagnation above the obstacle. This can be interpreted as indication of incipient closed streamline formation. It occurs in our model and for our particular smooth obstacle at \( z = 0 \) and for \( Ro < 1 \), right of the centre of the obstacle looking downstream, when the perturbation quantities become of the same order as the basic state, i.e. when \( \gamma/|Ro| \geq 1 \rightarrow HN/|u| \geq 1 \). The critical number \( HN/|u| \geq 1 \) is the same as that used by Huppert in a similar but more general criterion for smooth obstacles in a stratified unbounded fluid.

All the fields displayed in Figs. 1(a) to (c) are at \( z = 0 \); their dependence on height shows a vertical decay, as \( z' \) increases with a typical dimensional scale \( af/N \) in agreement with the scaling of the vertical coordinate and without changes of shape as marked as those in the next case where the Rossby number is larger.

The static stability role, as can be deduced from the analytic dimensional solutions, is evident in governing the dynamics of such a system. Remembering Eqs. (14), (15), (16) and
(32), it is easy to recognize that the Brunt-Väisälä frequency $N$ acts as an amplitude factor for all the orographically induced perturbations, except the vertical velocity, as well as a decay factor with height. For example, at $z' = 0$ the horizontal velocity perturbation is simply proportional to the Brunt-Väisälä frequency, while for values of $z' > 0$ its total amplitude as a function of $N$ (Ro constant) shows a maximum for a certain value of $N$, say $N_m$, and the value of $N_m$ decreases when $z'$ increases. These considerations are valid also for the following solutions.

6. The Inertial Case

We are now interested in the case $Ro < 1$, $E \to 0$, for values of $Ro$ always less than unity but not so small. The corresponding choice for $n$ and $l$ in the truncated expansion is $n = 3$, $l \gg 4$. This means that we have retained terms until the third order in $Ro$. This solution allows us an inspection of the inertial effects for increasing values of $Ro$; it allows us also to explore the validity of the pure quasi-geostrophic approximation as far as the advection effects are concerned. Similar techniques, although not restricted to the linearization assumption, have been already proposed in the past literature; non-geostrophic contributions have been taken into account, in these works, to improve the description of atmospheric phenomena without including gravity-inertia waves. See, for example, the balance approximation, Charney (1962), and the semi-geostrophic model, Hoskins (1975).

Because the solutions for zero and first orders are the same as in the reference case, and the third order is immediately known as soon as the second order is, we have to solve only the Poisson equation

$$\nabla^2 p^{(2)} = -\partial^2 p^{(0)}/\partial x^2 \partial z^2$$

where $p^{(0)}$ is given by (48), with the b.c. (28), (29) and $\partial^2 p^{(2)}/\partial x \partial z = 0$, at $z = 0$, obtained from (46) by considering that $l \gg 4$ because $E \to 0$.

Eq. (49) can be solved by Fourier transforming twice with respect to $x$ and $y$ and solving the second-order linear differential equation in $z$ by means of the arbitrary constants variation method. All the results are in the appendix.

Solutions in which $Ro = 0.7$ are shown in Figs. 3 (a to f). The numerical validity of the solutions for such a high value of the Rossby number is doubtful because of the asymptotic nature of the series.*

We would like to point out here that the choice for such a large Rossby number is only due to the need to show readily visible modifications with respect to the previous strictly quasi-geostrophic case.

The velocity field (Fig. 3(a)) shows a maximum $u$-component on the top of the obstacle, while stagnation points tend to form in two regions on the right of the $x$-axis (parallel to the main flow and passing through the centre of the obstacle) one in front and one behind the obstacle. The horizontal streamlines (Fig. 3(b)) are now more asymmetric with respect to the $x$-axis than in the quasi-geostrophic case. On the left (looking downstream) the clockwise curvature becomes very sharp, while on the right it changes its sign with respect to the quasi-geostrophic case, becoming anticlockwise.

It is interesting to compare the pressure fields in Figs. 1(c) and 2(c). To the pressure high generated by the lowest-order solution a pressure pattern has been superimposed which

* As suggested by a referee, we have tried to evaluate the successive degree of accuracy of our solutions as a function of the maximum order retained, $l_{\text{max}}$, and of the Rossby number. There is, once $l_{\text{max}}$ is fixed, a maximum value of $Ro$ for which the error committed approximating the solutions with their asymptotic expansion diminishes for increasing values of $l$ until it reaches $l_{\text{max}}$. If one uses a value of $Ro$ greater than $Ro_{\text{max}}$ the numerical error is larger than the error one would commit without considering the term for $i = l_{\text{max}}$, as usual in asymptotic expansions. Our numerical computations show that $Ro_{\text{max}} \approx 0.3$ for $l_{\text{max}} = 3$; only if $Ro \leq 0.3$ the accuracy of the solutions increases retaining terms until the third order. A critical behaviour of the flow when $Ro$ approaches 1 can be inferred from these results.
Figure 3. Inertial case. $Ro = 0.7; \gamma = 0.7; E = 0$.
(a), (b), (c), (d) on plane $z = 0$. (e), (f) on plane $z = 1$ (top of mountain).
For more specifications see Fig. 1.
resembles the one associated with a classical non-rotating flow around a cylinder, with two (front and rear) maxima and the two lateral minima that produce the non-geostrophic acceleration around the obstacle. The pressure pattern is no longer similar to the $\eta$-pattern, at least near the obstacle, and neither can represent a stream function, although the $\eta$-lines are still approximate trajectories. In Fig. 3(d) the vorticity field at $z = 0$ is plotted; two regions of positive (cyclonic) vorticity appear in front of and to the rear of the obstacle, while the negative lobe just above it is reinforced and deformed. This behaviour is qualitatively in agreement with the considerations of the previous section regarding vortex tube deformation.

Figs. 3(e) and (f'), representing velocity and vorticity fields at $z = \gamma$ (top of the mountain), show that the vertical structure of the perturbations is no longer a simple decrease with height, as in the previous case. The two-lobe vertical velocity field at $z = 0$ splits into a four-lobe structure, in good agreement with an analytical solution by Edelmann (1972b) for a ridge. Strong derivatives of the vertical displacement and of vorticity (and so strong divergence) take place only over the mountain; the flow aloft is only weakly disturbed (see Figs. 3(e) and (f')).

Something can be said concerning the vertical scale: an easy way to deduce it is to introduce wave solutions into Eq. (27) and to calculate the dispersion relation between the three wavelength components (steady case).

If the horizontal wavelengths are of the same order of magnitude, as in our case, and if we go back to dimensional quantities, it can be deduced that

$$H_0 = (af/N)(1 - Ro^2)\frac{1}{4}, \quad Ro < 1$$

(50)

where $H_0$ is the dimensional internal vertical decay scale (imaginary vertical wavelength). This means that the approximation $H_0 = af/N$ is good only for $Ro \ll 1$; $H_0$ tends to zero when $Ro$ tends to 1. For $Ro > 1$ the solutions become essentially of the inertial-gravity wave type with real vertical wavelengths.

7. THE FRICTIONAL CASE

In this case we look at a problem for which

$$Ro \ll 1, E^\pm = O[Ro^2]$$

(51)

A vertical suction of order $Ro^2$ is, in this case, sufficient to modify the horizontal flow of order $Ro$ because $O[vertical \ motion] = Ro O[horizontal \ motion]$. The values for $n$ and $l$ are $n = 1, l = 2$. This solution allows an inspection of the modification that an Ekman boundary layer (E.b.l.) of small thickness of order $Ro^2 \sim E^\pm$ would produce on the quasi-geostrophic case solutions. The thinness of the E.b.l. is consistent with our linear approach, i.e. b.c. applied at zero instead of at the top of the E.b.l.; see also Charney and Eliassen (1949).

Because the solution for the zero order is the same as in the reference case, we have to solve only the Laplace equation $\nabla^2 p^{(1)} = 0$, with the b.c. (28), (29) and (46) with $l = 2$. This equation can be solved using again Fourier transform techniques. The results are in the appendix.

Figs. 4(a)–(d) display a solution for which $Ro = 0.4, E = 0.025$. The value of the Rossby number does not fulfill the first of conditions (51) in a very satisfactory way, considering that only the zero and first orders in $Ro$ have been retained in this solution. Nevertheless, as we have calculated a solution in which the viscous effects act on the first-order terms, we decided, as in the previous inertial case, to choose a Rossby number big enough to produce an easily visible effect.
The most impressive difference with respect to the previous cases is the front-rear asymmetry of all the perturbation fields. The horizontal velocity (Fig. 4(a)) and the streamlines (Fig. 4(b)) show that the fluid bends towards the right in the lee of the obstacle. This behaviour is in very good agreement with the experimental and numerical results with smooth obstacles and comparable values of $Ro$ and $E$; see for example Moore, Saffman and Maxworthy (1969), Vaziri and Boyer (1971), and Davies (1972).

The related vertical velocity shows a net downward velocity in response to the zero-order anticyclonic vorticity, giving a downward motion in the lee that is stronger than the upward motion on the upwind side, with the maximum elevation reached before the top of the mountain. This is also qualitatively in agreement with numerical results obtained, in the case of a ridge, by Mahrer and Pielke (1975).

The $\zeta$-lines (Fig. 4(b)) show that the zero and first-order $u$-components act in opposite senses. The zero order, as in the quasi-geostrophic case, accelerates the flow on the left side of the obstacle and decelerates it on the right. The first order, on the contrary, produces
acceleration on the right and deceleration on the left, and this effect becomes dominant for large positive values of $x$, giving a horizontal velocity shear in the lee. The vorticity pattern (Fig. 4(d)) shows a positive cyclonic tail in the lee of the mountain, which does not vanish for $x \to +\infty$. This positive vorticity is created by vortex tubes stretching in the downward motion behind the obstacle, that overcompensates the shrinking before it, and is then simply advected downwind. The contribution of the next order, if included in the solutions, would probably have been a correction to the vorticity field that would have damped it in the lee with a horizontal non-dimensional scale of the order of $Ro/E^4$, as it comes from simple scaling considerations.

8. GEOPHYSICAL APPLICATION: FLOW OVER AND AROUND THE ALPS

There is little doubt about the relevance of the Alps in European meteorology. It has been often suggested in the literature, see for example Radinovic (1965 a, b) and Buzzi and Rizzi (1975), that the high occurrence of cyclogenesis in northern Italy and surrounding areas can be explained in terms of small scale (500-1000 km) baroclinic disturbances triggered by the interaction of the large-scale flow with the Alps. The full time-dependent, baroclinic problem can be tackled only numerically, see Egger (1972) and Trevisan (1976). There are, nevertheless, examples of quasi-steady flow over the Alps in which baroclinicity seems not to play a very important role. Our solutions can be compared with such situations.

Typical atmospheric values of the non-dimensional numbers relevant to the problem of flow over the Alps are

$$Ro = \frac{\bar{u}}{(f\bar{a})} \sim \left[10^{-1} : 1\right]; \frac{\gamma}{Ro} = HN/\bar{u} \sim \left[10^{-1} : 1\right]; E \equiv \Gamma/H_0^2 \sim \left[10^{-3} : 10^{-1}\right].$$

These ranges are satisfactorily compatible with those for which analytical solutions have been obtained, though not all the possible values of $Ro$, $\gamma/Ro$ and $E$ are included; and some extreme values like $\gamma/Ro = 1$ are at least at the limit of the linearization assumption. Fig. 5 shows a pressure-surface analysis in a quasi-steady situation of southerly flow across the Alps that can be compared with our solutions. It shows a typical high-low pressure pattern.

Figure 5. Surface analysis 16 September 1975, 1500 GMT. The thick black arrow indicates the direction of 700 mb flow over the Alps at the same time; and the dotted area presents a rough profile of the Alps.
induced by the mountain; a similar but reversed pattern forms with northerly flow, independently of the eventual occurrence of cyclogenesis. This behaviour can be interpreted in terms of our solutions. In particular the ‘high’ standing upwind, rather than directly above the mountain range, suggests the importance of the inertial terms. Although the isobars in Fig. 5 are far from being streamlines near the Alps, we have computed isentropic trajectories at low levels in similar conditions (see also Buzzi and Rizzi 1973) showing sharp curvature in the anticyclonic sense on the left side of the Alps, looking downstream, in accordance with Fig. 3(b). The behaviour in the lee seems, on the other hand, to be dominated by the friction which induces the pressure trough. Comparing our theoretical solutions for the frictional case (see Fig. 4) with the actual observations, the conclusion is that, although we underestimate quantitatively the effects of friction, we are able to reproduce the qualitative general feature of the real flow, hence the high-low pressure pattern (or negative-positive vorticity pattern). The thick black arrow in Fig. 5 sketches the direction of the 700 mb flow (otherwise not shown), which does not present any appreciable perturbation, at least on the available resolution. This rather shallow vertical scale of the perturbation is in agreement with the vertical scale of the analytic solution for the inertial case. Remembering Eq. (50) we obtain, for the Alps, a maximum value of $H_0$ of about 5 km.

One cannot a priori exclude the occasional occurrence of a closed anticyclonic circulation over the Alps on a small scale unresolved by the available synoptic network. Nevertheless this circulation is not expected to have a large vertical extent mainly because of the effects of both stratification and relative largeness of $Ro$. It could be concluded that the atmospheric data for the flow over and around the Alps show at least qualitative similarities with characteristics appearing in both inertial and viscous cases. It is unfortunate that the technique that we have developed here is not suitable for finding a solution in which both effects are included simultaneously. More effort is needed in this direction.

These results can be of guidance for numerical modelling, as they demonstrate the importance of non-geostrophic effects and of the adequate parameterization of the boundary layer, even though they do not include representation of explicit gravity-inertia waves.

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APPENDIX

The quasi-geostrophic case can be obtained from the inertial case omitting second and third order or from the viscous case setting $C = 0$.

Inertial case:

zero order $u^{(0)} = yr^{-3}$

$v^{(0)} = -xr^{-3}$

$w^{(0)} = -3x(z+1)r^{-5}$

$p^{(0)} = r^{-1}$

$z^{(0)} = r^{-5}(r^2 - 3(z+1)^2)$;

first order $u^{(1)} = r^{-5}(r^2 - 3x^2)$

$v^{(1)} = -3xyr^{-5}$

$w^{(1)} = p^{(1)} = z^{(1)} = 0$;
second order
\[ u^{(2)} = \frac{3}{2} y r^{-7} \{ r^2 - 5x^2 - 5(z + 1)r^{-2}(r^2 - 7x^2) \} \]
\[ v^{(2)} = -\frac{3}{2} x r^{-7} \{ 3r^2 - 5x^2 - 5(z + 1)r^{-2}(3r^2 - 7x^2) \} \]
\[ w^{(2)} = \frac{1}{2} x z r^{-9} \{ 4r^2 - 7y^2 + 7(z + 1)^2 r^{-2}(2r^2 - 9x^2) \} \]
\[ p^{(2)} = -\frac{1}{2} r^{-5} \{ r^2 - 3x^2 + 3z(z + 1)r^{-2}(r^2 - 5x^2) \} \]
\[ \zeta^{(2)} = \frac{1}{2} r^{-7} \{ r^2 - 5x^2 - 5(z + 1)(4z + 1) + 35(z + 1)r^{-2} \{ z(z + 1)^2 + x^2(4z + 1) - 9x^2z(z + 1)^2 r^{-2} \} \}; \]

third order
\[ u^{(3)} = \frac{3}{2} r^{-9} \{ r^4 - 30x^2 r^2 + 35x^4 - 15z(z + 1)r^{-2}(r^4 - 14x^2 r^2 + 21x^4) \} \]
\[ v^{(3)} = -\frac{1}{2} x yr^{-9} \{ 3r^2 - 7x^2 - 21z(z + 1)r^{-2}(r^2 - 3x^2) \} \]
\[ w^{(3)} = p^{(3)} = \zeta^{(3)} = 0. \]

Viscous case:
zero order as inertial case;

first order
\[ u^{(1)} = -Cy(z + 1)r^{-3}(r^2 - x^2)^{-2} \{ x(r^2 - x^2) + 2r^2(r + x) \} + r^{-5}(r^2 - 3x^2) \]
\[ v^{(1)} = -Cy(z + 1)r^{-3} - 3xy r^{-5} \]
\[ w^{(1)} = Cy^{-5} \{ r^2 - 3(z + 1)^2 \} \]
\[ p^{(1)} = -C(z + 1)(r^2 - x^2)^{-1}(1 + x r^{-1}) \]
\[ \zeta^{(1)} = C(z + 1)(r^2 - x^2)^{-3} [8(z + 1)^2(1 + x r^{-1}) - 6(r^2 - x^2)(1 + x r^{-1}) + 4x(z + 1)^2 r^{-3}(r^2 - x^2) - 3x r^{-5} (r^2 - x^2)^2 \{ r^2 - (z + 1)^2 \} ] \]

where \( C = \frac{1}{2} E^4 Ro^{-2} \) and \( r \) is defined in Eq. (48).