Updating prediction models by dynamical relaxation: an examination of the technique

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SUMMARY

A study is undertaken of a particular method of updating numerical prediction models. The method constrains the time development of a subset of the model fields toward a prescribed space-time estimate of those fields, while the other model variables are allowed to evolve in an explicitly unconstrained fashion. It represents an attempt to update the model by a form of dynamical relaxation.

An analysis of the method is carried out in the context of the primitive equations linearized about an isothermal basic state of no motion. It is shown that, for a particular form of the scheme and the availability of the time history of the mass field (or the wind field) on an \( \beta \)-plane, all the time- and space-scale error fields in the initial specification suffer an amplitude reduction of at least the order \( e^{-1} \) on the timescale of one day. On an equatorial \( \beta \)-plane it is shown that the same rate of amplitude reduction can be achieved if an accurate time history is known for both the mass field and the zonal-wind field. Numerical experiments performed with the nonlinear shallow-water equations for a mid-latitude \( \beta \)-plane geometry support these results and demonstrate that the technique compares favourably with the conventional direct insertion update method. Consideration is also given to the possible effects of errors in the prescribed fields and it is shown that the relaxation scheme can to some extent be tuned to offset the effect of this particular source of error.

This study of the relaxation update scheme, although not comprehensive, is nevertheless sufficient to indicate its potential. However, it is stressed that a trenchant assessment of the scheme's usefulness should be based at least in part upon its performance under more testing and realistic conditions.

1. INTRODUCTION

A specification of the atmospheric state at a given instant, say, the distribution in space of the wind and pressure fields and possibly other ancillary meteorological variables, constitutes requisite initial data for short- and medium-range numerical weather prediction models. The quality and quantity of the observational data required will obviously depend upon the sensitivity of the model forecasts to inaccuracies in the specification of the initial state. Two gross deleterious effects have been attributed to the inadequacy of the initial specification. Incompatibility in the numerical model between the assigned wind and pressure fields can lead to the generation of spurious large-amplitude inertia-gravity waves. This was the root cause of the failure of L. F. Richardson's pioneering attempt at numerical weather prediction (Platzman 1967). Again, numerical simulation experiments have demonstrated forcibly that errors associated with the initial data can amplify with a doubling time of the order of a few days and severely modify the large-scale meteorological constituent of the flow field.
Thus the acquisition of accurate and model-compatible data is one of the central objectives in the endeavour to attain better predictions. The re-emphasis of this objective has provided a spur to two related activities. Data density has been increased by the establishment of additional stations to supplement the existing meteorological observational network, and complemented substantially with satellite derived data. The increased or portended increase in asynchronous data has been modelled or utilized in a range of numerical simulation studies. These studies were undertaken following the attractive proposal (see, for example, Charney et al. 1969) that the dearth of instantaneous data with which to specify the initial state for a prediction model could be alleviated by the use of the available historical data sets.

It was suggested that trade-offs existed between the variables, whereby a knowledge of the time history of the temperature or wind field was sufficient to determine the instantaneous spatial distribution of both these meteorological variables. Verification of these and related suggestions has generally been sought by recourse to simulation experiments with numerical models. In these experiments the 'true' model field is compared with the evolution of a simulated flow field that had an initial error but later in the simulation run is allowed to be influenced by some additional data in a prescribed manner in space and time.

The degree of adaptation of the latter, 'updated' field toward the true state is a measure of the success of the procedure. A veritable profusion of experiments have been conducted to examine various facets of the update procedure. These include the dependence of the method upon the sparsity of data in time and space, the technique of data insertion, the effect of error in the inserted data, the influence of asynchronous insertion corresponding to satellite derived data, the role of the spatial and temporal difference schemes of the model and repercussions arising from the dissimilarity between the physics of the atmosphere and the model.

A range of theoretical studies have been made to complement and provide an analytical foundation for the understanding and interpretation of the simulation experiments. In particular the adaptive process accompanying periodic updating of some subset of the meteorological variables has been studied in the context of a simple linear shallow-water model by Williamson and Dickinson (1972), and further elaborated by Blumen (1975) using a barotropic quasi-geostrophic model. Considerable theoretical progress has also been made in developing sequential analysis techniques to provide the statistically best estimate of the true field at an update event. A series of reviews of four-dimensional data assimilation methods given by Kasahara (1972), Jastrow and Halem (1973), and more recently by Bengtsson (1975) and McPherson (1975) give a comprehensive picture of the development, problems and status of research in this field.

In this paper we undertake a limited analytical study of a particular update technique. We examine, albeit with a linear model, the rate and degree of adaptation to be expected with the given update procedure. We regard the results as useful information that provides a theoretical guide upon which a series of numerical experiments can be planned and their results examined.

Details of the technique are developed in the next section. For the analysis we assume that a complete four-dimensional time history is known of some prescribed subset of the meteorological variables. The rate of adaptation of the flow variables toward the true state is determined in the first instance for a linearized f-plane model. Some consideration is later given to mid-latitude and equatorial β-plane models. An analysis of a scheme requiring such glaringly unobtainable observational data might seem merely academic. This is not so. In context, the derived results represent upper bounds to the rate of adaptation that can be achieved with the stipulated form of the update procedure. In a practical implementation
of the scheme it is to be expected that data paucity, observational errors, model approximations and perhaps nonlinear effects would reduce the rate of adaptation.

In sections 3, 4 and 5 we examine, with linear and nonlinear models, some of the shortcomings and certain aspects of these debilitating effects in order to supplement and corroborate our earlier analysis. The purpose of our study is primarily to establish, in principle, the effectiveness of the scheme under consideration. Thus we are content to supply only a provisional assessment of the various effects that would arise in a practical implementation of the scheme.

2. THE UPDATE SCHEME

(a) Outline

It is conceptually helpful first to formulate the update process in rather general terms. The prognostic primitive equations are of the form

$$\frac{\partial u}{\partial t} = L(u) + F(r, t)$$

where $u = u(r, t)$ is the vector of the prognostic dependent variables, the operator $L$ is nonlinear and a function of the dependent variables, and $F$ represents externally prescribed forcing functions. The updating is performed with a numerical representation of the continuous equations that may be written

$$\frac{\partial \tilde{u}}{\partial t} = \{L(u) + F\} + \{L^*(u) + F^*\} + G(r, t, \tilde{u}^*, E).$$

The starred terms represent defects arising from truncation and imperfect parameterization of physical processes, whilst the updating procedure is represented by the last term. The overbarred $\tilde{u}^*$ variables represent the measured data and may be incomplete or inaccurate estimates of the 'desired' fields, $\tilde{u}$, whilst $E$ denotes some measure of the local fidelity of the observations. The first update experiments involved the direct insertion of observational data to replace the model values. Thus for this technique the update term $G$ would be represented by a finite set of delta functions that are sparse in space and time.

A measure of the inaccuracy or error of the model state is given by the norm $R$, $R = \|u - \tilde{u}\|$, that measures the deviation of the model values $u$ from the desired values $\tilde{u}$. The time evolution of this norm will be determined by the model defects, the interaction of the error fields and the true fields, and the modifying effect of the updating procedure. Thus the process leading to assimilation may be viewed as a control problem in which we seek to minimize $R$ by a judicious formulation of the update terms. The choice of the update terms should be based upon the nature and quality of the data and the known deterministic and statistical properties of the real and model systems.

In this paper we shall examine a special form of the following update formulation: $G = \mathcal{M}(u - \tilde{u}^*)$, where $\mathcal{M}$ is as yet an undefined linear operator. The effect of the update terms is thus to constrain or relax the model variables to the prescribed variables $\tilde{u}^*$. The nature and degree of constraint are related to the precise form of the operator. Schemes of this form have been proposed by Hovermale and others. Kistler (1974) and Anthes (1974) performed numerical experiments to examine aspects of the scheme and obtained encouraging results. The latter authors applied the procedure to all the prognostic equations with whatever data were known or assumed known. Here we shall assume for the most part only the availability of an estimate of the time history of the mass field. Thus only some of the update terms, $G$, in the prognostic equations will be non-zero.
(b) The linear model

We now consider the form of the primitive equations, incorporating the relaxation update formulation, for flow on an $f$-plane. The equations are linearized about a basic isothermal, motionless state. Then the horizontal equations of motion take the form

\begin{align}
    u' - fu' &= -(p'/\rho_0)_{x} + \mathcal{M}_1(u' - \bar{u}^*) \\
    v' + fu' &= -(p'/\rho_0)_{y} + \mathcal{M}_2(v' - \bar{v}^*)
\end{align}

(3)

and the hydrostatic approximation implies that

\begin{align}
    (p'/\rho_0)_{z} - S(p'/\rho_0) &= g(\theta'/\theta_0).
\end{align}

(5)

The linearized form of the mass continuity, thermodynamic and ideal-gas equations are written as follows:

\begin{align}
    \rho' + \rho_0\{(\nabla \cdot v') - (S + \gamma/c^2_0)w'\} &= \mathcal{M}_3(\rho' - \bar{\rho}^*) \\
    (\theta'/\theta_0) + Sw' &= \mathcal{M}_3(\theta'/\theta_0 - \bar{\theta}^*/\theta_0)
\end{align}

(6)

(7)

and

\begin{align}
    (p'/\rho_0) &= c^2_0(\theta'/\theta_0 + \rho'/\rho_0).
\end{align}

(8)

Dashed quantities are the perturbation variables and zero subscripts refer to the isothermal basic state $(\rho_0, \rho_0, \theta_0, T_0)$ with $c^2_0 = \gamma RT_0$ and $S = (\ln \theta_0)_{z}$.

We remarked previously that the $\bar{u}^*$ fields are merely estimates of the desired fields $\bar{u}$. It is now appropriate, in view of some impending assumptions, to comment further upon this statement. The desired fields $\bar{u}$ for a perfect model atmosphere would represent the true atmospheric state. For a given prediction model the desired $\bar{u}$ field at the commencement of the prediction would correspond to the specification of the initial state that produced the minimum error in the resulting prediction. Thus the desired $\bar{u}$ fields should be an adequate representation of the true atmospheric state that is also compatible with the prediction model. The specification of the $\bar{u}^*$ field should ideally be subject to dynamical and statistical constraints that seek to minimize in some sense the difference between the prescribed $\bar{u}^*$ fields and the desired $\bar{u}$ fields.

We eschew these problems in the analysis of this section by assuming a perfect model and a knowledge of the true time history of a subset of the variables. In the succeeding sections we relax some of these assumptions and examine some of the implications.

Returning to the perturbation equations we proceed as in conventional atmospheric tidal theory (see e.g. Lindzen 1971) to reduce the system of Eqs. (3) to (8) to a single equation in a single unknown. Our assumptions imply that $\bar{u}^* = \bar{u}$, and further, that the $\bar{u}$ fields represent true solutions to the perturbation equations. Thus it is convenient to develop the solution of the equations in terms of the variables $\bar{u}''$, the perturbation of the variables away from the true values, i.e. $u'' = u - \bar{u}$. We shall assume that the operators $\mathcal{M}_i$ ($i = 1, 2, 3$) are not functions of the height co-ordinate ($z$); then straightforward manipulation of Eqs. (5)-(8) yields the relationship

\begin{align}
    (\partial/\partial t + \mathcal{M}_3)[(p''/\rho_0)_{z} - (S + \gamma/c^2_0)(p''/\rho_0)_{z}] - (gS)(u''_x + v''_y) = 0.
\end{align}

(9)

Solutions to the system of Eqs. (1), (2) and (9) may then be sought with the method of separation of variables. We assume a separation of the form

\begin{align}
    \begin{pmatrix}
    u'' \\
    v'' \\
    p''/\rho_0
    \end{pmatrix} &= \begin{pmatrix}
    u(x, y, t) \\
    v(x, y, t) \\
    gh(x, y, t)
    \end{pmatrix} A(Z)
\end{align}

A(Z)
whereupon the system of equations reduces to

\[ u_t - f v = - g h_x + \mathcal{M}_1(u) \quad . \quad (10) \]
\[ v_t + f u = - g h_y + \mathcal{M}_2(v) \quad . \quad (11) \]
\[ h_t + H(u_x + v_y) = \mathcal{M}_3(h) \quad . \quad (12) \]

where \( H \), the separation constant, is defined by

\[ (g S)^{-1} \{(p''/\rho_0)_{xx} - (S + g/\rho_0)(p''/\rho_0)_x\} = -(gH)^{-1}(p''/\rho_0). \quad (13) \]

Thus the perturbation equations have been reduced to the conventional shallow-water system modified by the presence of the update terms. We note that Eq. (13) is unaffected by the update terms, and hence the vertical structure of the perturbations and the value of the separation constant are also unaltered by these terms.

The Eqs. (10)-(12) possess the following integral relationship for the time change of the perturbation energy,

\[ \frac{\partial}{\partial t} \int_S \frac{1}{2} \{H(u^2 + v^2) + gh^2\} \, ds = \int_S \left[ H\{u_0\mathcal{M}_1(u) + v\mathcal{M}_2(v)\} + gh\mathcal{M}_3(h) \right] \, ds \quad (14) \]

where the integral is over a channel domain \( S \), bounded by latitudinal walls to the north and south, and over a length scale in the \( x \)-direction corresponding to the flow periodicity. Our task may be restated as follows: we seek, subject to the data available, a form for the operators \( \mathcal{M} \) such that the perturbation energy or, in effect, the error energy will decrease as rapidly as possible.

The system of Eqs. (10)-(12) reduces to the following single equation in the single unknown \( v \),

\[ \mathcal{L}_3(\mathcal{L}_1 \mathcal{L}_2 + f^2)v - (g H)(\mathcal{L}_2 v_{xx} + \mathcal{L}_1 v_{yy}) = 0 \quad . \quad (15) \]

where \( \mathcal{L}_i = (\partial/\partial t - \mathcal{M}_i) \) with \( i = 1, 2, 3 \).

We now explicitly assume that only the time history of the mass field is known and correspondingly set \( \mathcal{L}_1 \equiv \mathcal{L}_2 = \partial/\partial t \). A similar analysis can be undertaken when the wind field alone is assumed known, and for this case we would set \( \mathcal{L}_3 \equiv \partial/\partial t \) and \( \mathcal{L}_1 \equiv \mathcal{L}_2 \). Further we restrict the coefficients in the operator \( \mathcal{M}_3 \) to be constants and we are then at liberty to assume solutions to Eq. (15) of the form \( \exp(\sigma' t + i(kx + ly)) \). Substitution of this form into Eq. (15) yields the equation

\[ \{(\sigma' - \mathcal{M}_3)(\sigma' + f^2) + (g H)^2(k^2 + l^2)\} v = 0. \quad . \quad (16) \]

We proceed heuristically and define the \( \mathcal{M}_3 \) operator as follows

\[ \mathcal{M}_3 = \{ - \{ c \partial/\partial t + K_N - K_D(\partial^2/\partial x^2 + \partial^2/\partial y^2) \} \}. \quad . \quad (17) \]

The first term on the r.h.s. acts to modify the effective equivalent depth of the model system and concomitantly change the phase speeds of the wave solutions. The second term guides the model flow fields towards the specified field in a manner reminiscent of Newtonian cooling, and the third term acts to diffuse horizontally the model fields towards the specified fields. Hereafter these terms will be referred to collectively as the relaxation terms. We assume that the tendency coefficient \( (c) \), the Newtonian relaxation coefficient \( (K_N) \) and the diffusive relaxation coefficient \( (K_D) \) are positive constants. Then on adoption of this form for the operator, Eq. (16) reduces to the cubic frequency equation

\[ (1 + c)\sigma^3 + K_N \sigma^2 + \{(1 + c) + \varepsilon^2(m^2 + n^2)\} \sigma + K_D \sigma^2 = 0 \quad . \quad (18) \]

where \( \sigma^2 = (1 + \varepsilon^2(m^2 + n^2)) \), and the variables have been non-dimensionalized such that
\[ \sigma = \sigma'f, \ K = K_N[f], \ \text{with} \ v^2 = K_D(2^2K_N), \ \varepsilon^2 = (gH)(fa)^2, \ \text{and} \ (m,n) = a(k,l), \ \text{where} \ a \ \text{is the radius of the earth.} \]

If \( \varepsilon^2 \) is specified then we have an equation for the eigenfrequencies of the problem posed by Eqs. (10)-(12). In particular we recall that for our purpose we desire that the polynomial equation (18) have only roots with negative real part corresponding to a decay of the error modes of the flow field. Examination of Eq. (18) or, alternatively, the form of Eq. (14) with \( \mathcal{A} \) defined as in Eq. (17), confirms that this criterion has been satisfied.

It remains to determine the optimum values for the coefficients \( c, K_N \) and \( K_D \). We shall consider separately the following three cases:

\[
\begin{align*}
\text{A:} & \quad c \equiv K_D = 0, \ K_N \neq 0, \\
\text{B:} & \quad c \equiv 0, \ K_N \neq 0, K_D \neq 0, \\
\text{C:} & \quad K_D = 0, \ c \neq 0, K_N \neq 0.
\end{align*}
\]

**Case A:** This case involves Newtonian relaxation alone, and Eq. (18) reduces to

\[ \sigma^3 + K_\sigma^2 + (1 + \varepsilon^2(m^2 + n^2))\sigma + K = 0. \]

For \( E = \{1 + \varepsilon^2(m^2 + n^2)\} \gg 1 \) the roots of this cubic are reasonably separated and are given quite accurately by the approximate expressions

\[ \sigma \approx -K/E, \ \text{...} \ (19a) \]

and

\[ \sigma \approx -\frac{1}{4}K[1 \pm (1 - 4E/K^2)^{1/2}], \ \text{...} \ (19b) \]

Eqs. (19a, b) denote respectively the values of the complex frequencies of the geostrophic and the two inertia-gravity modes that correspond to perturbations of the flow field away from the specified true mass field. If the scheme is to be efficient then \( \sigma^* (= -\sigma) \) must be large and positive (i.e. \( \sigma^* \gg 1 \)) for all realistic values of \( \varepsilon^2, m \) and \( n \). From these formulae we note that the damping of the perturbation geostrophic modes is least for large wave-numbers and a large rotational Froude number (\( \varepsilon^2 \)). Hence this method will not in general satisfy our efficiency criterion. The damping of the inertia-gravity waves is scale-independent if \( E \gg \frac{1}{2}K^2 \), and also independent of \( \varepsilon^2 \). The latter property implies that both external and internal modes are equally influenced by the scheme.

**Case B:** This scheme involves both Newtonian and diffusive relaxation and we now have the frequency equation

\[ \sigma^3 + (K\sigma^2)\sigma^2 + E\sigma + K\sigma^2 = 0. \]

Approximate expressions for the roots of this cubic are given by

\[ \sigma \approx -K\sigma^2/E, \ \text{...} \ (20a) \]

\[ \sigma \approx -\frac{1}{4}K\sigma^2[1 \pm (1 - 4E/(K\sigma^2)^2)]^1/2]. \ (20b) \]

We note that the introduction of the diffusive relaxation term has altered the scale dependency of the damping of the geostrophic mode. In particular for large wave-numbers the damping rate tends to a non-zero value, i.e. \( \sigma \approx -K\varepsilon^2/\varepsilon^3 \) as \( (m^2 + n^2) \rightarrow \infty \). Thus for this simple model the combined effect of Newtonian and diffusive relaxation of the mass field appears to provide a means of appreciably damping all the system's modes at all wavelengths. This assertion is corroborated by the results portrayed in Fig. 1. The decay rate of the error, \( \sigma^* (= -\sigma) \), is plotted against the zonal wavenumber \( m \) for an assumed square-wave profile \( n \equiv m \) with \( \varepsilon^2 = 1 \). The scripts 'v' and 'g' associate portions of the curve with the corresponding geostrophic and gravity-wave modes. These are identified by reference to the non-relaxed system.
Figure 1. Non-dimensional decay rate $\sigma^*$ plotted against zonal wavenumber ($m$) with the rotational Froude number ($v^2$) equal to 1. Values of $(K, v^2)$ equal (10, 0), (4, 0), (1, 0) and (0-1, 0) in diagram (a) and (10, $10^{-2}$) in diagram (b).

In Fig. 1(a) the decay rates corresponding to case A (purely Newtonian relaxation) are shown for a range of values of $K$. In accord with the approximate expression of Eq. (19a), the geostrophic mode is seen to be only weakly damped at synoptic and smaller length scales for all values of $K$. We also note that for $(m \sim 1, K \leq 1)$ there is a tendency for multiple roots. These roots increase in value with $K$ until $K \sim 1$, and thereafter are bifurcated with a rapid decrease in the value of the double root. Further calculations confirm this tendency.

In Fig. 1(b) the roots corresponding to case B with $(K, v^2)$ values of (10, $10^{-2}$) are shown with $v^2 = 1$ again. The beneficial effect of diffusive relaxation is evident. The decay rate of the geostrophic perturbation modes rapidly approaches an asymptotic value at the synoptic length scale, while one gravity mode is vigorously damped and the decay rate of the other increased slightly above its value for Newtonian relaxation alone. From Eqs. (20a,b) one can obtain a useful preliminary estimate of the optimum values of $K$ and $v^2$. For example with $\sigma^2 = 1$ we infer from these equations that $K \sim v^2 \lesssim 1$ would yield roots such that $\min |\sigma^*| \gtrsim 1$. Further calculations confirm this result; but for larger values of $v^2$ there would be a deleterious effect at small wavenumbers. These comparatively high values for the relaxation coefficients indicate that the time evolution of the height field is dominated by the contribution of the relaxation terms unless the model height field is close to the true height field.

This analysis reveals that, in the context of our simple linear model and given the time history of the mass field, a suitable combination of Newtonian and diffusive relaxation damps both the error gravity wave motion and the error geostrophic motion. Moreover there is no implied simultaneous damping of the true flow field, and hence the true flow state is approached with time. The efficacy of the technique is apparent on noting that a non-dimensional decay rate of 0-5 corresponds approximately at mid-latitude to an
effective 'e^{-1}-folding time' of one day for the error flow. We comment further on this scheme in the next section.

*Case C:* Now we allow the tendency coefficient, c, to be non-zero and retain only the Newtonian relaxation term. Eq. (18) reduces to

\[(1 + c)\sigma^2 + K\sigma^2 + \{(1 + c) + c^2(m^2 + n^2)\}\sigma + K = 0,\]

and approximate expressions for the roots are as follows,

\[\sigma \approx -K/(c + E) \tag{21a}\]

and

\[\sigma \approx \frac{1}{4}\left[-K \pm \sqrt{K^2 - 4(1 + c)(E + c)^2}\right]/(1 + c) \tag{21b}\]

In particular we note that for \(c \gg E\) and \(K \ll c\) the expressions reduce to

\[\sigma \approx -K/c, \quad \sigma \approx (-K/2c, -K/2c).\]

Thus with a suitable choice for the values of the coefficients \(K\) and \(c\) we again have an efficient relaxation scheme, capable of reducing the error appreciably on the timescale of one day. These results suggest that the schemes of both case B and case C may be useful in updating prediction models.

It is, however, important to note the limitations of our analysis and of the technique itself. We reiterate that we have hitherto assumed the availability of a practically inordinate amount of data, and the data themselves have been assumed free of error. It is to be expected that data paucity and errors will almost inevitably reduce the efficiency of the method and we examine some implications of this problem in the next section. Again we have considered an \(f\)-plane model. Extension of the analysis to a mid-latitude \(\beta\)-plane will produce only slight modifications to our results, but a separate analysis is required for an equatorial \(\beta\)-plane. This is performed in section 4.

3. A CONSIDERATION OF ERRORS IN THE SPECIFIED FIELDS

A central feature of the relaxation update scheme is the continuous forcing of a subset of the model variables. This forcing is proportional to the departure of the model variables from the specified \(\tilde{u}^*\) variables and in effect constrains the model variables to these postillion-like fields. It follows that to some extent the model variables will be forced to respond to errors in the postillion fields.

We examine this, conceivably catastrophic, response in the context of the linear model used in the previous section. Now we assume a time-history estimate of the mass field is available, such that \(\tilde{\rho}^* = \tilde{\rho} + \rho_E\), where \(\rho_E\) represents the error in the density field.

The system corresponding to Eqs. (10)–(12) for this situation takes the form

\[u_t - fu = -gh_x,\]

\[v_t + fu = -gh_y,\]

\[h_t + H(u_x + v_y) = \mathcal{M}_3(h - h_E).\]

We will again adopt the relation (17) for \(\mathcal{M}_3\). Eliminating \(u\) and \(v\) from the system we obtain the following differential equation for \(h\),

\[\left(\frac{\partial}{\partial t} - \mathcal{M}_3\right)\left(\frac{\partial^2}{\partial t^2} + f^2\right)h - \left(gH\right)\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)h = -\left(\frac{\partial^2}{\partial t^2} + f^2\right)\mathcal{M}_3 h_E \tag{22}\]

We can decompose the error field \(h_E\) in terms of the normal mode solutions of the homogeneous equation:
If the model's response to the forcing by the error field is similarly given by

\[ h = \mathcal{A}(\bar{\sigma}', k, l) \sum_{k} \sum_{l} \exp(i\bar{\sigma}'k) \exp[i(kx + ly)] \]  \hspace{1cm} (24)

then we wish to examine the amplitude response defined by

\[ R(\bar{\sigma}', k, l) = |\mathcal{A}|^2 / |\mathcal{A}|^2. \]

Utilizing expressions (23) and (24) in Eq. (22), we find that

\[ R = \frac{\bar{\sigma}^2 c^2 + (K\alpha^2)}{[\bar{\sigma}^2(c + E)^2 + (K\alpha^2)^2]}, \hspace{1cm} (25)\]

where \( \bar{\sigma} = \bar{\sigma}'/f \). In the previous section we sought values of \((c, K, \nu)\) that yielded large decay rates for all the perturbation (error) modes of the system. We would now like to impose a further constraint: that the amplitude response, \( R \), to the forcing by the error in the specified field also be small. An examination of the form of Eq. (25) for the schemes of cases B and C of the previous section reveals that the former is preferable when this additional constraint is also a stipulation. For this scheme, i.e. case B, Eq. (25) can be written as

\[ R_B = \left( 1 + \bar{\sigma}^2 E^2/(K\alpha^2)^2 \right)^{-1} \hspace{1cm} (26) \]

and if \( \nu^2 \ll \alpha^2 \) it follows that

\[ R_B \ll \left( 1 + \bar{\sigma}^2/K^2 \right)^{-1}. \hspace{1cm} (27) \]

The implications of these results are encouraging. High-frequency error modes in the forcing field (\( \bar{\sigma}^2 \gg 1 \)) will induce only a small amplitude response if the values of \( K \) and \( \nu^2 \) are chosen such that \( K \sim 1 \) with \( \nu^2 \ll \alpha^2 \). Moreover a restriction of \( K \) and \( \nu^2 \) to meet these stipulations is also consonant with the restraints imposed by the results of the previous section. Admittedly, low-frequency error modes in the forcing field (\( \bar{\sigma}^2 \ll 1 \)) will induce a comparable response in the model flow fields, but the amplitude of these components of the forcing fields may be comparatively small since it is to be expected that the observational error will be less for the larger timescales.

These results indicate that, at least for this linear model, the relaxation update scheme can be tuned to reduce the model's response to high frequency noise in the specified postillion fields. It is possible to extend the analysis slightly by allowing \( K_N \) and \( K_D \) to be functions of space and time. In particular we note that if \( K_N \) and \( K_D \) are set to zero in regions where no estimate of the postillion fields is available then the relaxation scheme acts to consume error energy propagating out of the no-data region and simultaneously constrain the boundary of that region toward the postillion fields. The latter effect will also enable the effect of the postillion fields to be propagated into the region of no-data.

The considerations of this section allay somewhat the possible shortcomings of the relaxation scheme when used in conjunction with inaccurate data, but of course we must also stress that our model is overtly naive.

4. An analysis for an equatorial \( \beta \)-plane system

The design and specification of data assimilation schemes for the tropics remains a predominantly unresolved problem (McPherson 1975). It is argued that, since the mass and motion fields are observed to be substantially decoupled, observations may be required of both the mass and wind fields. The need to supplement mass-field information with wind data in the sub-tropics was confirmed in model experiments (e.g. Gordon et al. 1972).
Further experiments have been conducted with various general circulation models in an attempt to elucidate the details of the data requisites for successful model updating in the tropics.

It is against this background that we proffer the simple analysis of this section. We return to the linearized system represented by Eqs. (10)–(12) but now assume explicitly that $f = \beta y$ corresponding to an equatorial $\beta$-plane. Further, submitting to considerations of mathematical expediency, we restrict our analysis to schemes where the update terms are of the simplified form

$$M_i = -(c_i \partial/\partial t + K_{N(i)})$$

and the $(c_i, K_{N(i)})$ values are constants, but not necessarily non-zero.

Subject to these stipulations our system of equations reduce to the form

$$L_3(L_1L_2 + f^2)v - (gH)(L_2v_{xx} + L_1v_{yy}) - (gH)\beta v_x = 0$$

where $L_i = (\partial/\partial t - M_i)$.

We seek solutions of the form

$$v = V(y)\exp(\sigma't + ikx).$$

Substitution of this form in Eq. (28) leads to the differential equation

$$d^2V/dy^2 - (\alpha_1\sigma + K_1)^{-1}\{\lambda^2 + (\alpha_2\sigma + K_2)e^{-2}\psi^2\}V = 0$$

where

$$\lambda^2 = (\alpha_2\sigma + K_2)[(\alpha_1\sigma + K_1)(\alpha_3\sigma + K_3)e^{-2} + m^2] - im,$$

and the variables have been non-dimensionalized such that $\sigma = \sigma'/(\beta a)$, $K_i = K_{N(i)}/(\beta a)$, $y^* = y/a$, and $\epsilon^2 = gH/(\beta a)^2$, with $\alpha_i = (1 + c_i)$, $i = 1, 2, 3$.

We shall now consider two special cases of Eq. (29).

Case (i). We assume that only an estimate of the north–south velocity field, $v$, is available and set $\alpha_1 = \alpha_3 = 1$, and $K_1 \equiv K_3 = 0$.

For this case the solutions of Eq. (29) that vanish as $y$ tends to infinity, and are therefore in accord with the equatorial $\beta$-plane approximation, are given by

$$V = \exp(-\frac{1}{\delta}\xi^2)H_n(\xi) \quad n = 0, 1, 2, \ldots$$

where $H_n(\xi)$ is the $n$th Hermite polynomial, $\xi = y^*/(\epsilon^2)^{1/2}$, and the integer $n$ is such that

$$n = (\alpha_2\sigma + K_2)(\sigma^2 e^{-2} + m^2) - im/\sigma = (2n + 1).$$

This last expression may be rearranged to the form

$$\sigma^3 + K_2^* e^{\sigma^2} + (\epsilon G)\sigma + \epsilon^2(K_2^* m^2 - im) = 0$$

where

$$G = \epsilon m^2 + (2n + 1)/\alpha_2, K_2^* = K_2/\alpha_2.$$

For $n \geq 1$ the roots of this equation are given to a good approximation by the expressions

$$\sigma \approx -\epsilon(K_2^* m^2 - im)/G, \quad (32a)$$

$$\sigma \approx -\frac{1}{\delta}K_2^*\left[1 \pm \sqrt{1 - 4(\epsilon G)/(K_2^* m^2)}\right]$$

We note that, if $\alpha_2$ is chosen such that $\epsilon m^2/(2n + 1) > \alpha_2^{-1}$ then all these error wave modes, except the symmetric modes, $m \equiv 0$, suffer appreciable damping on the timescale of one day if $\epsilon^2 \geq 1$ and $K_2^* \sim 1$. The wave modes corresponding to $n = 0$ normally require a separate treatment but, with the above restrictions, we can infer directly that they too possess comparable properties. Contrarywise it is apparent that the so-called Kelvin wave
characterized by $v = 0$ will not be influenced by this form of the update scheme. Thus our analysis suggests that an accurate knowledge of the time history of merely
the north–south component of the wind is sufficient to ensure the appreciable damping of all the error modes, other than the Kelvin ($n = -1$) waves and the symmetric ($m = 0$)
waves, if $\varepsilon^2 \gg 1$. This is an interesting result but its usefulness is obviously limited by the
listed restrictions.

**Case (ii).** In this case we consider the hybrid scheme based upon the assumption of
the availability of a time history of both the zonal velocity field and the mass field. We
choose the coefficients in the update terms such that

$$\alpha_2 = 1, K_2 \equiv 0, \text{ and } (\alpha_1 \alpha + K_1) = \gamma(\alpha_3 \alpha + K_3).$$

It follows that we again have permissible solutions of the differential equation (29) of
the form of Eq. (30) but with $\zeta$ redefined such that $\zeta = \gamma / (\varepsilon^2 \gamma)$. The resulting frequency
equation takes the form

$$\sigma^3 + 2K_3 \sigma^2 + \{\varepsilon^* F + (K_3^*)^2\} \sigma + \varepsilon^* \{K_3^*(2n + 1) - \imath m \sigma^*\} = 0 \quad . \quad (33)$$

where $F = \varepsilon^* m^2 + (2n + 1)$, with $K_3^* = K_3 / \alpha_3$, $\varepsilon^* = \varepsilon / (\alpha_3)^{1/2}$. The corresponding approximate roots of the cubic for $n \geq 1$ are given by

$$\sigma \approx -\varepsilon^* \{K_3^*(2n + 1) - \imath m \sigma^*\} / \{\varepsilon^* F + (K_3^*)^2\} \quad . \quad (34a)$$

and

$$\sigma \approx -K_3^* \pm \imath (\varepsilon^* F)^{1/2}. \quad . \quad (34b)$$

For this case we note that if the free parameter $\alpha_3^{1/2}$ is such that $\varepsilon^* m^2 / (2n + 1) \ll 1$, then
Eqs. (34a, b) indicate that all these error modes suffer appreciable damping on the one-day
timescale if $K_3^* \sim 1$. In addition we need to consider the special cases excluded from the
above analysis. The $n = -1$ modes, corresponding to the westerly Kelvin waves, are
modified by this relaxation scheme such that

$$\sigma = -\left[ K_3^* \pm \imath (\varepsilon^* m^2)^{1/2} \right]$$

and there is a slight modification of the relaxation-free latitudinal structure of the wave
arising from the fact that $\sigma$ has both real and imaginary parts. We also infer that the
$n = 0$ modes, corresponding to the mixed Rossby waves, are damped at rates comparable
with those given by Eqs. (34) for the other modes.

Thus the analysis indicates that the relaxation update scheme of case (ii) will sub-
stantially damp all the permissible equatorial $\beta$-plane wave modes on a one-day timescale,
provided the coefficients of the relaxation update operators are chosen appropriately and
the mass and zonal flow fields are known.

In view of our meagre understanding of the mechanics of assimilation in the tropics
and equatorial regions we regard this definitive result to be of some significance despite the
limitations of the analysis.

Data coverage in tropical latitudes is far from comprehensive and hence the effects of
errors in the estimates of the field and the repercussions of data gaps are of considerable
interest and concern. In this respect it is pertinent to recall the results of our study in
section 3 of the effects of errors in the forcing fields. Those results suggest that the efficiency
of a scheme with the form of that considered in case (ii) may be impaired by high-frequency
errors in the forcing fields. This may not be crucial since the exclusion of high-frequency
components from the postilion field would be a trivial procedure and in reasonable accord
with observations of the synoptic-scale meteorological fields in low latitudes. Moreover
we further note from Eq. (34a) that a discrete localized error in the specification of the
low-frequency meteorological field would be subject only to a non-dispersive westward propagation with the group velocity, $u_G$, 

$$u_G \approx -e^*/[e^*(2n + 1) + (K_H^*)^2].$$

Thus the error would remain longitudinally localized and its influence would not pervade the whole domain.

It is also worth noting that we excluded the diffusive relaxation terms in our analysis merely on the grounds of mathematical expediency. Their inclusion in the analysis of the $f$-plane model was found to be preferable to the tendency-modification term when consideration was given to the effect of errors in the postillion fields. A programme of numerical experimentation could determine whether the same conclusions apply for the subtropical regions. Finally we note that our previous remarks concerning the relaxation scheme and data gaps remain apropos of the low-latitude regions.

5. **Update experiments**

(a) **Preliminaries**

In the previous sections we undertook an analytical investigation of the potential of the relaxation update scheme in the context of linear $f$- and $\beta$-plane models. This has enabled us to glean some of the basic properties of the scheme. In particular the results indicate that if limited time-history estimates of a particular subset of the dependent variables is available then the scheme can effect a substantial reduction in the error modes on the timescale of one day.

A definitive statement of the technique's operational utility would of necessity be based upon its performance with real data in a multi-level prediction model. This is not attempted in this study. However, it seems prudent to subject the relaxation scheme to a further examination to determine whether the main conclusions of the analysis remain valid when one removes some of the major assumptions, constraints and shortcomings of the model. In this section we seek to substantiate with a series of numerical experiments that the relaxation scheme operates with comparable efficiency when the linearity assumption is discarded and the data demand is reduced to requiring a knowledge of only a succession of instantaneous historical snapshots of the pre-specified data fields. While this does not constitute a comprehensive examination of the scheme and all features of the theoretical analysis, it is nevertheless sufficient to indicate the potential of the method.

(b) **Model features**

The nonlinear equations representing the hydrostatic flow of shallow water on a $\beta$-plane constitute the basic framework for our model-update experiments. These equations, which can sustain Rossby and gravity waves, are also the basis for the one-level, primitive equation, barotropic prediction models, and thus are a natural choice for an initial examination of the dynamical relaxation technique.

On a $\beta$-plane, the equations of motion and mass continuity assume the form

$$u_t + uu_x + vu_y - fu = -gh_x,$$
$$v_t + uv_x + vv_y + fu = -gh_y,$$

$$h_t + uh_x + vh_y + h(u_x + v_y) = -K_H(h - h_p) + K_D\nabla^2(h - h_p),$$

where $(u,v)$ are the zonal and meridional velocities in the $(x,y)$ directions, $gh$ is the geopotential, and $f$, the Coriolis parameter, varies linearly in the $y$-direction such that
\[ f = f_0 + \beta y. \] Here \( h_p \) denotes the externally specified height field toward which the model field is guided by a relaxation scheme of the form detailed in case B of section 2; we refer figuratively to \( h_p \) as the postillion field.

The flow domain was taken to be rectangular with rigid walls to the north and south, and cyclic boundary conditions to the east and west. The north–south and east–west dimensions were chosen to approximate respectively the pole-equator distance and half the circumference of the earth at mid-latitudes. Corresponding to this domain, a grid of \((40, 31)\) points is assigned with a grid length of 300 km, and \( f_0 \) and \( \beta \) have respectively the mid-latitude values of \( 1.3 \times 10^{-4} \text{s}^{-1} \) and \( 1.62 \times 10^{-11} \text{m}^{-1} \text{s}^{-1} \) such that \( f = f_0 \) corresponds to the mid-channel. A finite-difference scheme constructed from the momentum form of the equations is used for the spatial terms and corresponds in the interior to scheme B of Grammeltevdt’s (1969) classification. A leap-frog time-scheme was used and the Newtonian and diffusive relaxation terms are represented respectively by an implicit treatment and a DuFort–Frankel representation. We note that a form of the mass relaxation terms enters into all prognostic equations since they have been cast in momentum form.

\[ (c) \text{ The control field} \]

A control field has first to be generated from which the postillion field could be obtained for the relaxation method. The initial state for this field was obtained by a 2\( \frac{1}{2} \)-day integration of the non-relaxed equations starting from an analytically posed wind and height field. The latter field corresponded to a Rossby-Haurwitz wave with a ‘planetary’ latitudinal wavenumber of 6, superimposed upon a zonal shear flow giving westerlies in mid-latitudes and easterlies elsewhere. The frequency filter described by Asselin (1972) was incorporated in the time-differencing scheme during this integration.

Towards the end of the integration period the high-frequency gravity wave noise generated by the initial imbalance was substantially reduced but there remained a persistent small-amplitude oscillation with a period of approximately six hours.

The control field was then obtained by a further integration of the non-relaxed equations with a timestep of 600 s and without the frequency filter. Fig. 2(a) displays the initial height field for the control run. The pattern has a trough-ridge pattern of maximum amplitude in the model ‘middle-latitudes’, and a fairly uniform field with weak gradients in the ‘tropics’. There was a northward incursion of the ridges in the subsequent 48 hours and the troughs cut off to form closed lows (Fig. 2(b)).

\[ (d) \text{ Experimental procedure} \]

A series of experiments were undertaken with the control height field assumed known at every grid point at intervals \( \tau \) of 3, 6 or 12 hours. The time evolution of the postillion height field, \( h_p \), was then constructed from these fields. For simplicity the method adopted herein was merely linear interpolation between the data periods. The relaxation coefficients were also given a parabolic time dependency in each interval such that their value attained a zero minimum value at the mid-interval instant. Thus for the interval \( < t_0 - \tau/2, t_0 + \tau/2 > \) we set

\[ (K_N, K_D) = (K_N^*, K_D^*)(2/\tau)^2(t - t_0)^2. \]

For most experiments the initial wind state for the update integrations was specified to be geostrophic. This estimate is readily obtained from the height field and is a reasonable first approximation to the wind field. It is moreover useful for our experiment since it
generates a significant gravity-wave motion, and this reflects the fact that geostrophy is not an acceptable state for primitive equation models. In contrast the wind field was set identically zero in one set of experiments to test the resilience of the relaxation techniques, namely, its damping of large amplitude meteorological and high-frequency, error waves.

Our primary objective was an assessment of the dynamical-relaxation technique. Thus the update experiments cover periods of up to two days, and the results are contrasted with others obtained with conventional direct insertion of data at update times. In the latter experiments the leap-frog differencing was again employed except for a forward difference step used immediately after each data insertion to avoid the 'time-splitting' computational effect. Thus there is no explicit damping mechanism in our conventional update runs but adaptation can occur by error-energy interchange during the update interval.

A catalogue of the experiments is given in Table 1 and the indexing system established there is chosen to conform with the subsequent figures. The experiments were designed to examine the dependency of the technique on the interval (τ) separating the available data, the values of the relaxation coefficients, and the error in the initial velocity fields.
### Table 1. List of update comparison experiments conducted with nonlinear $\beta$-plane model.

DI, NR and N+DR denote direct insertion, Newtonian relaxation, and Newtonian and diffusive relaxation methods.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Data interval $\tau$ (hours)</th>
<th>Update period $\Delta$ (hours)</th>
<th>Initial wind field</th>
<th>Update method</th>
<th>$K_n(10^{-2}s^{-1})$</th>
<th>$K_D(10^4m^2s^{-1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3a</td>
<td>12</td>
<td>48+0</td>
<td>$g$ (geostrophic)</td>
<td>DI</td>
<td>3.00</td>
<td>0.0</td>
</tr>
<tr>
<td>b</td>
<td>12</td>
<td>48+0</td>
<td>$g$</td>
<td>NR</td>
<td>0.83</td>
<td>3.0</td>
</tr>
<tr>
<td>c</td>
<td>12</td>
<td>48+0</td>
<td>$g$</td>
<td>N+DR</td>
<td>3.00</td>
<td>0.0</td>
</tr>
<tr>
<td>4a</td>
<td>6</td>
<td>24+24</td>
<td>$g$</td>
<td>DI</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>6</td>
<td>24+24</td>
<td>$g$</td>
<td>NR</td>
<td>0.83</td>
<td>3.0</td>
</tr>
<tr>
<td>c</td>
<td>6</td>
<td>24+24</td>
<td>$g$</td>
<td>N+DR</td>
<td>3.00</td>
<td>0.0</td>
</tr>
<tr>
<td>5a</td>
<td>3</td>
<td>12+0</td>
<td>$g$</td>
<td>DI</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>3</td>
<td>12+0</td>
<td>$g$</td>
<td>NR</td>
<td>0.83</td>
<td>3.0</td>
</tr>
<tr>
<td>c</td>
<td>3</td>
<td>12+0</td>
<td>$g$</td>
<td>N+DR</td>
<td>3.00</td>
<td>0.0</td>
</tr>
<tr>
<td>6a</td>
<td>12</td>
<td>48+0</td>
<td>0 (zero field)</td>
<td>DI</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>12</td>
<td>48+0</td>
<td>0</td>
<td>NR</td>
<td>0.83</td>
<td>0.0</td>
</tr>
<tr>
<td>c</td>
<td>12</td>
<td>48+0</td>
<td>0</td>
<td>N+DR</td>
<td>0.83</td>
<td>3.0</td>
</tr>
</tbody>
</table>

(e) **Method of verification**

Customary techniques of verifying assimilation experiments involve the evaluation of the time history of the r.m.s. velocity or vector winds either globally or in latitude bands. In the context of the initialization problem Temperton (1973) indicated that these are not necessarily the most sensitive measures of the error field and he also employed the r.m.s. mass divergence. This measure was also usefully employed by Hayden (1973). We verified the update experiments with these various diagnostic measures, and also examined the performance of some of the update fields against the control fields during a subsequent 24-hour forecast.

![Figure 3. Time history of the wind and height field error measures, $\psi$ and $\Phi$ (see text for definitions), for expts. 3a, b, c with the nonlinear barotropic model. Solid, dash-dash and long-dash lines represent respectively the direct insertion (3a), Newtonian relaxation (3b), and Newtonian plus diffusive relaxation (3c) experiments.](image-url)
Curves for the time history of measures of the error wind and height field for the update period of the experiments listed in Table 1 are plotted in Figs. 3-6. The measures displayed are suitably related error norms for this flow system (Davies 1973) and represent
half the m.s. vector wind error and $2g\{\text{m.s.}(\sqrt{h_e} - \sqrt{h_u})\}$ where $h_e$ and $h_u$ denote respectively the true field and the height field of the update flow, and m.s. denotes the mean square. Hereafter we shall refer to these error measures by $\Psi$ and $\Phi$ respectively.

The solid line in these diagrams represents results obtained with the conventional update method of direct insertion. The particularly poor adaptation with this method in exp. 6 is attributable to the large-amplitude oscillation in the error fields. Data insertion every 12 hours is rendered ineffective because the error oscillation, essentially between kinetic and potential forms, has a comparable period.

![Graph](image)

Figure 6. Results for expts. 6a, b, c, with line markings as for Fig. 3.

It is seen that the relaxation technique reduces the wind error from 4.6 m$^2$s$^{-2}$ to less than 2.0 m$^2$s$^{-2}$ in experiments 3-5 and the technique heavily constrains the model height field. These results are consistently better than those with the conventional technique. For these simple experiments, the error reduction with the relaxation technique appears to be related to the number of update data sets rather than the update interval. Moreover, the results also tend to suggest that for this case Newtonian relaxation alone is comparable in efficiency to the full relaxation scheme. Indeed, better results are obtained with the simplified scheme in exp. 6. The results obtained in the latter experiments are also
encouraging in that they demonstrate the resilient and rapid adaptation of the flow fields from bad initial wind estimates.

The latitudinal distribution of the r.m.s. 'zonal' velocity error field was examined at the end of every experiment. Fig. 7 is a plot of this error measured at the end of exp. 5. There is a slight increase in the error near the northern boundary but the major errors occur near the southern end of the channel. These features are also present in the zonal

Figure 7. 'Latitudinal' distribution of the r.m.s. zonal wind error ($\bar{u}$) at the end of exp. 5 and depicted in Fig. 5. Line markings as for previous figures.

Figure 8. As for Fig. 5 (expt. 5c) but with a relaxation of the zonal velocity field in the lower third of the flow domain.
error fields at the end of the other experiments. This asymmetric error distribution is in keeping with our deductions based upon the linear analysis of the previous section, and the need for tropical wind data has already been stressed (Gordon et al.). Fig. 8 shows the time history of the measures of error energy for an experiment comparable with exp. 5 (cf. Fig. 5) incorporating in addition a zonal velocity Newtonian relaxation with the zonal wind component also assumed available every three hours but only in the lower-latitude third of the model domain.

The update period of exp. 4 formed a prelude to a 24-hour free run, or forecast, with the desired update field forming the initial conditions. These extended runs are, in essence, predictability runs. During the free-run period, the additive error measure \((\Psi + \Phi)\) for exps. 4a, b, c, changed as follows: 2·77 to 3·24, 2·04 to 2·17, 1·74 to 1·81. These values indicate a progressive and significant decrease in the growth of the error during the pseudo-forecast. Thus the improvement in the specification of the initial state, resulting from the use of the relaxation technique, also produced a measurable improvement in the accuracy of the subsequent forecast.

![Figure 9](image)

**Figure 9.** Time history of the r.m.s. mass divergence for a period straddling the transition \((t = 24 \text{ hours})\) from update to a free-run mode in exp. 2. Line markings \(a, \beta, \gamma,\) and \(\delta\) correspond respectively to the control run and exps. 2a, b, c, and \(e\) corresponds to a Newtonian relaxation run with \(K_N = 0·4 \times 10^{-3} \text{s}^{-1}\).

The behaviour of the flow fields during the transition from the update phase to the forecast or free-run phase is an indication of the degree of balance attained during the first phase. Curves for the time history of the r.m.s. mass divergence of the flow fields are shown in Fig. 9 for a period of 12 hours straddling the update to free-run transition. The control field itself \((a)\) is not steady during this period. Data insertion at \(t = 18\) and 24 hours instigates an oscillation in the flow field derived from the direct insertion method \((\beta)\). The relaxation technique induces a similar but somewhat more complex behaviour related to the amplitude and time variation of the relaxation coefficients during the update phase. During the free-run period there is little to choose between the results of the various runs. The runs with pure Newtonian relaxation are marginally steadier and attain a closer estimate to the true amplitude. It is clear that this particular measure is highly sensitive and none of the update runs provide a satisfactory estimate of the overall r.m.s. divergence amplitude.

Further experiments to determine the sensitivity of the relaxation technique to the values of the relaxation coefficients showed that for pure Newtonian relaxation with \(K_N\) in the range \(3 \times 10^{-4}\) to \(3 \times 10^{-3} \text{s}^{-1}\) the only appreciable quantitative difference lay in the value of the r.m.s. mass divergence. The technique gave improved results with \(K_N\) in
this range as $K_0$ increased to approximately $3 \times 10^6 \text{m}^2\text{s}^{-1}$ and, thereafter, the error fields increased rapidly.

In this section we have not attempted to verify all the deductions of our analytical work. We have not undertaken a systematic study of the tendency modification terms nor the effect of error in the postillion forcing fields, although some aspects of both occur implicitly in the experiments. Again we compared the relaxation scheme only with the conventional direct insertion method. Further experiments to investigate these effects are required and these experiments would obviously be more incisive in determining the practical efficacy of the relaxation scheme if they were performed with more sophisticated models and real data fields. Meanwhile we contend that the results reported here are at least useful indicators of the potential of the dynamical relaxation update scheme.

6. FURTHER REMARKS

In this paper we sought to outline and examine a dynamical relaxation technique for updating prediction models. The study was undertaken with the linear and nonlinear barotropic primitive equations. It can be shown (Charney et al.) that, for the shallow-water equations, a knowledge of the instantaneous mass distribution and its first and second time derivatives is sufficient to determine the wind field on an $f$-plane. The latter determination involves solving a modified form of the balance equation. This result lends support to the premise that the required initial fields can be obtained by the process of four-dimensional updating. Here we establish, in the context of a linear model, the fundamental result that the assimilation process can proceed successfully in a forward time integration of the governing equations when modified by the presence of suitable relaxation update terms. It is shown that all time and space scales of the error fields suffer at least an amplitude reduction of the order of $e^{-1}$ on the time scale of one day if a time history of the mass field (or the wind field) is known on an $f$-plane, or if a time history of the mass and zonal wind field is known on an equatorial $\beta$-plane. These definitive analytical results are corroborated by numerical experiments with the nonlinear shallow-water equations.

The magnitude of the perturbation amplitude reduction is significant since it is taken as axiomatic that this value must exceed the rate of error growth associated with the physical instability of the flow in more complicated models if successful assimilation is to take place. Again we note that with the relaxation scheme and the prescribed subset of the dependent variables our analysis shows that the true flow field is approached in time if the postillion fields are themselves accurate. Moreover, we also indicate that the update terms can to some extent be tuned to offset the effects of inaccurate estimates of the postillion fields.

It is appropriate at this point to comment on the related work of Kistler and Anthes. In essence they suggest relaxing all the update variables to the limited available external data in a manner similar to that described herein. In this study we have contended that the availability at periodic intervals of a particular subset of the meteorological variables may suffice to dynamically update a model. Thus these studies are complementary. In view of the incompleteness in practice of that data subset - global pressure distribution complemented with zonal winds in the tropics - it is clearly prudent to consider, as in the former studies, the incorporation of additional relaxation terms.

However, localized relaxation of a redundant amount of essentially prognostic fields renders the equations strongly quasi-parabolic in character and may unduly inhibit the necessary dynamical adjustment of the model flow fields towards a state of mutual balance. This remark will be particularly appropriate if the truncation errors are large in that region, or if the data are inaccurate.

The results obtained in this study are encouraging and warrant a further examination
of the relaxation scheme under more testing and realistic conditions. The extension and application of the technique to multi-level prediction models is straightforward. No additional problems are posed by the presence of irreversible processes in the model. However, it is to be expected that truncation errors and parameterization schemes will inevitably produce incompatibilities between the model fields and the external data. This incompatibility will in turn influence the effectiveness of the relaxation terms. An accurate specification of the postillation fields in the relaxation terms can be shown in linear models to partially offset errors arising in the phase speed due to truncation effects. On the other hand it may prove necessary, for instance, to weaken the effect of relaxation terms in the vicinity of poorly-resolved mountainous regions although good observational data might be available for the area. In effect one requires ideally some model-dependent sequential analysis scheme to establish a suitable local value for the coefficients of the relaxation update terms. This caveat serves to re-emphasize our previous comment that a trenchant assessment of the scheme's usefulness must per se be based at least in part upon the performance of the scheme under more realistic conditions.

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