Instability of planetary waves and zonal flows in two-layer models on a sphere

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Summary

A study is made of the stability of finite amplitude planetary waves by themselves and together with zonal flows in two-layer quasi-geostrophic models on a sphere. Critical amplitudes for incipient instability of baroclinic waves, growth rates, perturbation streamfunctions and momentum fluxes are obtained and are compared with the corresponding properties for barotropic waves in the nondivergent barotropic model. The change in the growth rates, perturbation streamfunctions, momentum and heat fluxes due to the superposition of basic planetary waves on zonal flow profiles is examined in two-layer quasi-geostrophic models. The presence of the long planetary waves is found to produce regions of preferential development of cyclones and anticyclones. The position of the most intense development, downstream from the long wave troughs or ridges depending on the basic flow profile, is shown to be related to Phillips’s criterion for incipient instability. It is also found that the largest of all the growth rates is increased in the presence of baroclinic waves but decreased in the presence of barotropic waves and it is concluded that baroclinic instability, rather than barotropic instability, is the most important factor in the unpredictability of large-scale atmospheric motions. Barotropic waves, however, produce larger changes in the disturbance streamfunctions, momentum and heat fluxes; in particular, when superimposed on solid body rotation zonal flow profiles, they may produce zonally averaged perturbation momentum fluxes with both poleward and equatorward components. The presence of planetary waves in the basic profile removes the short wave instability cutoff characteristic of two-layer quasi-geostrophic models with purely zonal flow basic profiles.

1. Introduction

By far the majority of studies of the stability or instability of atmospheric flows have dealt with basic flows which are steady, that is, independent of time. References to the literature are given, for example, in Baines and Frederiksen (1978), hereafter denoted BF. However, Lorenz (1972) studied the stability of time-dependent flow consisting of zonal flow and a Rossby wave in the nondivergent barotropic beta-plane model. He found that whereas zonal flow resembling the averaged tropospheric motion in mid-latitudes is usually barotropically stable, zonal flow with a superimposed Rossby wave may be unstable with respect to further perturbations. In particular, when the amplitude of the wave is sufficiently large or the wavenumber sufficiently high the flow is unstable, while for small amplitude or small wavenumber, the flow is stable due to the beta effect. He also found that the perturbation streamfunctions induced jet-like features in the Rossby wave motion and proposed that

(i) the prevalence of jet streams in the atmosphere is in part a manifestation of barotropic instability of Rossby waves; and
(ii) the barotropic instability of Rossby waves is largely responsible for the unpredictability of the atmosphere. This latter proposition was supported by Lilly (1973).

Further studies of the stability of Rossby waves in beta-plane barotropic models were carried out by Hoskins and Hollingsworth (1973), Gill (1974) and Duffy (1975). Gill found that when the local Rossby number is large, the nature of the instability is of Rayleigh type while when it is small the instability can be regarded as due to resonant wave interaction among a triad of barotropic Rossby waves. The stability properties of basic Rossby waves in spherical barotropic models were analysed by Hoskins (1973) and Baines (1976). They found that the finite size of the sphere and the discrete nature of the spectrum both tend to inhibit the instability. In fact, according to Hoskins's severely linearized theory, the example used by Lorenz is stable on the sphere. Baines further showed that all waves with total wavenumbers, \( \nu, \geq 3 \) are unstable if their amplitudes are sufficiently large and that the critical amplitude of the basic wave for incipient instability decreases with increasing \( \nu \).

The problem of the stability of basic pure baroclinic and mixed waves by themselves, and of these waves and pure barotropic waves together with zonal flow profiles having both vertical and horizontal shear, has received little attention. The stability of free pure baroclinic and pure barotropic waves in a two-layer beta-plane model has recently been examined by Yamagata (1976), but only in the limit of large local Rossby number. It seems however that the propositions (i) and (ii) of Lorenz should be studied in baroclinic models with basic states consisting of zonal flow and planetary waves of various types.

In this paper the stability properties of finite amplitude pure planetary waves and of combinations of planetary waves and zonal flows are analysed in idealized two-layer models on a sphere; both free and forced waves are considered. Since in two-layer models, a general planetary wave can be regarded as a linear combination of a pure barotropic and a pure baroclinic wave, we shall concentrate mainly on these two extreme cases in the basic states, although we do consider some free transient mixed waves and forced stationary upper layer waves. Here we are mainly interested in elucidating the effect of the barotropic and baroclinic components of basic planetary waves on the disturbance growth rate and phase speed spectra, streamfunctions and eddy heat and momentum fluxes. In future articles, the effect of waves forced by specific diabatic heating functions and topography will be examined in two- and multi-level models.

There are of course limitations to studies based on two-layer models as discussed for example by Lindzen, Batten and Kim (1968) and Frederiksen (1978). However, the qualitative results regarding the effect of planetary waves on growth rate spectra, etc., found here should provide at least a first approximation to the corresponding results for models of infinite and multi-layer atmospheres. We note that Geisler and Dickenson (1975) (see also Charney and Drazin 1961) have found that a number of different free neutral planetary waves, as well as forced waves, exist in models of infinite atmospheres. For realistic zonal wind profiles the free internal waves are trapped between the ground and the strong westerlies above.

In section 2 we examine the stability of basic states consisting of pure baroclinic waves in a spherical model used by Baer (1968, 1970) and which we denote the B model. For this model, baroclinic (and barotropic) waves, whose spatial variations are given by spherical harmonics, provide exact solutions of the free (frictionless and adiabatic) nonlinear equations. In this section we also examine some aspects of basic barotropic waves in the single-layer spherical nondivergent barotropic model.

Section 3 contains a study of the growth rates, perturbation streamfunctions, eddy momentum and heat fluxes in the B model for basic states consisting of solid body rotation and planetary waves. We consider both profiles containing forced stationary pure barotropic and pure baroclinic waves, and profiles containing free transient mixed waves.
The above properties of growth rates, etc., are examined in section 4 in the P model for basic states consisting of forced stationary planetary waves together with either solid body rotation or a jet profile. The P model, introduced by Lorenz (1960), formally extends the quasi-geostrophic two-layer equations to the whole of the sphere.

The conclusions are summarized in section 5 and details of the stability eigenvalue problems are given in the appendix. We use the same notation as in BF and appendix B of that article contains a list of important symbols.

2. BAROTROPIC AND BAROCLINIC PLANETARY WAVES

(a) The B model

The first model, with which we study the stability of basic baroclinic and barotropic planetary waves, is the two-layer B model which was discussed in detail in BF. The model has two layers, centred at 750 and 250 mb, and the dimensionless forms of the equations, obtained by taking \( a \) and \( \Omega^{-1} \) as length and time scales and \( a^2 \Omega^2 /bc_p \) as a temperature scale, are:

\[
\frac{\partial \nabla^2 \psi}{\partial t} = -J(\psi, \nabla^2 \psi + 2\mu) - J(\tau, \nabla^2 \tau) \quad . \quad (2.1a)
\]

\[
\frac{\partial \{(\nabla^2 - \Gamma) \tau\}}{\partial t} = -J(\psi, (\nabla^2 - \Gamma) \tau) - J(\tau, \nabla^2 \psi + 2\mu) \quad . \quad (2.1b)
\]

where

\[
\Gamma = 4\mu_0^2/\hat{\sigma} \quad . \quad (2.1c)
\]

Here the streamfunctions in the upper (\( \psi^1 \)) and lower (\( \psi^3 \)) layers are related to the average (\( \psi \)) and shear (\( \tau \)) streamfunctions through the relations:

\[
\psi = \frac{1}{4}(\psi^1 + \psi^3) \quad \text{and} \quad \tau = \frac{1}{4}(\psi^1 - \psi^3) \quad . \quad (2.2)
\]

The static stability parameter \( \hat{\sigma} \) (Eq. (3.10) of BF) measures the 250–750 mb potential temperature difference; \( \mu = \sin \phi \); \( \phi \) is latitude; and \( \mu_0 \) is the sine of a fixed latitude (usually chosen to be 45°). This model retains much of the analytical simplicity of Phillips's (1954) beta-plane model but has the additional advantage that no artificial boundary conditions are imposed; such artificial boundary conditions in the beta-plane model are known to affect the scale of the fastest growing modes (Stone 1969).

(b) The stability problem

In this section we shall be concerned with the stability of pure barotropic and baroclinic waves. Thus, for the basic states we take streamfunctions in the upper and lower layers

\[
\tilde{\psi}^j = -\text{Re}\{A_{\rho}^j P^j_0(\mu) \exp[i(\rho \lambda - \bar{\omega} t)]\}, \; j = 1, 3 \quad . \quad (2.3a)
\]

or, equivalently,

\[
\tilde{\psi} = -\text{Re}\{CP^0_0(\mu) \exp[i(\rho \lambda - \bar{\omega} t)]\} \quad . \quad . \quad . \quad (2.3b)
\]

\[
\tilde{\tau} = -\text{Re}\{DP^0_0(\mu) \exp[i(\rho \lambda - \bar{\omega} t)]\} \quad . \quad . \quad (2.3c)
\]

Here \( A_{\rho}^j \) (and \( C \) and \( D \)) are amplitude coefficients, \( P^j_0(\mu) \) are normalized Legendre functions (see Eq. (3.2) of BF); \( \lambda \) the longitude; \( t \) the time; \( \rho \) the zonal wavenumber; \( v \) the total wavenumber; and \( \bar{\omega} \) the angular frequency. Graphs of the associated Legendre functions are given in Jahnke and Emde (1945). As an example of these functions we show in Fig. 1 a contour map of the basic (dimensional) streamfunction \(-0.00473a^2\Omega P_0^1(\mu) \cos \lambda\), which we shall study in detail in subsequent sections. The amplitude coefficient 0-00473 has been chosen so that the maximum, taken over all latitudes, of the variance of the corresponding
zonal wind component is 32.5 m s\(^{-2}\); this is the value given, for the year, at 200 mb and
35°N in Table B1b of Oort and Rasmusson (1971).

Now the streamfunctions in Eqs. (2.3) provide exact solutions of the free nonlinear
B model equations (2.1) provided that
\[
either A^4_{pv} = A^3_{pv} \quad (D = 0) \quad \text{and} \quad \ddot{\omega} = -2\rho/(\nu(v+1)) \quad . \quad (2.4a)
\]
or
\[
a^4_{pv} = -A^3_{pv} \quad (C = 0) \quad \text{and} \quad \ddot{\omega} = -2\rho/(\nu(v+1)+\Gamma) \quad . \quad (2.4b)
\]
where
\[
C = \frac{1}{2}(A^4_{pv} + A^3_{pv}), \quad D = \frac{1}{2}(A^4_{pv} - A^3_{pv}) . \quad (2.4c)
\]
When Eq. (2.4a) or (2.4b) is satisfied the Jacobians involving products of basic fields in
Eqs. (2.1) vanish. The solution (2.4a) corresponds to a purely barotropic wave, while
(2.4b) corresponds to a purely baroclinic wave.

In order to consider the stability of the basic planetary waves to small disturbances, we
linearize Eqs. (2.1) by replacing each of the fields by a basic field (denoted by a bar), as in
Eqs. (2.3), and a perturbation field (which, for convenience, is denoted by the original
symbol). We retain in the equations for the perturbation fields only quantities which are first
order in the perturbation fields. As usual (cf. Eq. (A.2) of BF) we consider general dis-
turbances of the form
\[
\begin{pmatrix} \psi \\ \tau \end{pmatrix} = \text{Re} \sum_{a=1}^{\infty} \sum_{k=-n}^{n} \left( \psi_{kn}(\mu)P_n^m(\mu) \exp[i(k\lambda - \omega t)] \right) . \quad (2.5)
\]
Here \(\psi_{kn}\) and \(\tau_{kn}\) are amplitude coefficients.

(c) Triad interactions

Before embarking on the solution of the stability problem for general perturbation
streamfunctions we consider first the triad interaction case. In this approximation, the
stability of the basic field to disturbances for which only one amplitude coefficient appears
on the right hand side of Eq. (2.5) for each of the two perturbations, is considered.

We see from Eqs. (2.1) that there are three distinct triad interactions to consider: (i) a
basic \(\bar{\tau}\) field interacting with an \(\psi\) and a \(\tau\) perturbation component with all other \(\psi\) and \(\tau\)
coefficients zero; (ii) a basic \(\bar{\psi}\) field interacting with two \(\psi\) disturbance components; and
(iii) a basic \(\bar{\psi}\) field interacting with two \(\tau\) perturbation components.

Here we are mainly concerned with case (i), and we examine when the baroclinic wave
first becomes unstable. For the sake of symmetry, we denote the zonal and total wave-
numbers of the basic wave by \(m_3\) and \(n_3\) (\(p = m_3\), \(v = n_3\)), of the perturbation \(\tau\) component
by \(m_1\) and \(n_1\), and of the perturbation \(\psi\) component by \(m_2\) and \(n_2\). Further, the amplitude
coefficient of the basic $\bar{r}$ field is $D (= A^4_{24} = -A^2_{20})$ and for the perturbation $\tau$ and $\psi$ fields the amplitude coefficients are denoted $\tau_1$ and $\psi_2$ respectively. The equations for the perturbation fields are given by Eqs. (2.1), linearized about the basic field, and the equations for the amplitude coefficients are then obtained by multiplying by $P_{\nu}(\mu)e^{-im\lambda}$ and integrating over the surface of the sphere.

By considering the selection rules for the interaction coefficients (Eq. (A.5a) of BF) it may be seen that there are three possible subcases to study: (a) $m_3 = m_1 + m_2$, (b) $m_2 = m_1 + m_3$ and (c) $m_1 = m_2 + m_3$. Now, it happens that with $m_3 = |m_3| > 1$ the triad which first causes instability as $D$ is increased from zero has $m_1 = |m_1|$, $m_2 = -|m_2|$ (or $m_1 = -|m_1|$, $m_2 = |m_2|$) and satisfies $|m_3| = |m_1| + |m_2|$. The equations for this stability problem are

$$\omega d_1 \tau_1 + 2|m_1|[1-d_1/d_3]\tau_1 - \frac{1}{2}K(c_2-d_3)D\psi_2 = 0 \quad \text{(2.6a)}$$

$$\omega c_3 \psi_2 - 2|m_2|[1-c_2/d_3]\psi_2 - \frac{1}{2}K(d_1-d_3)D\tau_1 = 0 \quad \text{(2.6b)}$$

$$|m_3| |m_2| |m_1| \quad \text{(2.7a)}$$

where

$$K = K_{n_3 n_2 n_1} \quad \text{(2.7b)}$$

$$c_j = n_j(n_j+1), \quad j = 1, 2, 3 \quad \text{(2.7b)}$$

$$d_j = c_j + \Gamma \quad \text{(2.7c)}$$

The interaction coefficient in Eq. (2.7a) is defined in Eq. (A.5a) of BF.

The eigenvector–eigenvalue problem (2.6) has the solution

$$\omega = c_2^{-1}d_1^{-1}[c_2|m_1|(d_1/d_3-1)+d_1|m_2|(1-c_2/d_3)\pm$$

$$\pm \sqrt{[(c_2|m_1|(d_1/d_3-1)-d_1|m_2|(1-c_2/d_3))^2-\frac{1}{4}D^2(d_3-c_2)(d_3-d_1)c_2d_1K^2]]} \quad \text{(2.8)}$$

and the condition for the basic state to be unstable is that the term inside the square root must be negative. If we define the modified wave numbers $l_j$ by

$$l_j(l_j+1) = d_j, \quad j = 1, 2, 3 \quad \text{(2.9)}$$

then this implies that

$$l_1 < l_3 < n_2 \quad \text{or} \quad n_2 < l_3 < l_1 \quad \text{(2.10)}$$

and, of course, $n_2$ must be an integer. For growing disturbance modes, $\omega$ is the solution in Eq. (2.8) which has positive growth rate ($\omega = \omega_1 + i\omega_i; \omega_1 > 0$) and for a given amplitude coefficient $\tau_1$ (which may be chosen without loss of generality to be unity), $\psi_2$ may be obtained by solving Eq. (2.6a). This constitutes the solution of the general triad interaction stability problem.

Next, we consider when, as $D$ is increased from zero, a particular basic baroclinic wave first becomes unstable to one pair of the possible disturbance components making up the triad. To find the critical value, $D_c$, the equation obtained by setting the square root in Eq. (2.8) equal to zero must be solved for all $m_1, n_1, m_2, n_2$ subject to inequality (2.10). For example, taking the wave

$$\bar{r} = -\text{Re}\{DP_{\nu}(\mu)\exp(i4\lambda - \bar{\omega}t)\} \quad \text{(2.11)}$$

(proportional to that shown in Fig. 1, at $t = 0$) and a realistic atmospheric value of $\Gamma = 200$ (corresponding for example to $\bar{\sigma} = 0.01$ and $\mu_\beta = \frac{4}{3}$ as in BF) we find that $D_c = 2.088 \times 10^{-5}$. The corresponding perturbation components which first cause instability have $m_1 = 3, n_1 = 12(P_{12}^4)$ and $m_2 = 1, n_2 = 10(P_{10}^4)$. We shall find in the subsequent subsections that these perturbation components provide a good approximation to the disturbance streamfunctions for the complete stability problem just above the instability threshold.
In Table 1 we show the critical amplitude coefficients \( D_e \), for which the square root in Eq. (2.8) vanishes, for different basic baroclinic waves \( P_{\nu}^\nu \), and as well, the two components of the corresponding unstable triad; a realistic value of \( \Gamma = 200 \) has been chosen.

For case (ii), which is the same as the barotropic triad case discussed by Baines (1976) for the nondivergent barotropic model, the equations for triad interaction are again Eqs. (2.6) to (2.10) but with the replacements \( \tau_1 \rightarrow \psi_1, \Gamma \rightarrow 0 \) (i.e. \( l_j \rightarrow n_j, d_j \rightarrow c_j \)) and \( D \rightarrow C \). Thus, in the barotropic case \( l \) is replaced by \( n \) in Eq. (2.10) and this is consistent with Fjørtoft’s (1953) result for the nonlinear barotropic (triad) case that energy will flow into or out of the component whose total wavenumber, \( n \), is of intermediate value.

For case (i) in the B model, Baer (1970) has generalized Fjørtoft’s result for nonlinear triads to show that energy will flow into or out of the component whose modified wave-number is of intermediate value; that is, Eq. (2.10) must be satisfied for the case under consideration. An important point to notice about Eq. (2.10) is that, depending on the value of \( \Gamma \), it may be possible to have both \( n_1 \) and \( n_2 \) either less than or greater than \( n_3 \), unlike in the barotropic problem.

For the barotropic triad interaction case, the critical amplitudes of the vorticity coefficient are given in Table 3 of Baines (1976)*; the streamfunction amplitude coefficients may be obtained from these by dividing by \( v(v+1) \).

From the results in Table 1 and in Table 3 of Baines it may be seen that for given \( \rho \) and \( v \), the threshold for instability is in general lower for the baroclinic case and decreases with increasing \( \Gamma \) (or decreasing static stability parameter); (although not shown here, the critical amplitude coefficients for \( \Gamma = 400 \) were also calculated). There are, however, exceptions to this, notably the barotropic waves with \( \rho = 5, v = 7 \) and \( \rho = 9, v = 9 \) for which the critical amplitude coefficient is zero. Further, for given \( \rho \) and \( v \), as \( \Gamma \) increases the total wavenumber, \( n \), of the two perturbation components, increases. Taking the basic

* A misprint occurs in this table; for \( M = 2, N = 3 \) the two perturbation components of the corresponding triad are \( P_{1j}^2 \) and \( P_{1j}^3 \).
streamfunction $P_2^3$ as an example (cf. Eq. (2.11)) we note that for the barotropic case the critical streamfunction amplitude is $1.34 \times 10^{-3}$ and the two components are $P_2^1$ and $P_2^3$, while when $\Gamma = 200$ we have $2.088 \times 10^{-5}$ with $P_2^3$ and $P_1^1$ and when $\Gamma = 400$ we have $2.388 \times 10^{-6}$ with $P_1^1$ and $P_3^3$. We also note that the decrease in the critical amplitude with increasing $\nu$ shown in Table 3 of Baines for the vorticity amplitude (and which is even more evident for the streamfunction amplitudes), is not as noticeable or consistent in Table 1.

For case (iii) above, it is found that with $\Gamma = 200$ and, for the range of $\rho$ and $\nu$ considered in Table 1, the basic $\bar{\psi}$ field is stable to interactions with two $\tau$ components. The difference in the stability of the basic $\bar{\psi}$ and $\bar{\tau}$ fields is to be expected since the mean kinetic energy is small compared with the available potential energy (cf. Eq. (2.12)). In fact for $\nu < 14$ the available potential energy exceeds the mean kinetic energy, for the same amplitude coefficient of the $\bar{\psi}$ and $\bar{\tau}$ fields (cf. Baer 1970).

(d) Critical amplitude coefficients

Next, we study the stability of basic waves to general perturbations of the form (2.5). The complete eigenvalue problem is formulated in the appendix. We consider both basic baroclinic waves in the B model and basic barotropic waves in the single-layer non-divergent barotropic model. This latter model is given by Eq. (2.1a) with the Jacobian involving $\tau$ put to zero and the single layer is taken as the whole depth of the atmosphere.

For these eigenvalue problems, the size of the matrices required to produce convergence to the correct solutions may be reduced by using the selection rules depicted in Fig. 2 of Baines (1976). A further reduction may be achieved by noting that, in the stability problem for the basic planetary wave $P_2^3$, the equation for the perturbation component $\psi_1^2$ involves only the other components $\psi_{2m}^k$ for which $m = k + \rho$ (cf. Eq. (A.5)). Thus the subset of components for which $m = k \pm \rho$ where $r$ is a non-negative integer, forms a separate eigenvalue problem. The use of this additional selection rule, which effectively increases the resolution, (especially for the larger $\rho$) and our different truncation, may account for some of the discrepancies (discussed later) between some of the barotropic results presented here and those of Baines. Baines used an essentially triangular truncation in which 30 nonzero terms were kept while a rhomboidal truncation is used here with up to 60 components of the eigenvector retained in the barotropic problem, and up to 120 in the baroclinic problem. However, the critical amplitude coefficients for incipient instability were obtained using half these numbers of components. In general, as the amplitude of the basic planetary wave increases, and as $\Gamma$ increases, increasing numbers of components need to be retained to obtain convergence.

For the sake of reducing the number of cases to be considered, we have restricted our attention to basic planetary waves with antisymmetric streamfunctions and thus symmetric

<table>
<thead>
<tr>
<th>$P_0^s$</th>
<th>$s$</th>
<th>$k$</th>
<th>$D_e$</th>
<th>$D_e$ triad</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_2^4$</td>
<td>0</td>
<td>all</td>
<td>$7.95 \times 10^{-5}$</td>
<td>$8.19 \times 10^{-5}$</td>
</tr>
<tr>
<td>$P_3^5$</td>
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<td>odd</td>
<td>$9.40 \times 10^{-5}$</td>
<td>$7.35 \times 10^{-5}$</td>
</tr>
<tr>
<td>$P_3^5$</td>
<td>1</td>
<td>$1 \pm 3j$</td>
<td>$7.48 \times 10^{-5}$</td>
<td>$7.35 \times 10^{-5}$</td>
</tr>
<tr>
<td>$P_2^4$</td>
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<td>odd</td>
<td>$3.44 \times 10^{-5}$</td>
<td>$2.95 \times 10^{-5}$</td>
</tr>
<tr>
<td>$P_2^4$</td>
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<td>$1 \pm 4j$</td>
<td>$2.09 \times 10^{-5}$</td>
<td>$2.09 \times 10^{-5}$</td>
</tr>
</tbody>
</table>
zonal velocities, although both symmetric and antisymmetric perturbation streamfunctions will be used. A streamfunction $P_{l+2r+s}^s$, where $r$ is a non-negative integer, is antisymmetric (symmetric) if $s = 1$ ($s = 0$).

We find that the critical amplitude coefficients for the complete eigenvalue problem in the barotropic case agree with those of Baines. In terms of the streamfunction amplitude, $C$, they are 0.00410 for $P_1^4$, 0.0335 for $P_2^4$, 0.01080 for $P_3^4$ and 0.00134 for $P_4^4$. In addition we have calculated the critical amplitude for $P_2^4$ which is 0.00103 and, as for $P_3^4$, $P_4^3$ and $P_5^4$, it agrees well with the triad critical amplitude.

For the B model the critical amplitudes for baroclinic waves are shown in Table 2 and are compared with the corresponding triad values. The agreement with the triad values is again quite good, particularly for $P_3^4$ and $P_4^4$. Note also that $P_1^1$ is no longer completely stable as it is in the barotropic model.

Figures 2(a) and (b). Nondimensional growth rates, $\omega_1$, of the fastest growing disturbances and of the disturbances which first become unstable for the case of basic barotropic waves $P_{l+2r+s}^s$. The abscissa shows the ratio of the amplitudes of the basic wave to the critical amplitudes of section 2(d). The disturbance streamfunctions are symmetric (antisymmetric) if $s = 0$ ($s = 1$). Also shown are the zonal wavenumbers, $k$, of the subsystem forming the most destabilizing disturbance; $j$ is a positive integer.
(e) Growth rates

In Figs. 2(a) and (b) are shown, for the barotropic model, the growth rates of the fastest growing modes, and of the modes which first become unstable, for increasing values of the amplitude coefficient of the basic wave. Comparing Fig. 2 with Fig. 3† of Baines, we see that the main discrepancy is in the growth rate of \( P_3^2 \) interacting with antisymmetric perturbation streamfunctions of odd zonal wavenumber \( k \). Why this discrepancy should occur is unclear; it would seem, however, that with an amplitude coefficient of only two or three times the critical amplitude, the stability problem should still be dominated by a triad and that the growth rate should therefore be monotonically increasing. The results in Fig. 2(b) are largely of interest only in so far as they show that waves with \( \nu \geq 3 \) are unstable if their amplitudes are sufficiently large, since the amplitudes shown are considerably larger than those typical of atmospheric planetary waves. For \( P_3^4 \), the range of amplitude coefficients shown in Fig. 2(a) would seem to be reasonable on the basis of the observed results (cf. Eliassen 1958; Eliassen and Machenhauer 1969; van Loon and Jenne 1972; Oort and Rasmusson 1971).

![Figure 3](image_url)

Figure 3. Growth rates, \( \omega_n \), of the fastest growing disturbances corresponding to the basic baroclinic waves \( P_s^p \) whose critical amplitudes appear in Table 2. The abscissa is the amplitude, \( D \), of the basic wave in nondimensional units. Also shown for the basic wave \( P_1^4 \) is the contribution to the modulus of the disturbance streamfunction in the upper (and lower) layer from the dominant zonal wavenumber \( k^* = 1 \), plotted as a function of latitude for different values of \( D \).

† A misprint occurs in this figure; ‘\( P_1^2 \rightarrow P_1^4 \) + zonal flow’ should be ‘\( P_1^2 \rightarrow P_1^4 \rightarrow P_4^1 + \text{zonal flow} \)’. 
We show in Fig. 3 the growth rates of the fastest growing modes in the B model for the basic baroclinic planetary waves depicted in Table 2, for increasing values of the amplitude coefficient $D$. As well as having a much smaller critical amplitude than the barotropic waves, the growth rate for a given amplitude coefficient is much larger. In particular, for $P_3^*$ when the amplitude coefficient is 0.00473, as in Fig. 1, the growth rate is 0.0118 in the barotropic case and 0.0738 in the baroclinic case, with $\Gamma = 200$.

These results seem to cast doubt on Lorenz's (1972) suggestion that barotropic instability is the most important immediate factor in the unpredictability of large-scale atmospheric flow. (We have, of course, not excluded the possibility that for larger values of the zonal and total wavenumbers than considered here barotropic growth rates may be considerably larger. Indeed the growth rate seems to increase with increasing $\nu$, as discussed in section 1.)

(f) Streamfunctions

Unlike the case of a purely zonal basic flow profile, in the case of the stability problem for basic planetary waves, the fastest growing perturbation mode does not have a single zonal wavenumber (cf. Eq. (A.5)). It is therefore of interest to examine the distribution of the total energy as a function of the zonal wavenumber for the fastest growing mode. In the barotropic model, the total energy in a given zonal wavenumber is just the mean kinetic energy

$$K_M(k) = \pi \sum_{n=|k|}^{\infty} n(n+1)\psi_{kn}\psi_{kn}^*$$

(2.12a)

while for the B model it consists of the mean kinetic energy and as well the shear kinetic energy and available potential energy, which are given by

$$K_s(k) = \pi \sum_{n=|k|}^{\infty} n(n+1)\tau_{kn}\tau_{kn}^*$$

(2.12b)

$$A(k) \equiv \pi \sum_{n=|k|}^{\infty} \{ (1/\sigma)\theta_{kn}\theta_{kn}^* \}$$

(2.12c)

$$= \pi \sum_{n=|k|}^{\infty} \Gamma_{kn}\tau_{kn}\tau_{kn}^*$$

(2.12d)

Here $\theta_{kn}$ are the spectral components of the potential temperature. The last equality, (2.12d), is valid in the B model where $\theta = 2\mu_0\tau$; for the P model, studied in section 4, only (2.12c) is valid.

Tables 3a and b show the energy spectrum of the fastest growing perturbation modes for the case of the basic barotropic and baroclinic waves $P_3^*$ with amplitude coefficient 0.00473 (as shown in Fig. 1) in the barotropic and B models respectively. We notice that in the barotropic model, by far the majority of the energy resides in zonal wavenumbers 1 and -3 (and their conjugates 1 and 3) with the dominant zonal wavenumber $k^*$ (the wavenumber for which the energy is a maximum) being 1. Further, most of the contribution to the energy at $k = 1$ comes from $n = 3$ and at $k = -3$ from $n = 7$, as in the case of triad interaction. In fact, for the range of values of the amplitude considered in Fig. 2, the streamfunction of the most unstable mode is dominated by these two components of the triad. A similar situation occurs for the other basic waves in Fig. 2 with the notable exception of $P_1^*$ for which the dominant contributions come from $P_3^*$ and a superposition of zonal flow modes.

Table 3b shows that, in contrast to the barotropic case, there is quite a significant spread of energy to high zonal wavenumbers in the B model, but with the dominant zonal
TABLE 3. ENERGY SPECTRUM OF THE MOST UNSTABLE DISTURBANCE FOR THE BASIC WAVE $P_3$ WITH AMPLITUDE $0.00473$ IN THE CASE OF (a) A BASIC BAROTROPIC WAVE IN THE BAROTROPIC MODEL AND (b) A BASIC BAROCLINIC WAVE IN THE B MODEL WHERE $\Gamma = 200$. THE ENERGY IN A GIVEN ZONAL WAVE NUMBER IS SHOWN AS A PERCENTAGE OF THE ENERGY IN THE DOMINANT ZONAL WAVE NUMBER WHICH IN BOTH CASES IS $k^* = 1$; THE DISTURBANCE STREAMFUNCTIONS ARE SYMMETRIC (ANTISYMMETRIC) IF $s = 0$ ($s = 1$)

<table>
<thead>
<tr>
<th>$P_3$ barotropic ($s = 0$)</th>
<th>$C = 0.00473$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k^*$</td>
<td>$\omega_1$</td>
</tr>
<tr>
<td>1</td>
<td>0.0118</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$P_3$ baroclinic ($s = 1$)</th>
<th>$D = 0.00473$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k^*$</td>
<td>$\omega_1$</td>
</tr>
<tr>
<td>1</td>
<td>0.0738</td>
</tr>
</tbody>
</table>

wavenumber $k^*$ again being 1. Further, the contribution to the total energy at zonal wavenumber 1 is not dominated by one particular value of $n$ when $D = 0.00473$. This is illustrated in Fig. 3 where the contribution to the modulus of the disturbance streamfunction in the upper (and lower) layer from $k = 1$ is graphed for various values of $D$ for the basic baroclinic wave $P_3$. Near the instability threshold, this contribution has the same shape as the modulus of $P_{10}$. In fact, just above the instability threshold the average and shear streamfunctions are dominated by the triad components $P_{12}$ and $P_{10}$ of subsection (c), both of which have largest amplitude near the pole. As $D$ increases, the area under the curve in Fig. 3 begins to fill in until at $D = 0.00473$ the number of turning points has decreased and the structure is quite different from at threshold, with a maximum in mid-latitudes. This qualitative behaviour is quite similar to that shown in Fig. 2 of BF where the structure of the modulus of the fastest growing mode in the upper layer of the P model is studied, for increasing amplitudes of the basic zonal flow streamfunction $P_1$.

For the other basic waves whose largest growth rates are shown in Fig. 3, the perturbation modes have a similar qualitative behaviour as described for the case of $P_3$.

(g) **Momentum Fluxes**

As mentioned in section 1, Lorenz (1972) proposed that basic barotropic waves superimposed on quite general zonal flows will, through instability, produce momentum fluxes with both poleward and equatorward components which then produce or reinforce a jet structure. This hypothesis will be tested in subsequent sections. It is, however, of interest to examine the perturbation momentum flux due to pure basic baroclinic and barotropic waves. With the nondivergent contributions to the zonal and meridional velocities defined by

$$u = -(1 - \mu)^{\pm} \partial \psi / \partial \mu, \quad v = (1 - \mu^2)^{\pm} \partial \psi / \partial \lambda,$$

the zonally averaged momentum flux is given by

$$\overline{uv} = (1/2\pi) \int_0^{2\pi} u v d\lambda.$$

It is found that the momentum fluxes, produced by the phase variations of the perturbation streamfunctions, show a considerable variability depending on the particular basic
wave and on its amplitude, both in the barotropic and baroclinic models. This variability, in the baroclinic case, might be expected on the basis of the results in BF where a similar sensitivity was found for different zonal flow profiles.

Fig. 4 shows the zonally averaged eddy momentum fluxes for the basic barotropic and baroclinic waves $P_4^4$ with amplitude coefficient of 0.00473. In both cases the fluxes have poleward and equatorward components.

3. ZONAL FLOW AND PLANETARY WAVES IN THE $P_4$ MODEL

(a) Basic profiles

In this section we consider the stability of combinations of zonal flow and planetary waves in the B model. We shall be mainly concerned with profiles consisting of stationary forced barotropic and baroclinic planetary waves superimposed on solid body rotation. That is, we take as basic profiles average and shear streamfunctions:

$$\bar{\psi} = -A_1 P_2^1(\mu) - \text{Re}\{CP_2^1(\mu) \exp(i \rho \lambda - \bar{\omega} t)\} \quad \cdots \quad (3.1a)$$

$$\bar{\tau} = -B_1 P_2^0(\mu) - \text{Re}\{DP_2^0(\mu) \exp(i \rho \lambda - \bar{\omega} t)\} \quad \cdots \quad (3.1b)$$

where $A_1$ and $B_1$ are related to the amplitude coefficients for the corresponding upper and lower layer streamfunctions through

$$A_1 = \frac{1}{2}(A_{01}^0 + A_{01}^0), \quad B_1 = \frac{1}{2}(A_{01}^1 - A_{01}^1) \quad \cdots \quad (3.1c)$$

and $C$ and $D$ satisfy Eq. (2.4c). For stationary waves $\bar{\omega} = 0$ and for pure barotropic (pure baroclinic) waves $D = 0$ ($C = 0$). As discussed in the appendix, basic states consisting of zonal flow and single planetary waves will in general only be solutions of the nonlinear equations in the presence of suitable forcing. Here, the effect of the forcing on the perturba-
tions is neglected and the conditions under which this is valid are discussed in the appendix where the eigenvalue stability equations are also formulated.

We shall also examine the stability of exact solutions of the free nonlinear equations, which are provided by basic profiles consisting of solid body rotation and planetary waves. For Eqs. (3.1) to provide a solution of the free B model equations (2.1) the following equations must be satisfied:

\[ \bar{\omega} \nu(v+1)C = \rho C \left[ -2 - \alpha(2 - \nu(v+1)) \right] - \rho D \left[ 2 - \nu(v+1) \right] \beta \]  
\[ \bar{\omega} \nu(v+1) + \Gamma \right] D = \rho D \left[ -2 - \alpha(2 - \nu(v+1)) \right] - \rho C \left[ 2 - \nu(v+1) \right] \beta + \rho \Gamma (D \alpha - C \beta) \]  

where \( \alpha = \sqrt{(\frac{3}{4})} A_1, \beta = \sqrt{(\frac{3}{4})} B_1. \) Note that the Jacobians involving products of planetary waves vanish in Eqs. (2.1). Thus, the nonlinear equations satisfied by the finite amplitude waves in Eqs. (3.1) are formally the same as the usual perturbation equations obtained by linearizing Eqs. (2.1) about the zonal flow profile in Eqs. (3.1) (the same applies for the corresponding multi-level and continuous B models and beta-plane models).

Now letting \( \kappa = \nu(v+1) \) and \( \Delta = D/C \) (when \( C \neq 0 \)), it follows that for given \( \alpha \) and \( \beta \), \( \Delta \) and \( \bar{\omega} \) must satisfy the equations

\[ \Delta = \Delta_\pm \equiv \frac{\Gamma(1 + \alpha) \pm \sqrt{[\Gamma(1 + \alpha)]^2 - \beta^2(\kappa - 2)(\kappa + \Gamma)\kappa(\kappa - \Gamma - 2)}}{\beta(\kappa - 2)(\kappa + \Gamma)} \]  
\[ \bar{\omega} = \bar{\omega}_\pm \equiv \rho \kappa^{-1} \left[ -2 + (\kappa - 2)\alpha + \Delta_\pm(\kappa - 2)\beta \right] \]  

and \( \Delta_\pm \) are real if and only if \( \Delta_\pm \) are. (Note that Eq. (3.4) is just Eq. (3.14) of BF.)

We shall be principally interested in the case \( \alpha = \beta \) corresponding to zero zonal flow in the lower layer and then the condition for the vanishing of the square root in Eq. (3.3) is

\[ \alpha = \alpha_0 \equiv \Gamma \left[ \sqrt{(\kappa - 2)(\kappa + \Gamma)\kappa(\Gamma + 2 - \kappa)} \right]^{-1} \]  

Further, we take \( \alpha = \beta = 0.03 \) so that the zonal velocity in the upper layer corresponds to profile 1 in Fig. 5 of BF. That is, for profile 1,

\[ A_{01}^1 = 0.0490, \quad A_{01}^3 = 0.0 \]  

Then, with \( \Gamma = 200 \) we find that \( \alpha_0 \) is greater than 0.03 for \( 1 < \nu \leq 5 \) but less than 0.03 when \( 6 < \nu \leq 13 \) while for \( \nu \geq 14 \) the square root in Eq. (3.4) is always real. In particular for \( 1 < \nu \leq 5 \), \( \Delta_\pm \) are real, and for a given amplitude coefficient \( A_{pr}^1 \), the upper layer, the coefficient in the lower layer is given by

\[ A_{pr}^3 = A_{pr}^1 (1 - \Delta)/(1 + \Delta) \]  

where \( \Delta = \Delta_- \) corresponds to the wave with the larger barotropic component of the two, denoted by (−) and \( \Delta = \Delta_+ \) corresponds to that with the larger baroclinic component, denoted by (+).

There is considerable variability in the amplitude and nature of atmospheric planetary waves (cf. Eliassen 1958; van Loon and Jenne 1972). To be specific, we shall, however, choose the values of the amplitude coefficients in the upper layer, for both stationary and transient waves, based on the 200 mb values of the variance of the zonal wind component, resulting from stationary eddies for the year, \([\bar{u}^2]^3\), given in Table B1b of Oort and Rasmusson (1971). We shall, in the following two sections, restrict our attention mainly to \( P_3^0 \) and to a lesser extent to \( P_2^0 \), with a brief mention of the corresponding results for \( P_3^1 \) and \( P_2^1 \). There are a number of reasons for this selection of waves. First, as discussed by Palmén and Newton (1969), section 6.4, although the mean winter chart shows a predominance of waves with zonal wavenumber 3, this is a statistical result of the fact that the long-wave
troughs are most often particularly well developed at certain longitudes, and in fact there are most often four or five waves around the northern hemisphere. A further reason for concentrating on $P_4^1$ is that the size of the matrix needed to ensure convergence of the stability eigenvalue problem is considerably reduced for the higher values of the zonal wavenumbers of the basic wave, as discussed in section 2.

Now, normalizing the amplitude coefficients of the basic waves in the upper layer so that the maximum, taken over all longitudes, of the variance of the zonal wind component is equal to the maximum of $[u^2]$ at 200 mb, for the year (i.e. 32.5 m$^2$s$^{-2}$), we find that

$$A_{15}^1 = 0.00473, \quad A_{24}^1 = 0.00555 \quad (3.8a)$$

For the pure barotropic and pure baroclinic waves we then have:

pure barotropic \hspace{1cm} A_{pv}^3 = A_{pv}^1

pure baroclinic \hspace{1cm} A_{pv}^2 = -A_{pv}^1 \quad (3.8b)

while for the mixed (−) waves

$$A_{25}^{-} = 0.00163, \quad A_{34}^{-} = 0.00306 \quad (3.8c)$$

and for the mixed (+) waves

$$A_{25}^{+} = -0.00115, \quad A_{34}^{+} = -0.00290. \quad (3.8d)$$

For the mixed (+) waves, the angular frequency, Eq. (3.4), is small and so they may also be regarded as forced quasi-stationary waves.

In Fig. 5 we show the difference between the zonal velocities in the upper and lower layers, $\bar{u}_1 - \bar{u}_3$, when $\cos \rho \lambda = 0, \pm 1$ for profile 1 and for this profile together with the pure

![Figure 5](image_url)

Figure 5. The difference of the basic zonal velocities in the upper and lower layers, $\bar{u}_1 - \bar{u}_3$, for profile 1, (1), and this profile together with the following basic waves: the baroclinic wave $P_4^1$, (A); the upper layer wave $P_4^1$, (B); the baroclinic wave $P_4^b$, (C); the upper layer wave $P_4^b$, (D). The signs ± refer to $\cos \rho \lambda = \pm 1$ respectively. Also shown are Phillips's stability criterion in the P model (P) and in the B model (PB).
baroclinic and pure barotropic waves discussed above (and, as well, upper layer waves studied in section 4). The presence of barotropic waves does not of course change the vertical shear. Also shown is Phillips's (1954) (heuristic) criterion for incipient instability in the B model:

\[ \bar{u}_1 - \bar{u}_3 = \bar{a} a \Omega \cos \phi / \sin^2 \phi_0 \]  \hspace{1cm} (3.9)

For solid body rotation, Eq. (3.9) may, for suitable $\bar{a}$, be an exact result in the B model. To see this we note that the minimum value of $a_\kappa$ in Eq. (3.5) is $\bar{a}/2\sin^2 \phi_0$ and occurs when $\kappa(\kappa - 2) = \frac{1}{4} (\Gamma^2 + 2\Gamma)$. However, since $\kappa = \nu(\nu + 1)$, where $\nu$ is an integer, the above equality can only be satisfied exactly for suitable values of $\Gamma$ and in general Eq. (3.9) is an approximation. It differs from Phillips's heuristic criterion for the P model (Eq. (4.4)) in that $\sin^2 \phi_0$ replaces $\sin^2 \phi$ since the Coriolis parameter varies only in the advective terms. For the cases involving $P^4_3$, the streamfunctions (3.1) may be constructed by adding to the contour map in Fig. 1 a term proportional to $\sin \phi$ (or $P^4_3(\mu)$).

For the remainder of this article we shall, as in BF, restrict our attention to the even perturbation modes (antisymmetric streamfunctions).

(b) Growth rates

We show in Fig. 6 the growth rates of the fastest growing even waves for profile 1 and, as well, the corresponding growth rates for this profile together with the pure barotropic, pure baroclinic and mixed waves for both $P^2_3$ and $P^4_3$. The growth rates have been associated with a dominant zonal wavenumber $k^*$ for the subsystem concerned, where, as in section 2, the dominant zonal wavenumber corresponds to the maximum total energy. For profile 1

![Figure 6](image)

Figure 6. Growth rates as a function of $k$ for the fastest growing even modes in the B model for profile 1 (solid line) and this profile plus the pure barotropic wave $P^4_3(\Theta)$; the pure baroclinic wave $P^4_3(\ast)$; the pure barotropic wave $P^4_3(\Delta)$; the pure baroclinic wave $P^4_3(\bigtriangledown)$; the mixed ($+$) wave $P^4_3(+)$; the mixed ($-$) wave $P^4_3(\times)$; the mixed ($+$) wave $P^4_3(\diamond)$; the mixed ($-$) wave $P^4_3(\odot)$.

and each of the pure barotropic and baroclinic waves $P^4_3$, we show, in Table 4, the energy spectrum as a function of zonal wavenumber and the growth rates of the fastest growing modes, for the subsystem of zonal wavenumbers coupled to $k = 9$. The corresponding results for the wave $P^4_3$ are similar, but the coupling between the different wavenumbers is somewhat stronger. The number of zonal wavenumbers $k$ retained in the truncation is as shown in Table 4 and has been chosen to obtain convergence for the fastest growing mode; the corresponding largest total wavenumber is $|k| + 23$. As far as the fastest growing modes are concerned there is little coupling between the modes of positive and negative zonal wavenumbers.
TABLE 4. Growth rates $\omega_k$ and total energy spectrum of fastest growing disturbances for profile 1 plus each of the pure baroclinic and pure barotropic waves $P_k^s$ in the B model. The energy is expressed as a percentage of the total energy of the dominant zonal wave number $k^*$

<table>
<thead>
<tr>
<th>$k^*$</th>
<th>$\omega_k$</th>
<th>$k = 1$</th>
<th>5</th>
<th>9</th>
<th>13</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>0.119</td>
<td>1.5%</td>
<td>25.6%</td>
<td>100%</td>
<td>2.3%</td>
<td>0.17%</td>
</tr>
</tbody>
</table>

Profile 1 + $P_1^s$ barotropic wave ($k^* = 9, s = 1$)

<table>
<thead>
<tr>
<th>$k^*$</th>
<th>$\omega_k$</th>
<th>$k = 1$</th>
<th>5</th>
<th>9</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>0.143</td>
<td>0.03%</td>
<td>0.71%</td>
<td>15.4%</td>
<td>100%</td>
</tr>
</tbody>
</table>

We see from Fig. 6 (and Table 4) that both the pure baroclinic waves and the mixed (,) waves increase the growth rates of the fastest growing waves of intermediate dominant zonal wavenumber ($k^*$ ranging from 7 to 10), relative to that for solid body rotation, while the pure barotropic waves do the reverse. The mixed (,) waves have little effect on the growth rates. Note also that the coupling between the various zonal wavenumbers as evidenced by the energy spectrum is not in general as strong for the pure baroclinic wave cases as for the cases involving pure barotropic waves. Similarly, the (,) waves which cause a larger change in the growth rate are associated with stronger coupling than the (,) waves.

Table 4 shows that for the pure barotropic wave $P_1^s$ plus zonal flow, the growth rate at $k^* = 9$ is 0.119 and the energy at $k = 5$ in this mode is 25.6% of the maximum. For the pure barotropic wave $P_1^s$ plus zonal flow, the corresponding growth rate at $k^* = 9$ is 0.112 and the energy at $k = 6$ is 47.0% of that at $k = 9$. Since these growth rates are greater than that of the fastest growing mode with $k = 5$ and 6, respectively, for solid body rotation, this means that the growth rates at $k = 5$ and 6, respectively, are effectively increased by the presence of the barotropic waves.

From Table 4 it may also be seen that although the contribution to the various modes from the large zonal wavenumbers is small, it is nevertheless nonzero, and thus the presence of the pure baroclinic and pure barotropic waves removes the sharp short-wave cutoff that is characteristic of two-layer quasi-geostrophic models in which the basic profile is pure zonal flow. The same may be said of the (,) and (,) waves.

For solid body rotation and the planetary waves $P_1^2$ and $P_2^2$ (normalized as described in subsection (a)) our results confirmed the above qualitative conclusions, although it did not appear that the solutions had completely converged.

An explanation of why the barotropic and baroclinic waves should alter the growth rates in different ways may be proposed as follows: A pure baroclinic wave produces regions of increased vertical shear as well as regions of decreased vertical shear. Now, it is well known that increased vertical shear increases the growth rates of the fastest growing waves (Green 1960). Moreover, as discussed in BF, the regions of increased vertical shear are the regions where the perturbation waves are growing to their maximum amplitude and these regions are expected to have the largest effect on the growth rate. This effect is evidently the dominant one for the pure baroclinic and mixed (,) waves. In contrast, a pure barotropic wave does not alter the vertical shear. It does, however, produce a coupling of the perturbation streamfunctions of different zonal wavenumbers and a flow of energy into the shorter and longer wavelengths, at the expense of the intermediate wavelengths, with a subsequent reduction in the growth rate of the fastest growing wave. For the pure barotropic waves this is the dominant effect while for the mixed (,) waves the two effects very nearly balance.
(c) *Streamfunctions, momentum and heat fluxes*

In the case of pure solid body rotation the fastest growing mode occurs at $k = 9$ and for the even mode the perturbation streamfunctions in the upper and lower layers may be normalized to

\[
\psi^1 = \text{Re}\{P_{20}^2(\mu) \exp(i(\lambda - \omega t))\} \quad \quad \quad (3.10a)
\]

\[
\psi^3 = \text{Re}\{(0.4075 - 0.5958)P_{10}^0(\mu) \exp(i(\lambda - \omega t))\} \quad (3.10b)
\]

When the basic planetary waves $P_k^4$ and $P_2^2$ are also present, the contribution to the modulus of the perturbation streamfunction at the dominant zonal wavenumber 9 does not change greatly. We do however notice that the ratio of the moduli of the streamfunctions in the lower layer to that in the upper layer is slightly bigger for the pure baroclinic basic wave compared with the situation for the pure barotropic basic wave. Further, the pure barotropic basic wave produces a latitudinal spreading of the modulus of the perturbation streamfunction, reflecting the stronger coupling.

![Figure 7](image1.png)

*Figure 7.* Disturbance streamfunction $\psi^1$ in the upper layer of the B model for the fastest growing mode with $k^* = 9$ and for a basic flow consisting of profile 1 plus the pure baroclinic wave $P_k^4$. The streamfunction satisfies the relationship $\psi^1(\lambda + \pi, \mu) = -\psi^1(\lambda, \mu)$. Units shown correspond to $10^4 m^2 s^{-1}$ as in Fig. 1.

Figure 7 shows the fastest growing perturbation streamfunction in the upper layer, for solid body rotation and the pure baroclinic wave $P_k^4$. We notice that the zonal wavenumber 4, of the basic wave, modulates the primarily zonal wavenumber 9 pattern. This is very similar to the phenomenon of beats in acoustics (see Sears and Zemansky 1963) with contributions from wavenumbers 13 and 5 combining to form a beat wavenumber $\frac{4}{5}(13 - 5)$ modulating wavenumber $\frac{4}{5}(13 + 5)$. The structure of the mode shown in Fig. 7 may, to a large extent, be related to the positions where the excess shear, the difference between

![Figure 8](image2.png)

*Figure 8.* As Fig. 7 for profile 1 plus the pure barotropic wave $P_k^4$. 
the vertical shear and Phillips's criterion (for the B model) is a maximum. From Figs. 5 and 7 we notice that slightly downstream of the position where the excess shear is a maximum \((\cos \rho \lambda = 1)\) corresponding to the trough in the basic shear streamfunction, the disturbance lows and highs are most intense.

With the pure barotropic wave \(P^4_5\) present (and with the \((+\)) and \((-\)) waves) the wavenumber 4 is also evident in the perturbation streamfunction pattern of the fastest growing mode as shown in Fig. 8; we leave a discussion of the differences produced by barotropic and baroclinic waves to section 4 where these are discussed in detail for the P model. Again, zonal wavenumber 3 modulates the perturbation streamfunction pattern when the basic waves \(P^4_3\) appear in the basic profile.

From Fig. 8 it will also be seen that there are regions where the phase of the streamfunction changes in such a way as to produce both equatorward and poleward components of the momentum flux. This is shown in detail in Fig. 9 where the zonally averaged perturbation momentum fluxes in the upper and lower layers, \(\overline{u_1v_1}\) and \(\overline{u_3v_3}\), for \(k^* = 9\) are graphed. The four cases involving \(P^4_5\) and the fluxes for zonal flow and the pure barotropic and

Figure 9. Momentum fluxes in the upper layer \(\overline{u_1v_1}\) (solid curves) and the lower layer \(\overline{u_3v_3}\) (broken curves) in the B model at \(k^* = 9\) for profile 1 plus each of the pure barotropic wave \(P^4_5(A)\); the pure baroclinic wave \(P^3_4(B)\); the pure barotropic wave \(P^4_5(C)\); the pure baroclinic wave \(P^4_2(D)\); the mixed \((+\)) wave \(P^4_5(E)\); and the mixed \((-\)) wave \(P^4_5(F)\). The fluxes in the upper layer have been normalized to unity.
baroclinic waves $P_3^2$ are shown. We note that the pure barotropic and (—) waves produce momentum fluxes with poleward and small equatorward components (the same is true of the (—) wave $P_3^1$) while the pure baroclinic waves yield only equatorward fluxes. The (—) wave $P_3^2$ combines the properties of both barotropic and baroclinic waves with both poleward and equatorward fluxes while the (—) wave $P_3^1$ (not shown), being relatively more baroclinic, has nearly purely equatorward fluxes. The momentum fluxes in Fig. 9 have been normalized such that their maximum absolute values in the upper layer is unity. However, normalizing the eigenvectors in Eq. (A.5) such that the sum of their moduli squared equals unity, the maximum values of the momentum fluxes in the upper layers are approximately 1.46 and 1.74 times larger in the pure barotropic cases compared with the pure baroclinic cases for $P_3^2$ and $P_3^1$ respectively.

These results are consistent with Lorenz’s proposition that barotropic instability of planetary waves is an important mechanism in determining the eddy momentum fluxes and may be partly responsible for the prevalence of jet streams in the atmosphere.

Since $\theta = 2\mu_0 \tau$ in the B model, the potential temperature flux $\frac{1}{2} [\theta \Phi + \Phi \theta] \varepsilon$ at $k = 9$ is easily obtained from Eqs. (3.10) and (2.13) for solid body rotation. When the pure baroclinic waves are present, there is little change in this (normalized) flux while with the pure barotropic waves the main effect is a latitudinal spread reflecting the corresponding spread in the streamfunction.

4. ZONAL FLOW AND PLANETARY WAVES IN THE P MODEL

(a) The P model

With the same scaling and two-layer structure as described for the B model in section 2, the dimensionless forms of the P model equations (discussed in detail in BF) are

$$\frac{\partial \theta}{\partial t} = -J(\psi, \tau) + \bar{\sigma} \nabla^2 \chi$$  \hspace{1cm} (4.1a)

$$\frac{\partial \nabla^2 \psi}{\partial t} = -J(\theta, \nabla^2 \psi + 2\mu) - J(\tau, \nabla^2 \tau)$$  \hspace{1cm} (4.1b)

$$\frac{\partial \nabla^2 \tau}{\partial t} = -J(\psi, \nabla^2 \tau) - J(\tau, \nabla^2 \psi + 2\mu) + \nabla 2\mu \nabla \chi$$  \hspace{1cm} (4.1c)

$$\nabla^2 \theta = \nabla 2\mu \nabla \tau$$  \hspace{1cm} (4.1d)

where $\theta$ is the potential temperature and $\chi$ is the velocity potential in the lower layer. These equations formally extend quasi-geostrophic theory to the whole sphere and thus provide a more realistic model than either the B model or the beta-plane model.

For the zonal flow contribution to the basic state we consider profile 1 in Eq. (3.6) and the jet profile 2 of BF which is defined by

$$A_{01}^1 = 0.01372, \quad A_{03}^1 = 0.00628,$$

$$A_{05}^1 = -0.00358, \quad A_{03}^3 = A_{03}^5 = 0$$  \hspace{1cm} (4.2)

For the basic planetary waves we again concentrate on $P_3^2$ and $P_3^1$ and take the amplitude coefficients in the upper layer as given in Eq. (3.8a) and $\bar{\sigma} = 0.01$ as in BF. We consider stationary forced pure barotropic, pure baroclinic and pure upper layer waves for which

$$\bar{\omega} = 0$$  \hspace{1cm} (4.3a)

pure barotropic

$$A_{\mu \nu}^3 = A_{\mu \nu}^1$$  \hspace{1cm} (4.3b)

pure baroclinic

$$A_{\mu \nu}^3 = -A_{\mu \nu}^1$$  \hspace{1cm} (4.3c)

pure upper layer

$$A_{\mu \nu}^3 = 0$$  \hspace{1cm} (4.3d)
Figure 10. Contour map of the (dimensional) streamfunction in the upper layer for profile 2 and the planetary wave $P_4^4$. The units shown correspond to $10^4 \text{m}^2\text{s}^{-1}$.

The corresponding potential temperature fields are obtainable from Eq. (A.1b).

Fig. 10 shows the (dimensional) basic streamfunction in the upper layer for profile 2 together with the planetary wave $P_4^4$; notice the resemblance to the schematic four-wave pattern in Fig. 6.5 of Palmén and Newton (1969). Shown in Fig. 11 are the differences of the zonal velocities in the upper and lower layers, $\bar{u}_4 - \bar{u}_3$, when $\cos \rho_1 = 0, \pm 1$ for each of the cases mentioned above. Also shown is Phillips's criterion for incipient instability

$$\bar{u}_4 - \bar{u}_3 = \bar{\sigma} a \Omega \cos \phi / \sin^2 \phi$$

in the P model (and in the B model, Eq. (3.9)). The corresponding results for profile 1 are given in Fig. 5.

As in section 3, the effect of forcing on the perturbations is neglected and the eigenvalue stability equations are formulated in the appendix. Again, in the following discussion, we shall restrict our discussion to the even disturbance modes.

Figure 11. As Fig. 5, for profile 2.
Figure 12. Growth rates as a function of $k$ for the even modes in the P model for profile 1 by itself (solid curves) and for this profile together with the barotropic wave $P^4_1$ (○); the upper layer wave $P^4_2$ (□); the baroclinic wave $P^4_3$ (◇); the barotropic wave $P^4_4$ (△); the upper layer wave $P^4_5$ (●); the baroclinic wave $P^4_6$ (▽).

Figure 13. As Fig. 12, for profile 2.

(b) Growth rates

In Figures 12 and 13 we show the growth rates for profiles 1 and 2, respectively, and, as well, the largest growth rates for these profiles together with the planetary waves $P^4_1$ and $P^4_2$ of the three types shown in Eq. (4.3). Again, the largest growth rates have been associated with a dominant zonal wavenumber $k^*$ for which the total energy is a maximum. We note that the pure baroclinic waves enhance the growth rates of intermediate dominant zonal wavenumber while the pure barotropic waves cause a decrease in the growth rates. The effect of the upper layer waves depends on the zonal wavenumber in question; for the shorter waves shown ($k^* = 8$ and 9) they cause an increase, and for the intermediate waves a slight decrease, in growth rates. These results are in general agreement with those of section 3.

For basic states consisting of zonal flow and the planetary waves $P^4_1$, we show in Tables 5 and 6 the total energy spectrum of perturbation waves for the subsystem of zonal wavenumbers coupled to $k^* = 7$; Table 6 also shows the energy spectrum of the second- and third-fastest growing waves for profile 2 plus $P^4_2$ and the associated growth rates. The number of zonal wavenumbers retained is as shown in these tables and the corresponding largest total wavenumber is $|k| + 23$. Again, there is little coupling between the eigenvectors of positive and negative values of the zonal wavenumber, for the fastest growing modes. We note, however, that for the slower growing modes the convergence is not as good.

We see that when $k^* = 7$, the amount of coupling to the other zonal wavenumbers, as shown by the energy spectrum, decreases as we go from the cases involving basic barotropic waves through the upper layer waves to the baroclinic waves. Again the sharp short-wave cutoff characteristic of two-layer quasi-geostrophic models with pure basic zonal flow is removed. With the waves $P^4_1$ present in the basic states, the coupling of the other zonal wavenumbers to $k^* = 7$ is stronger than for $P^4_2$: for example, the percentages of the maxi-
TABLE 5. Total energy spectrum of the fastest growing disturbance at the dominant zonal wavenumber $k^* = 7$ for profile 1 plus $P_s^3$ in the P model. The energy is expressed as a percentage of the total energy at $k^* = 7$.

Profile 1 + $P_s^3$ barotropic wave ($k^* = 7, s = 1$)

<table>
<thead>
<tr>
<th>$k$</th>
<th>3</th>
<th>7</th>
<th>11</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.9%</td>
<td>25.6%</td>
<td>100%</td>
<td>6.6%</td>
<td>1.8%</td>
</tr>
</tbody>
</table>

Profile 1 + $P_s^3$ baroclinic wave ($k^* = 7, s = 1$)

<table>
<thead>
<tr>
<th>$k$</th>
<th>3</th>
<th>7</th>
<th>11</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>0.19%</td>
<td>7.8%</td>
<td>100%</td>
<td>1.4%</td>
</tr>
</tbody>
</table>

Profile 1 + $P_s^3$ upper layer wave ($k^* = 7, s = 1$)

<table>
<thead>
<tr>
<th>$k$</th>
<th>3</th>
<th>7</th>
<th>11</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4%</td>
<td>10.7%</td>
<td>100%</td>
<td>5.1%</td>
<td>0.11%</td>
</tr>
</tbody>
</table>

The minimum energy at $k = 4$ are 73.6%, 23.0% and 27.1% for the barotropic, baroclinic and upper layer waves respectively together with profile 2. With the waves $P_s^3$ and $P_s^1$ present (normalized as described in section 3(a)) 120 components of the eigenvector did not appear to be sufficient for convergence although the results we obtained with this truncation had the same qualitative behaviour as for $P_s^3$ and $P_s^4$.

TABLE 6. As for Table 5 for profile 2 plus $P_s^3$ in the P model. In addition the second- and third-fastest-growing modes are shown together with their growth rates, $\omega$, and dominant zonal wavenumbers, $k^*$. The energy is expressed as a percentage of the total energy at the dominant zonal wavenumber.

Profile 2 + $P_s^3$ barotropic wave ($s = 1$)

<table>
<thead>
<tr>
<th>$k^*$</th>
<th>$\omega$</th>
<th>$k = -1$</th>
<th>3</th>
<th>7</th>
<th>11</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.0399</td>
<td>15.8%</td>
<td>44.0%</td>
<td>100%</td>
<td>11.8%</td>
<td>4.3%</td>
</tr>
<tr>
<td>3</td>
<td>0.0206</td>
<td>37.5%</td>
<td>100%</td>
<td>7.6%</td>
<td>2.4%</td>
<td>9.9%</td>
</tr>
<tr>
<td>7</td>
<td>0.0102</td>
<td>34.2%</td>
<td>88.6%</td>
<td>100%</td>
<td>35.11%</td>
<td>13.0%</td>
</tr>
</tbody>
</table>

Profile 2 + $P_s^3$ baroclinic wave ($s = 1$)

<table>
<thead>
<tr>
<th>$k^*$</th>
<th>$\omega$</th>
<th>$k = -1$</th>
<th>3</th>
<th>7</th>
<th>11</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.0728</td>
<td>0.96%</td>
<td>13.3%</td>
<td>100%</td>
<td>1.8%</td>
<td>0.24%</td>
</tr>
<tr>
<td>3</td>
<td>0.0456</td>
<td>29.2%</td>
<td>100%</td>
<td>14.8%</td>
<td>0.5%</td>
<td>0.02%</td>
</tr>
<tr>
<td>3</td>
<td>0.0344</td>
<td>44.0%</td>
<td>100%</td>
<td>32.5%</td>
<td>0.73%</td>
<td>0.31%</td>
</tr>
</tbody>
</table>

Profile 2 + $P_s^3$ upper layer wave ($s = 1$)

<table>
<thead>
<tr>
<th>$k^*$</th>
<th>$\omega$</th>
<th>$k = -1$</th>
<th>3</th>
<th>7</th>
<th>11</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.0633</td>
<td>1.5%</td>
<td>15.0%</td>
<td>100%</td>
<td>8.5%</td>
<td>0.33%</td>
</tr>
<tr>
<td>7</td>
<td>0.0200</td>
<td>4.7%</td>
<td>33.3%</td>
<td>100%</td>
<td>22.0%</td>
<td>2.4%</td>
</tr>
<tr>
<td>3</td>
<td>0.0152</td>
<td>49.5%</td>
<td>100%</td>
<td>10.4%</td>
<td>2.7%</td>
<td>1.2%</td>
</tr>
</tbody>
</table>
(c) Streamfunctions

Shown in Figs. 14 to 17 are the even perturbation streamfunctions with the maximum growth rate at \( k^* = 7 \), for the various representative cases described in the captions. We consider first the disturbance streamfunctions in the upper layer in the case of basic states consisting of profiles 1 and 2 plus the purely baroclinic wave \( P_4^b \), as shown in Figs. 14 and 15. As in section 3, the \( k = 7 \) structure is modulated by the zonal wavenumber of the basic planetary wave. When \( \cos \rho \lambda = -1 \) in Figs. 5 and 11, the excess shear, the difference between the vertical shear and Phillips's criterion for the P model, is a maximum and the modulus of the streamfunction is increased. Further, since the horizontal shear is increased at the same place the modulus of the streamfunction is also more confined in the meridional direction there. In contrast, when \( \cos \rho \lambda = 1 \) there is an increased spreading and a decrease in the intensity of the lows and highs. Note also that the positions of the centres of the lows and highs change slightly in the latitudinal direction, reflecting the corresponding change in the position of the maximum excess shear with the longitudinal phase of the basic wave. Moreover, we see that the positions where the lows and highs are least (most) intense are slightly to the east, i.e. downstream, of the positions of the long wave troughs where \( \cos \rho \lambda = 1 \) (long wave ridges where \( \cos \rho \lambda = -1 \)).

Fig. 16 shows the disturbance streamfunction in the upper layer for a basic state consisting of profile 2 and the purely barotropic wave \( P_4^t \). The effect of the zonal wavenumber of the basic waves in modulating the wavenumber \( k = 7 \) structure is again evident. The effect of the coupling to the other zonal wavenumbers, particularly 3 and 11, as shown in Table 6, is now more noticeable than for the cases involving pure baroclinic waves.

![Figure 14](image14.jpg)

**Figure 14.** Disturbance streamfunction \( \psi^i \) in the upper layer of the P model for the fastest growing mode with \( k^* = 7 \) and for a basic flow consisting of profile 1 plus the baroclinic wave \( P_4^b \). Here \( \psi^i(\lambda + \pi, \mu) \neq -\psi^i(\lambda, \mu) \). The corresponding zonal component of the vertical shear at \( \cos \rho \lambda = \pm 1, 0 \) may be seen from Fig. 5.

Units as in Fig. 1.

![Figure 15](image15.jpg)

**Figure 15.** As Fig. 14, for profile 2 plus the baroclinic wave \( P_4^b \). Again \( \psi^i(\lambda + \pi, \mu) \neq -\psi^i(\lambda, \mu) \). The corresponding zonal component of the vertical shear at \( \cos \rho \lambda = \pm 1, 0 \) may be seen from Fig. 11.
Figure 16. As Fig. 14, for profile 2 plus the barotropic wave $P_2^4$.

Figure 17. As Fig. 14, for profile 2 plus the upper layer wave $P_2^4$.

In Fig. 17 we show the perturbation streamfunctions in the upper layer for profile 2 and the purely upper layer wave $P_2^4$ as the basic state. As expected, the upper layer basic wave has an effect on the perturbation streamfunctions which is intermediate between that of pure baroclinic and pure barotropic waves.

For the cases involving pure baroclinic and barotropic basic waves, the disturbance streamfunctions in the lower layer are quite similar to those in the upper layer. However, basic barotropic and baroclinic waves have quite a different effect on the relative magnitude of the streamfunction in the two layers. This is illustrated in Fig. 18 where the contribution to the modulus of the disturbance streamfunction from $k = 7$ is graphed for the basic wave $P_2^4$ together with profile 2. We see that the barotropic wave causes a relative decrease of the perturbation streamfunction in the lower layer and, as well, an increased meridional spreading while the baroclinic wave has little effect. For the basic upper layer wave, the relative magnitude of the streamfunction in the two layers is intermediate between the results for the baroclinic and barotropic waves shown.

Comparing the results of this subsection with those of section 3(c), we see that while the development of cyclones and anticyclones is most intense downstream of the long wave ridges in the P model, for the B model with solid body rotation and basic planetary waves, the most intense development occurs downstream of the long wave troughs. For both models, however, the results for the cases involving baroclinic and upper layer waves are consistent with the following hypothesis: At least to a first approximation, the most intense development occurs slightly downstream of the position where the excess shear, above Phillips's criterion for the particular model, is a maximum. The difference between the two models for the cases involving solid body rotation arises because of the different behaviour near the
Figure 18. Modulus of the contribution to the disturbance streamfunction from \( k = 7 \) in the upper (solid curves) and lower (broken curves) layers for profile 2 (2); and for this profile together with the barotropic wave \( P \frac{1}{4} \) (A); and the baroclinic wave \( P \frac{3}{4} \) (B). The upper layer moduli have been normalized to unity.

equator of Phillips's criterion in the two models as shown in Fig. 5. For the P model, the maximum excess shear occurs when \( \cos \rho \lambda = -1 \), near 45° latitude, while for the B model it occurs when \( \cos \rho \lambda = 1 \), at the equator.

Assuming that the above hypothesis is valid in general, we may make some predictions for more general profiles. First, we see from Fig. 11 that for profile 2 and any of the pure baroclinic or upper layer waves \( P \frac{1}{4} \) and \( P \frac{3}{4} \) in the B model, the maximum shear excess occurs near 45° latitude when \( \cos \rho \lambda = -1 \). Thus for these (our most realistic profiles) both P and B models yield most intense development downstream of the long wave ridges. However, for a sharp single jet with a maximum equatorward of 30° latitude, such as is characteristic of the winter atmosphere, most intense development in both models may occur downstream of the long wave troughs. This may be seen from the fact that the zonal velocities associated with the waves \( P \frac{1}{4} \) and \( P \frac{3}{4} \) vanish at 30° latitude and slightly equatorwards of 30° latitude, respectively, as shown in Figs. 5 and 11. Thus, provided the jet profile decreases to zero sufficiently rapidly poleward of 30°, the maximum shear excess will occur when \( \cos \rho \lambda = 1 \).

For the (continuum of) transition profiles between those for which the development is a maximum downstream of the long wave ridges and those for which this occurs downstream of the troughs, a small change in the profile due to a small change in the external forcing may change the subsequent flow in either direction. Thus for the transition profiles one would expect that the skill of predicting the evolution of the flow would be a minimum.

We conclude that the regions of most intense development (like the momentum fluxes discussed in section 6(d) of BF) are very sensitive to the basic flow characteristics and the models used.

In the winter atmosphere, it is found that while cyclones commonly form downstream from upper air, long wave ridges (particularly in western Canada, western Europe and in the southern Australian region) these generally do not achieve the intensity of cyclones forming downstream from upper air, long wave troughs (see, for example, Palmén and Newton 1969). Of course, the models used here consider only small amplitude perturbations and therefore can at best apply only to the initial development of cyclones and anticyclones, with non-geostrophic and latent heat effects becoming increasingly important as the disturbances grow to larger amplitude. Further, in linear theories, no distinction is made between the prevalence and intensity of the highs and lows, while in the atmosphere,
anticyclones often appear as sluggish passive systems that fill the spaces between the far more active cyclone systems. However, occasionally an anticyclone may develop in the region of an upper air, long wave ridge (see, for example, Green 1977); quite frequently this is also accompanied by cutoff cyclones in neighbouring regions (see, for example, Fig. 12.8.6 of Petterssen 1956).

(d) Momentum and heat fluxes

The variability with latitude and longitude of momentum fluxes in the upper layer may be seen from the variability in the shape of the eddies in Figs. 14 to 17. This variability is greatest and least systematic for the cases involving basic barotropic waves.

Figure 19 shows the zonally averaged momentum fluxes, \( \overline{u_1 v_1} \) and \( \overline{u_3 v_3} \), for profile 2 by itself and together with the basic wave \( P_1^* \) for the three cases. These fluxes have been normalized such that the maximum flux in the upper layer is unity. However, with the eigenvectors in Eq. (A.5) normalized such that the sum of their moduli squared equals unity, the ratios of the maximum momentum flux in the upper layer for the cases with baroclinic, upper layer and barotropic waves to that with no wave are 0.87, 0.87 and 0.49 respectively. As in section 3, the pure barotropic wave has the largest effect on the magnitude of the momentum.
The barotropic wave, and to a lesser extent the upper layer wave, also produce increased meridional spreading. Further, the barotropic wave produces a relative decrease in the flux in the lower layer to that in the upper layer, reflecting the behaviour of the disturbance streamfunction.

The same qualitative behaviour described above is found for the other combinations of profiles 1 and 2 and $P^4_2$ and $P^4_4$. In particular, it is interesting to note that the superposition of the barotropic waves $P^4_3$ and $P^4_4$ on the solid body rotation profile 1, produces small equatorward components in the momentum fluxes as shown in Fig. 20. Thus, for both $P$ and $B$ models the pure barotropic waves produce the greatest change in the momentum fluxes and our results support Lorenz’s hypothesis (i) stated in section 1.

The normalized potential temperature fluxes $\frac{1}{\bar{\nu}_2} \bar{v}_2 \bar{\theta} + \frac{1}{\bar{\nu}_3} \bar{v}_3 \bar{\theta}$ are shown in Fig. 15 of BF for profiles 1 and 2. As in BF, the heat fluxes are less sensitive than the momentum fluxes to changes in the shear. We find here that the introduction of the baroclinic waves $P^4_4$ and $P^4_2$ produces little change in the normalized potential temperature fluxes while the upper layer and especially barotropic waves increase the meridional spread of the fluxes. Moreover, as may be expected from the relative magnitudes of the streamfunction amplitudes in the upper and lower layers, shown in Fig. 19, the barotropic waves reduce the contribution, $\frac{1}{\bar{\nu}_3} \bar{v}_3 \bar{\theta}$, to the heat flux in the lower layer relative to that in the upper layer, $\frac{1}{\bar{\nu}_1} \bar{v}_1 \bar{\theta}$.

In view of the changes in heat and momentum fluxes brought about by the presence of basic planetary waves, it would seem of interest to generalize the present study to multi-level models. It may be that some of the discrepancies between observed heat and momentum fluxes and those obtained from multi-level stability studies with zonal flow basic profiles (see, for example, Gall 1976 and Simmons and Hoskins 1977) may to some extent be alleviated by incorporation of planetary waves in the basic flow. This is not to suggest, however, that such a linear study could replace a complete nonlinear study of the evolution of baroclinic waves.

5. CONCLUSIONS

The stability of finite amplitude barotropic and baroclinic planetary waves and of combinations of zonal flows with superimposed planetary waves has been studied in two-layer quasi-geostrophic models on a sphere.

Critical amplitudes for incipient instability of basic baroclinic waves have been ob-
tained in the B model; they have been compared with both the corresponding triad interaction approximation and with the associated results for barotropic waves. It is found that the critical amplitudes for baroclinic waves are usually smaller than for the corresponding barotropic waves and decrease with decreasing static stability parameter. For both barotropic and baroclinic basic waves, the growth rates of the fastest growing perturbation modes usually increase with increasing amplitude of the basic waves, over the domain studied.

The change in the growth rates, perturbation streamfunctions, momentum and heat fluxes due to the superposition of planetary waves on zonal flow profiles has been examined in the two-layer quasi-geostrophic B and P models. We have concentrated largely on the basic waves $P_4^2$ and $P_4^4$ together with two zonal flow profiles, one consisting of solid body rotation (profile 1) and the other (profile 2) being a jet profile which approximates the observed annual mean flow. The superposition of the baroclinic waves on the zonal flows increases the largest of all the growth rates while the barotropic waves do the reverse. The presence of planetary waves also removes the short wave instability cutoff characteristic of two-layer quasi-geostrophic models with purely zonal flow basic profiles.

Both baroclinic and barotropic waves have the effect of producing regions of increased and decreased intensity of the disturbance streamfunction lows and highs as well as regions of decreased and increased meridional spreading of the amplitudes of the streamfunctions, the spreading being the greatest for the case of basic barotropic waves. For the cases involving basic baroclinic and upper layer waves, the position of the most intense development, downstream from the long wave troughs or ridges, depending on the basic flow, is found to be related to Phillips's criterion for incipient instability. The barotropic waves cause the greatest changes in the heat and momentum fluxes, the most noticeable again being the increased meridional spreading and decrease in the contribution to the heat fluxes from the lower layer relative to those from the upper layer. Moreover, when superimposed on profile 1, they produce zonally averaged perturbation momentum fluxes with both poleward and equatorward components.

We conclude that the results of this study support Lorenz's hypothesis (i) mentioned in the introduction but cast doubt on hypothesis (ii). This latter conclusion agrees with the findings of Lavery (1976) who examined the energy conversions in a nonlinear two-layer beta-plane model and found that baroclinic instability, rather than barotropic instability, is the most important factor in the unpredictability of large-scale atmospheric motions.

ACKNOWLEDGMENTS

It is a pleasure to thank P. G. Baines for his interest in this work and for numerous helpful suggestions. I am grateful to R. A. Plumb for an informative discussion, to E. K. Webb for his careful reading of the manuscript and to B. J. McAvaney for providing me with a contour map plotting routine. The assistance of R. C. Bell, P. J. Nelson and D. G. Reid with graphical presentation is gratefully acknowledged.

APPENDIX

THE EIGENVALUE EQUATIONS

We formulate here the eigenvalue equations for the P and B models for basic states consisting of a general zonal flow profile and a single planetary wave. For convenience, we choose here to write the spectral equations in terms of $\psi^1$ and $\psi^3$, the streamfunctions in the upper and lower layers, which are related to $\psi$ and $\tau$ through Eq. (2.2). In the P model, which we consider first, the basic states are then taken to be
\[ \tilde{\psi}^j = -A_{ON}^j P_N^j(\mu) - \text{Re}\{A_{ON}^j P_N(\mu) \exp i(\rho \lambda - \bar{\omega} t)\}, \quad j = 1, 3 \]  \hspace{1cm} (A.1a)

\[ \bar{\theta} = -C_{N+1} P_{N+1}(\mu) - \text{Re}\{(A_{pv}^1 - A_{pv}^3) [S(\rho, \nu - 1) P_{\nu-1}^0(\mu) + R(\rho, \nu + 1) P_{\nu+1}^0(\mu)] \exp i(\rho \lambda - \bar{\omega} t)\} \]  \hspace{1cm} (A.1b)

and

\[ \bar{\sigma} = \text{const.} \quad (= 0.01 \text{ as in BF}) \]  \hspace{1cm} (A.1c)

Here

\[ C_{N+1} = [R(0, N + 1)(A_{ON}^1 - A_{ON}^3) + S(0, N + 1)(A_{ON+2}^1 - A_{ON+2}^3)] \]  \hspace{1cm} (A.2)

and the various other functions needed above are defined in appendix A of BF. Also, \( A_{ON}^1 \) and \( A_{ON}^3 \) are related to \( A_N \) and \( B_N \) of BF by

\[ A_N = \frac{1}{4}(A_{ON}^1 + A_{ON}^3), \quad B_N = \frac{1}{4}(A_{ON}^1 - A_{ON}^3) \]  \hspace{1cm} (A.3)

In Eq. (A.1), summation over positive odd \( N \) up to an arbitrary odd \( N_{\text{max}} \) is implied, but we shall restrict our consideration to a single planetary wave.

The basic fields in Eq. (A.1) are to be regarded as forced oscillations of the two-layer quasi-geostrophic model. They satisfy Eq. (4.1d) but for the nonlinear prognostic equations (4.1a) to (4.1c) to be satisfied suitable combinations of heating, topography and friction must be assumed. We suppose that the effect of the forcing on the perturbations may be neglected: as discussed by Charney (1959) this is in general quite a good approximation. In particular, for completely inhomogeneous forcing, such as the inhomogeneous heating used by Phillips (1956) and Baer and Alyea (1971), the forcing does not affect the perturbations. Then the equations for the disturbances are given in Eq. (A.1a) and Eqs. (A.1c) to (A.1e) of BF but with the second last term in Eq. (A.1a) put to zero.

Now suppose we go into a reference system moving with the angular frequency \( \bar{\omega} \) of the planetary wave. Then the new longitude and time co-ordinates are related to the old through

\[ \lambda' = \lambda - (\bar{\omega}/\rho) t, \quad t' = t \]  \hspace{1cm} (A.4)

With each of the perturbation fields expanded in the general form shown in Eqs. (2.5) (the explicit expressions are given in Eq. (A.2) of BF with the replacements \( \psi \rightarrow \psi^1, \tau \rightarrow \psi^3 \)), we find on dropping the primes in Eq. (A.4) that the spectral equations, in dimensionless form, reduce to the eigenvalue problem for \( \omega \):

\[
\left( \omega + \frac{k \bar{\omega}}{\rho} + \frac{2k}{n(n+1)} \right) \psi_{kn}^1 + \sum_{l=|n-N|+1}^{n+N-1} D(k, 0, k, n, N, l) A_{ON}^1 \psi_{kl}^1 + \\
+ \frac{1}{4} \sum_{l=|n-N|+1}^{n+N-1} \left( D(k, \rho, k-\rho, n, l) A_{\rho \psi}^1 \psi_{k-l\rho}^1 + D(k, -\rho, k+\rho, n, l) A_{\rho \psi}^1 \psi_{k+l\rho}^1 \right) + \\
+ \frac{A}{2} A(k, n) \left( \omega + \frac{k \bar{\omega}}{\rho} \right) (\psi_{kn}^1 - \psi_{kn}^3) + \\
+ \frac{A}{2} B(k, n) \left( \omega + \frac{k \bar{\omega}}{\rho} \right) (\psi_{kn-2}^1 - \psi_{kn-2}^3) + \\
+ \frac{A}{2} C(k, n) \left( \omega + \frac{k \bar{\omega}}{\rho} \right) (\psi_{kn+2}^1 - \psi_{kn+2}^3) + \\
+ \frac{A}{4} \sum_{l=|n-N|-1}^{n+N+1} F(k, 0, k, n, N, l) (A_{ON}^1 + A_{ON}^3) (\psi_{kl}^1 - \psi_{kl}^3) + 
\]
\[
+ \frac{\Lambda}{8} \sum_{l=[a-v]-1}^{n+v+1} \{ F(k, \rho, k-\rho, n, v, l)(A_{\rho,v}^1 + A_{\rho,v}^3)(\psi_{k-\rho}^1 - \psi_{k-\rho}^3) \\
+ F(k, -\rho, k+\rho, n, v, l)(A_{\rho,v}^1 + A_{\rho,v}^3)(\psi_{k+\rho}^1 - \psi_{k+\rho}^3) \} + \\
+ \frac{4\Lambda}{8} \sum_{l=[a-N]-1}^{n+N+1} \{ F(k, k, 0, n, l, N)(A_{\rho,N}^1 - A_{\rho,N}^3)(\psi_{k+\rho}^1 + \psi_{k+\rho}^3) \} + \\
+ \frac{4\Lambda}{8} \sum_{l=[a-v]-1}^{n+v+1} \{ F(k, k, -\rho, k-\rho, n, l, v)(A_{\rho,v}^1 - A_{\rho,v}^3)(\psi_{k-\rho}^1 + \psi_{k-\rho}^3) \} \]
\]

\[
= 0. \quad \cdots \quad \cdots \quad \cdots \quad (A.5)
\]

In addition, we have the same equation with the interchange of the superscripts 1 and 3 wherever they occur. In the above equation the summation over \( l \) is in steps of 2 and additional summation over positive odd \( N \) is implied. The method of deriving Eq. (A.5) is similar to that described in appendix A of BF.

The functions \( A(k, n) \), \( B(k, n) \) and \( C(k, n) \) are given in Eq. (A.6) of BF while

\[
\Lambda = 4/\tilde{\sigma} \quad \cdots \quad \cdots \quad \cdots \quad (A.6a)
\]

\[
D(t, s, r, \gamma, \beta, \alpha) = \frac{\alpha(\alpha+1) - \beta(\beta+1)}{\gamma(\gamma+1)} K_{\gamma, \beta, \alpha}^{s, r} \quad \cdots \quad \cdots \quad \cdots \quad (A.6b)
\]

\[
F(t, s, r, \gamma, \beta, \alpha) = \left[ \frac{R(t, \gamma)}{(\gamma-1)\gamma} K_{\gamma-1, \beta, \alpha}^{t, r} + \frac{S(t, \gamma)}{(\gamma+1)(\gamma+2)} K_{\gamma+1, \beta, \alpha}^{t, r} \right] R(r, \alpha+1) + \\
+ \left[ \frac{R(t, \gamma)}{(\gamma-1)\gamma} K_{\gamma-1, \beta, \alpha}^{t, r} + \frac{S(t, \gamma)}{(\gamma+1)(\gamma+2)} K_{\gamma+1, \beta, \alpha}^{t, r} \right] S(r, \alpha-1) \quad (A.6c)
\]

and \( K_{\gamma, \beta, \alpha}^{s, r} \) is given in Eq. (A.5a) of BF. Note that \( D \) and \( F \) above reduce to \( D \) and \( F \) in Eqs. (A.6d) and (A.6f) of BF for pure zonal flow and that with \( E \) given in Eq. (A.6e) of BF

\[
F(k, k, 0, n, l, N) = -E(k, n, l, N). \quad \cdots \quad \cdots \quad \cdots
\]

With basic streamfunctions given in Eq. (A.1a) (and the corresponding potential temperature obtained from \( \tilde{\sigma} = 2\mu_0^2 \)) we find that for the B model in Eqs. (2.1) the spectral equations are as given in Eq. (A.5) but with the replacements

\[
A(k, n) \rightarrow \mu_0^2/\{n(n+1)\} \quad \cdots \quad \cdots \quad \cdots \quad (A.7a)
\]

\[
B(k, n) \rightarrow 0 \quad \cdots \quad \cdots \quad \cdots \quad (A.7b)
\]

\[
C(k, n) \rightarrow 0 \quad \cdots \quad \cdots \quad \cdots \quad (A.7c)
\]

\[
F(t, s, r, \gamma, \beta, \alpha) \rightarrow \frac{\mu_0^2}{\gamma(\gamma+1)} K_{\gamma, \beta, \alpha}^{s, r} \quad \cdots \quad \cdots \quad \cdots \quad (A.7d)
\]

Here \( \mu_0 \) is the sine of the latitude appearing in the (constant) Coriolis parameter.
REFERENCES


Eliaisen, E. and Machenhauer, B. 1969  On the observed large-scale atmospheric wave motions, Ibid., 21, 149–165.


Jahneke, E. and Emde, F. 1945  Tables of functions with formulae and curves, Dover, New York.

Lavery, T. F. 1976  The role of baroclinic and barotropic instability in atmospheric predictability, in conference on Simulation of Large-Scale Atmospheric Processes, Hamburg, 228–232.


Lorenz, E. N. 1960  Energy and numerical weather prediction, Tellus, 12, 364–373.


