On the use of an analytic solution in estimating eddy viscosity distribution and water vapour flux in a mature hurricane

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SUMMARY

An analytic solution, in $p$ coordinates, of the dynamical equations of a mature hurricane system is used to determine the $p$ dependence of eddy viscosity coefficients for an assumed quasi-stationary and axisymmetric phase. The solution is based on: (i) eddy viscosity coefficients $K_1$ and $K_2$, describing respectively horizontal and vertical transfers of momentum, expressed as general functions of $p$; and (ii) the premise that the radial variation of the magnitudes of the tangential and radial components of velocity are of similar form. This leads to an integral equation for the tangential velocity. Explicit expressions for $K_1$ and $K_2$ are finally obtained by choosing power law forms, in the $p$ variable, which lead to good agreement of the tangential velocity solution with observed distributions. By using this (inverse) method of calculating $K_1$ and $K_2$, the whole eddy system, ranging from eddies created by the very strong velocity shear close to the sea surface to those connected with violent cloud convection, is described by continuous mathematical functions. The associated temperature and condensation heating distributions are then calculated from the hydrostatic relation and the thermodynamic equation. To complete the model an integral water vapour flux condition relates the sea surface temperature to prescribed humidity conditions at the outer boundary of the model.

I. INTRODUCTION

A basic general problem in formulating general circulation models and weather prediction systems is that of relating sub-grid flow systems (or eddies) to grid scale model flow by a system of parametrizing (or eddy) coefficients. The problem is compounded by the effect of gridpoint errors (depending on the time of integration). Numerical experiments (e.g. Searle and Davies 1975) show that the characteristics of computed flows depend critically on the system used to represent vertical and horizontal eddy transports. It is, therefore, of considerable interest and practical importance to determine what functional forms these coefficients should take in any available analytic solution representing broadly the main characteristics of some significant atmospheric flow problem.

The specific case provided by the dynamical structure of hurricane systems presents the possibility of development of analytic solutions and their exploitation to determine the spatial distribution of eddy viscosities. The essential mechanism consists of the surface frictionally controlled turbulence which reduces the tangential (swirl) velocity in the lower region. The radial acceleration of the fluid is reduced below a balance with the radial inward pressure gradient resulting in a surface, radially inwards, component of velocity feeding the necessary water vapour fuel into the inner ring of condensation heating and consequent energy source. Appraisals of progress in appreciating the function and details of the component links are given by, for example, Anthes (1974) and in the GARP Report No. 13 (1976) on The study conference on the development of numerical models for the tropics; the problems involved in constructing numerical models are discussed by, for example, Sundqvist (1970). In particular, see for example Moss and Rosenthal (1975), observational knowledge of turbulence flux magnitudes is fragmentary.

Turbulence in a hurricane can be divided into two regions: there is the intensive frictionally induced turbulence close to the sea surface; and there is the large scale convective cloud turbulence in the core of the storm. A mathematical representation covering the

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whole vertical range of the dynamically intense region of the hurricane cell by a continuous function would, however, avoid the difficulty of dividing the spatial vertical range artificially into two zones.

A mature, quasi-stationary state is analysed, as it is generally accepted that in the dynamically active hurricane core assumed axisymmetry in the mean residual eddy distributions following temporal and azimuthal averages gives a good working model to study the main characteristics. An analytic solution for this state by, for example, Carrier et al. (1971) is not, however, helpful in this context as lateral turbulence is totally neglected; this is likely to be a serious omission in the light of hurricane eddy structures. We find, in fact, (section 3) that the distribution of the tangential velocity component strongly depends on the ratio of horizontal to vertical turbulence coefficients. The vertical eddy viscosity is also assumed by Carrier et al. to be independent of height, in contradiction to analyses of observational results. What is needed is a simple form of solution, incorporating both components of eddy viscosity, so that numerical experiments can be quickly carried out to test the validity of various parametrisations, and studies conveniently made of the relative importance of various forcing factors on the storm dynamics.

The problem of formulating appropriate boundary conditions is conveniently expressed in cylindrical coordinates. The inner cylindrical model surface is taken at a radial distance \( r = r_0 \) at which the radial and tangential velocities are assumed to be zero; physically, a non-zero value of \( r_0 \) is a consequence of the approximate conservation of angular momentum leading to an inverse proportionality between tangential velocity and radial distance, and the kinetic energy of an inward swirling air particle having a limited value (see for example Kuo 1958).

The outer cylindrical boundary is placed at a sufficiently large radius to enclose the dynamically active region; the mean observed atmospheric temperature stratification can then be imposed at this radius with confidence. The upper boundary condition, in \( p \) coordinates, is taken at about 100 mb by adjusting the analytic solution so that the velocities decay exponentially rapidly at greater heights. Specification of the lower (sea surface) boundary condition is extremely difficult in hurricane models as the numerical value of the drag coefficient is not known from observation in the range of the very high winds of these storms. However, our inverse method of calculation does not require a knowledge of the surface drag coefficient.

This method consists of obtaining an analytic solution for the velocity components, based on: (i) coefficients of eddy viscosity, \( K_1 \) and \( K_2 \) (describing respectively horizontal and vertical eddy transfers of momentum), expressed initially as general functions of pressure; and (ii) the premise that the magnitudes of radial and tangential components of velocity have similar dependences on radial distance. Explicit expressions for \( K_1 \) and \( K_2 \) are then obtained by choosing power law forms in the \( p \) variable, which lead to good agreement of model and observed tangential velocity components.

2. THE MODEL EQUATIONS AND ANALYTIC SOLUTION METHOD

(a) The model equations

The describing equations are expressed in pressure, \( p \), and radial, \( r \), coordinates. The tangential and radial velocity components are denoted by \( v \) and \( u \), and \( dp/dt \) by \( \omega \). The tangential equation is

\[
\begin{align*}
u \frac{\partial u}{\partial r} + \omega \frac{\partial v}{\partial p} + u v / r + f u &= K_1(p) \frac{\partial}{\partial r}(v u / r + v / r) + \frac{\partial}{\partial p} \{K_2(p) v u / \partial p\} \\
\end{align*}
\] (1)
f being the Coriolis parameter. $K_1(p)$ and $K_2(p)$, the eddy viscosity coefficients concerned with lateral and vertical eddy flux of momentum, are both likely to vary much more sharply with height above the surface than with radial distance, through the surface region of frictionally created turbulence and the higher regions of convective turbulence, so they are taken to depend on $p$ only. A similar mathematical formulation can be carried through if $K_1$ also depends separately on radial distance, $r$. However, this does not alter the functional dependences on $p$, which are shown to be very marked and are the main objectives of this paper.

The continuity equation is

$$r \partial u / \partial r + u + r \partial \omega / \partial p = 0. \tag{2}$$

The radial momentum equation (retaining the nonlinear terms) is written in the form

$$\partial \phi / \partial r = v(f + v/r) - u \partial u / \partial r - \omega \partial u / \partial p + \frac{\partial}{\partial r} \left[ K_1(p) (\partial u / \partial r + u/r) \right] + \frac{\partial}{\partial p} \left[ K_2(p) \partial u / \partial p \right], \tag{3}$$

where $\phi$ is the geopotential.

The hydrostatic equation, in terms of the potential temperature, $\theta$, and $\Pi = (p/p_0)^{R/c_v}$, is (in a conventional notation)

$$\partial \phi / \partial p = -(R \Pi / p) \theta. \tag{4}$$

The heat transfer equation is taken in the form

$$u \partial \theta / \partial r + \omega \partial \theta / \partial p = Q/[(\Pi_c) + K_{H1} \nabla^2 \theta] + (q/\Pi_c) \frac{\partial}{\partial p} (K_{H2} \partial \theta / \partial p), \tag{5}$$

where $Q$ denotes the diabatic heating per unit mass and unit time, and $K_{H1}$ and $K_{H2}$ denote coefficients of eddy transfer of heat in the lateral and vertical directions and are identified with $K_1$ and $K_2$, respectively.

(b) Solution of the dynamical variables

The method of solution consists of first eliminating $\omega$ from Eqs. (1) and (2), leading to

$$r \partial u / \partial r + u + r \frac{\partial}{\partial r} \left[ \frac{1}{\partial v / \partial p} \left( \frac{\partial}{\partial p} (K_2 \partial v / \partial p) + K_1 \frac{\partial}{\partial r} (\partial v / \partial r + v/r) - fu - u(\partial v / \partial r + v/r) \right) \right] = 0. \tag{6}$$

We now look for a solution in which the dependences of both $u$ and $v$ on $r$ are similar, i.e. we write

$$u = z(r) u(p), \tag{7}$$

and

$$v = z(r) v(p). \tag{8}$$

Available observational data suggest that the general forms of variation of $u$ and $v$ with $r$ are approximately similar, with the positions of maxima in $u$ and in $v$ tending to appear at about the same value of $r$. Recent numerical modelling work by Peng and Kuo (1975) also displays this characteristic. This may not of course be exactly true of all actual hurricane structures, but the ensuing calculated forms giving the spatial distributions of $K_1(p)$ and $K_2(p)$ are unlikely to be seriously in error. Using Eqs. (7) and (8), Eq. (6) can be expressed in the form

$$A(r) Q(p) + B(r) S(p) + T(p) = 0, \tag{9}$$

where

$$A(r) = dz/\partial r + z/r, \quad B(r) = (1/z) \frac{d}{dr}(dz/\partial r + z/r),$$
\[ Q(p) = u(p) - \frac{d}{dp} \left( \frac{u(p)v(p)}{dv(p)/dp} \right), \quad S(p) = \frac{d}{dp} \left( \frac{K_1(p)v(p)}{dv(p)/dp} \right), \text{ and} \]
\[ T(p) = \frac{d}{dp} \left[ \frac{d}{dv(p)/dp} \left( K_2(p) dv(p)/dp \right) - fu(p) \right] \]

If we differentiate Eq. (9) partially with respect to \( r \), we have (indicating differentials by primes),
\[ A'(r) Q(p) + B'(r) S(p) = 0, \]
and provided \( A' \) and \( B' \) are nonzero,
\[ B'(r)/A'(r) = -Q(p)/S(p) = \alpha, \text{ a constant.} \]
Hence
\[ B(r) = \alpha A(r) + \beta, \beta \text{ being a further constant,} \quad (10) \]
\[ Q(p) = -\alpha S(p), \quad . \quad . \quad . \quad . \quad (11) \]
and
\[ T(p) = -\beta S(p). \quad . \quad . \quad . \quad . \quad (12) \]
The tangential velocity is taken to be inversely proportional to \( r^\gamma \), with \( \gamma = 1 \) at large values of \( r \). This choice of \( \gamma \) is mathematically convenient; it is also taken by other modellers (e.g. Sundqvist 1970) and does not affect the result obtained for the \( p \) dependence of \( K_1 \) and \( K_2 \).
Substitution of this form into Eq. (10) shows that the integration constant \( \beta = 0 \).
Consequently, Eq. (12) integrates to give
\[ u = (1/f) \frac{d}{dp} \left( C v(p) + K_2(p) dv/dp \right), \quad . \quad . \quad (13) \]
where \( C \) is an arbitrary constant. Equation (11) then integrates to the form
\[ K_2(p) v d^2 v/dp^2 + \frac{dK_2}{dp} (v dv/dp) - K_2 (dv/dp)^2 + D dv/dp - \alpha f K_1(p) v = 0; \quad (14) \]
\( v \) refers here to the tangential component depending on \( p \). On the upper boundary, \( p \to 0 \);
substitution of a condition that \( v \to 0 \) and \( K_2 \to 0 \) in Eq. (14) shows directly that \( D = 0 \).
Substituting then \( v = \alpha V \) and \( \alpha z = Z \) into Eqs. (10) and (14) leads to the two basic separated equations
\[ \frac{d}{dr} (dZ/dr + Z/r) = Z (dZ/dr + Z/r), \quad . \quad . \quad (15) \]
and
\[ K_2(p) V d^2 V/dp^2 - K_2(p) (dV/dp)^2 + (dK_2/dp) V (dv/dp) - f K_1(p) V = 0. \quad (16) \]
The form of (16) is interesting in clearly demonstrating the strong dependence of the distribution of \( V(p) \) on the ratio \( K_1(p)/K_2(p) \). Numerical solutions are shown and discussed in section 3. Writing \( y = rZ \) and \( r = e^\xi \), Eq. (15) reduces to the form
\[ \frac{d^2 y/d\xi^2}{(y+2) dy/d\xi} = 0. \quad . \quad . \quad (17) \]
Then writing \( w = y+2 \), the solution with no singularities is
\[ w = a (1 - Gr^\delta)/(1 + Gr^\delta), \quad . \quad . \quad (18) \]
where \( a \) and \( G \) are arbitrary constants with \( G \) necessarily positive. This leads to the solution for radial dependence of tangential and radial velocities,
\[ z(r) = ((a-2)-(a+2)Gr^\delta)/r (1 + Gr^\delta). \quad . \quad . \quad (19) \]
This can conveniently be expressed in the form

\[ z(r) = F [ (a - 2) - (a + 2) R^a ] / R (1 + R^a), \]

(20)

where \( F = G \) and \( R = Fr \). The structure of the function \( z(r) \) is determined by the parameter \( a \), the effect of \( F \) being to control the spread of \( z(r) \) and its magnitude. The zero value of \( z(r) \) is given by \( R^a = (a - 2)/(a + 2) \) and the turning points are given by \( R^a = \{ a^2 - 2 + a(a^2 - 3)^{1/2} \}/(a + 2) \). When \( 0 < a < \sqrt{3} \) there is no turning point; when \( \sqrt{3} < a < 2 \) there are two turning points but no zeros; when \( a > 2 \), \( z(r) \) has a zero at \( R^a = (a - 2)/(a + 2) \), one turning point at \( R = \{ (a^2 - 2 + a(a^2 - 3)^{1/2})/(a + 2) \}^{0.5} \), and tends to zero as \( r \to \infty \). This last case, \( a > 2 \), can therefore represent the storm. If \( r = r_0 \), satisfying \( (Fr_0)^a = (a - 2)/(a + 2) \), is the position of the zero of \( z(r) \), then the region \( 0 < r < r_0 \) can reasonably be taken to represent the eye of the model storm, \( r = r_{\text{max}} \) the turning point.

A conveniently computable solution for \( v(p) \) can be obtained by first writing \( K_1(p) \) in the form \( \hat{K}_1(p) V(p) \) for mathematical convenience. Equation (16) can then be expressed in an integral equation form,

\[ V(p) = -H \exp \left[ \int \frac{(B-f\hat{K}_1 dp)}{K_2(p)} dp \right], \]

(21)

where \( B \) and \( H \) are constants of integration. It is found at this stage to be mathematically and computationally convenient to express \( \hat{K}_1 \) and \( K_2 \) as power-law forms and to express the constant \( B \) in terms of \( p_* \), the pressure at which the tangential velocity is a maximum (a basic storm parameter), given by \( dp/dp_* = 0 \). Writing

\[ \hat{K}_1 = \hat{K}_1 p^n \quad \text{and} \quad K_2 = \hat{K}_2 p^a \]

(22)

in Eq. (21) then leads to the solution for the \( p \) dependence of tangential velocity in the form

\[ V(p) = -H \exp \left[ -\left\{ f\hat{K}_1(m+1)^n \hat{K}_2 \{ p^{m+1} \} / (n-1) + p^{m+1} \{ m+2-n \} p^{1-n} \right\} \right] \]

(23)

and the corresponding radial velocity solution becomes

\[ u(p) = (\hat{K}_2/f) D [(D/p^n)(p^{m+1} - p^{m+1})^2 + (C/\hat{K}_2 p^n)(p^{m+1} - p^{m+1}) - (m+1)p^m] V(p) \]

(24)

where \( D = f\hat{K}_1(m+1)\hat{K}_2 \).

To complete the dynamical solution, \( \omega(r, p) \) is given by Eq. (1) to be

\[ \omega(r, p) = -C + \left[ \frac{\{ K_1(p) v(p) - u(p) v(p) \}}{d v/d p} \right] (d v/d p + \omega z) ; \]

(25)

this is consistent with the model \( u \) field and the continuity equation (2).

(c) The temperature distribution consistent with the dynamical solution

The radial momentum equation (3) can now be used to determine an analytical form for the geopotential, \( \phi \). The spatial distribution of \( \theta \) can be calculated from Eq. (4); this is consistent with the dynamical solution, and is seen to reduce to the form

\[ \theta(r, p) = \alpha(p) - (p/r \Pi) \hat{\theta} , \]

where \( \alpha(p) \) is an arbitrary function of \( p \) and is determined in the model by using typical temperature values at the outer boundary, and

\[ \hat{\theta}(r, p) = \{ f \omega - \bar{w} u'' + \hat{K}_2(p^n u'') \} X(r) + 2w' Y(r) - \]

\[ -(\omega u'' + \omega u') S(r) - uu' z^2 - \hat{K}_1(p^n u')(dz/dr + z/r) , \]

(26)
where \( X(r) = \int z(r)dr, \ Y(r) = \int z^2(r)/r \, dr \), and \( S(r) = \int (z \, dz/dr + z^2/r) \, dr \), and primes denote derivatives with respect to \( r \). Equation (5) can then be used to compute the diabatic heating function, \( Q \).

(d) The upper and lower dynamical boundary conditions

The sea surface boundary condition normally employed in air-sea interaction calculations (at sub-storm velocities) is the so-called bulk aerodynamic formula
\[
K_2 \partial v / \partial p = - \rho g C_D \partial |v| / \partial r,
\]
with empirical values of \( C_D \) in the range \( 0.7 \times 10^{-3} \) to \( 2.0 \times 10^{-3} \). However, no observational data are available to determine a representative value for this coefficient in hurricane strength, mean surface winds. In addition, it is physically reasonable to anticipate that such a coefficient would apply strictly at the bottom of a very thin sub-layer immediately in contact with the vigorously waving sea surface; this layer is likely to be very differently structured from the general hurricane lower flow and to require a separate treatment in a more detailed model. Fortunately, this is not a requirement in our present inverse method of calculating eddy viscosity spatial distribution in the general flow. It is interesting, however, to note that when our values of \( K_2 \), \( v \) and \( \partial v / \partial p \), as the model surface is approached, are substituted into Eq. (27), the numerical value of \( C_D \) is about \( 0.5 \times 10^{-3} \). This is smaller than those available from observational data, but is reasonably consistent with our supposition that a smaller gradient of velocity exists immediately above a thin sub-layer which contains a very sharp velocity gradient leading to a correspondingly higher \( C_D \) value.

The upper boundary condition taken is that all flow variables decay rapidly with height above 100 mb.

(e) The vapour flux boundary conditions

At the outer boundary \( (r = r_1) \) the vertical distribution of water vapour is prescribed; this is taken from typical storm data and can, of course, be varied in numerical experiments. Inward flux of water vapour is then determined by the values of radial velocity which the model produces at this radial distance. There is no inward flux at \( r = r_0 \) in the model, as it is defined by zero radial velocity. At the upper boundary, at 100 mb, the water vapour amount is taken to be zero.

At the lower surface, the air is assumed to be saturated, as a reasonable first approximation, in the region of the model which produces significant condensation heating. Consequently, from \( r = r_0 \) to the value of \( r \) when the air is just not saturated, \( r = r_B \) (shown in

![Figure 1. Schematic vertical section of the model storm indicating 'rain' \((Q > 0)\) and 'dry' \((Q = 0)\) areas.](image)
The diabatic heating is taken to be wholly derived from condensation. Palmén and Newton (1969) suggest that for the tropical cyclone the rate of radiative cooling is 100 W m\(^{-2}\), so that, taken over a cylinder of the troposphere from 30 to 90 km radially, this gives about \(2 \times 10^{12}\) W, whereas the estimated rate of latent heat release (Malkus and Riehl) is \(3 \times 10^{14}\) W, implying that radiation effects can be neglected compared with condensation in the context of a hurricane system.

To calculate the distribution of \(Q\) over our model domain, \(K_{H1}\) in Eq. (5) is taken to be identical with \(K_1\), and \(K_{H2}\) to be zero; this follows Sundqvist (1970) and Peng and Kuo (1975). The constant \(C\) in Eq. (25) for the \(\omega\) distribution was chosen numerically to prevent positive \(\omega\) values at the lower boundary of the inner active zone and turned out to be only 1% of average \(\omega\) field values. The vertical distribution of \(\omega\) in the dynamically active core of the system and in the lower inflow region was then reasonably similar to typical available data, although for the cases computed the peak values tended to appear near \(r = r_0\) rather than at \(r = r_{\text{max}}\), which observational data suggest: this is a somewhat similar result to that obtained by Peng and Kuo. It does not affect, however, the determination of \(K_1\) and \(K_2\) as functions of \(\rho\), and it is possible that the inclusion of \(K_1\) as a function of \(r\) would improve the \(r\) distribution of the \(\omega\) field: this requires further study. The \(Q\) distribution is then calculated from Eq. (5), following substitution of the \(u, \omega,\) and \(\theta\) distributions. This was found to give high (realistic) numerical values in the inner region, and to decay rapidly with increasing \(r\). At \(r = r_B\) (Fig. 1), of the order of 32 km in the case studied, \(Q\) was found to have decreased effectively to zero and a completely ‘dry’ region (in the sense that \(Q = 0\)) was assumed to exist for \(r_B < r < r_1\).

An integral water vapour flux condition then applies to this ‘dry’ region of the model; this enables us to relate sea surface temperature to incoming humidity flux, etc. It is based on the conservation equation that the flux of water vapour out of the ‘dry’ region into the condensation region equals the input of water vapour across the outer radial surface (\(r = r_1\)) plus evaporation from the sea surface over \(r_B < r < r_1\); the flux into the condensation region is given by \((1/L)\int \rho Q(r, p) \, dr\), where \(L\) is the latent heat of water, \(\rho\) the vapour density, \(dt\) a volume element, and the volume integration extends over the ‘rain’ (\(Q > 0\)) region. The evaporation is taken to be described by the bulk formula

\[
\rho C_D [v(q_v - q_0)],
\]

(28)

[\(v\) being the surface wind speed magnitude, \(q_v\) the (unknown) humidity of the air immediately on the sea at sea temperature \(T_v\) and surface pressure \(p_v\), \(q_0\) is the surface air humidity; appropriate values of \(p_v\) are calculated from the computed \(\phi\) distribution. An observational value is substituted for \(C_D\), as the thin sub-layer will involve a small gradient of vapour in contrast to the very large gradient of velocity. Knowing the saturated humidity at \(r_B\), \(q_s\) can be calculated from \(r = r_B\) to \(r = r_1\) using the expression (in conventional notation)

\[
\delta q = (-c_p T/L)[\exp(-Lq/c_p T)(\delta p/2\delta\theta) + \delta\theta/\theta - (Lq/c_p T^2) \delta T]
\]

(29)
which is assumed to hold along the sea surface and is obtained from the Malkus and Riehl (1960) empirical relationship
\[ \delta p_e = -2.5 \delta \theta_e. \]  
(30)
The domain integral water vapour flux condition is then
\[
(2\pi/Lg) \int_{r_1}^{r_0} \int_{p=100}^{p_1} Q(r, p) \lambda r \, dp \, dr = (2\pi/g) r_1 \int_{100}^{p_1} q(p) u(r_1, p) \, dp + \\
+2\pi \rho C_D \int_{r_0}^{r_1} r \mid \varphi ((q_w - q_e) \, dr,
\]  
(31)
where \( \lambda = 1 \) for \( r_0 < r < r_B(p) \) and \( \lambda = 0 \) for \( r_B(p) < r < r_1 \). By a trial and error numerical method the appropriate value of \( q_w \) to satisfy Eq. (31) can be evaluated and from this the model sea surface temperature, \( T_w \), consistent with the model dynamics and heat transfer properties calculated, noting that in real storms some evaporation in the \( r_0 < r < r_B \) region could contribute to the outward flux in the upper part of the \( r = r_B \) surface and modify the \( T_w \) value.

3. Numerical results and discussion

(a) Radial variation of the model structure

The main aim of the calculation is to estimate the \( p \) dependence of \( K_1 \) and \( K_2 \) in their relation to \( v(p) \), but the radial variation is, of course, also a basic storm property. The radial variations in the radial and tangential velocity components of the model storm are controlled by the function \( \varphi (r) \) as defined by Eq. (20). The radial variation of the vertical velocity is controlled by the function \( (dz/dr + z/r) \). The three components of velocity have a simple functional form in that all three consist essentially of the product of two functions, one of which is a function of \( p \) only, the other a function of \( r \) only. The other functions, \( \phi, \theta, Q \), given by the model have a more complex structure and their spatial variation cannot be looked at as a simple combination of variations in two independent directions. However, the radial variations of all the functions in the model storm are controlled by the two parameters \( F \) and \( a \) in that they define the function \( \varphi (r) \).

From the results in section 2 a simple expression for a fundamental storm parameter is obtained; this is the ratio \( r_{\text{max}}/r_0 = [(a^2 - 2 + (a^2 - 3)^2)/(a - 2)]^{1/a} \). It is independent of \( F \) and has an interesting form, shown in Fig. 2. Asymptotically, the curve approaches the line \( a = 2 \) as \( r_{\text{max}}/r_0 \to \infty \), and as \( r_{\text{max}}/r_0 \to 1, a \to \infty \), the value of the ratio approaching 1 representing the case of a very intense storm. From an examination of data from actual storms, and pictures of composite storms, we can conclude that although the ratio \( r_{\text{max}}/r_0 \) varies considerably, its value falls within the range \( 2.0 \leq r_{\text{max}}/r_0 \leq 5.0 \); the extreme values apply to hurricane Helene of 1958 (Palmén and Newton 1969, p. 481) and hurricane Cleo of the same year (La Seur and Hawkins 1963).

(b) Vertical variation of the model

Figure 3, taken from Palmén and Newton (1969, p. 512), gives an idealized picture of the vertical distributions of the components of the fluid velocity at a fixed radius. It was found that reasonably realistic forms for \( \psi(p) \) could be obtained using values for \( \bar{K}_1 \) and \( \bar{K}_2 \) corresponding to the eddy diffusion terms with the same maximum value as those used by Sundqvist: \( 5 \times 10^3 \text{m}^2\text{s}^{-1} \) for the horizontal diffusion coefficient and \( 0.15 \text{mb}^2\text{s}^{-1} \) for the vertical coefficient. Figure 4 is an example of the profiles obtained. \( \bar{K}_1 \)
Figure 2. (a) Variation of $r_{max}/r_0$ with $a$. (b) Variation of $r_{max}$ and $r_0$ with $a$.

Figure 3. Schematic representation of typical vertical profiles of radial ($u$), tangential ($v$) and vertical ($-\omega$) velocities (from Palmén and Newton 1969).
and $\bar{K}_2$ have values 1·5 and 0·15 respectively; these lead to the same maximum values for the diffusion terms as Sundqvist uses. The parameters $m$ and $n$ have the values 3·0 and 2·1 respectively. $C$ is given the value 1·0 and $p_1$ is 850 mb. The profiles obtained for $v(p)$, and to a certain extent $\omega(p)$, are encouragingly like those of Fig. 3. The radial velocity profile (scaled up by a factor 10 in the figure) is, however, not realistic; the maximum inflow and outflow velocities are about $1/100$ and $1/16$ of the maximum tangential velocity compared with about $1/5$ and $1/10$ for actual storms. The profiles of Fig. 4 are typical of the type of curves obtained with these values for $\bar{K}_1$ and $\bar{K}_2$, the major difficulty being to obtain a strong enough inflow.

(c) Values of the eddy diffusion coefficients

Typical values adopted by numerical modellers for the coefficients of eddy viscosity in gridpoint models are shown in Table 1.

The same values are used for these coefficients during the developing, mature and decay states of the model storm. Riehl and Malkus (1961), however, estimated from observational data that the vertical eddy diffusion coefficient, $K_2$, used for the mature stage of hurricane Daisy of 1958 was of the order $100\text{ m}^2\text{s}^{-1}$, and that the lateral coefficient, $K_1$, was of the order $10^3\text{ m}^2\text{s}^{-1}$. It is of importance, then, to know what values $K_1$ and $K_2$ take in an analytic (non-gridpoint) system.

The structure of Eq. (16) for $v(p)$ displays clearly an important result: $v(p)$ is crucially dependent on the ratio $\bar{K}_1/\bar{K}_2$, and not on the magnitudes of $\bar{K}_1$ and $\bar{K}_2$. Figure 5 shows profiles of $v(p)$ with various values of the ratio $\bar{K}_1/\bar{K}_2$, the other parameters being fixed, and the scaling factor is chosen so that the maximum value of the function is 10. Comparison with Fig. 3 (observational values) indicates that a choice of $\bar{K}_1/\bar{K}_2$ between 10 and 30 gives close agreement with actual storms.

**TABLE 1. LATERAL, $K_1$, AND VERTICAL, $K_2$, COEFFICIENTS OF EDDY DIFFUSION ADOPTED BY VARIOUS AUTHORS**

<table>
<thead>
<tr>
<th>Author</th>
<th>$K_1$(m$^2$s$^{-1}$)</th>
<th>$K_2$(m$^2$s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rosenthal</td>
<td>$10^4$</td>
<td>10</td>
</tr>
<tr>
<td>1970a</td>
<td>$10^3$</td>
<td>10</td>
</tr>
<tr>
<td>1970b</td>
<td>$5 \times 10^3$</td>
<td>10</td>
</tr>
<tr>
<td>Sundqvist</td>
<td>1970</td>
<td>$5 \times 10^3$</td>
</tr>
<tr>
<td>Peng and Kuo</td>
<td>1975</td>
<td>$5 \times 10^3$</td>
</tr>
</tbody>
</table>
In Figs. 6(a) and 6(b), various radial velocity, $u$, profiles are shown. In (a) $m = 3.0$, $n = 1.8$; in (b) $m = 20$, $n = 1.5$; and in both, the ratio $K_1/K_2$ was held at 20; the curves differ in the magnitude of $K_1$. It is seen that the strengths of both inflow and outflow increase with increasing $K_1$. It can also be seen that the depth of the atmosphere over which the inflow occurs also increases with $K_1$. Observed values (e.g. Hawkins and Rubsam 1968) suggest that the more realistic profiles are obtained with values of $K_1$ between 30 and 50.

The variation in calculated $u$, $v$ and $\omega$ profiles for different values of $K_1$, $K_2$, $m$ and $n$ are shown in Figs. 7(a), (b), (c), (d): the parameter $p_1 = 8.50$ mb, $C = 1$ and the scaling factor $F$ was chosen such that $\nu(p)$ has a maximum value of 10. It was found that the best profiles were obtained with values of $K_1$ and $K_2$ which correspond to maximum values of eddy coefficients of the order of those given by Riehl and Malkus (1961). It is interesting to note that the numerical values for $K_1$ and $K_2$, $m$ and $n$ used in Figures 7(c) and (d) lead to approximately zero values of $\omega$ at the surface. It is also interesting that the eddy diffusion coefficients required by an analytic model are in agreement with the calculated values obtained by Riehl and Malkus.

The main defect of all the profiles obtained in comparison with those of Fig. 3 is the smoothness of the radial velocity profile. In Fig. 3, radial motion is to all intents and purposes confined to two distinct regions: inflow between 800 and 1000 mb and outflow between 100 and 400 mb. The expression for $u(p)$ does not lend itself to this structure. In all our profiles it was found that the function was such that outflow and inflow regions covered most of the vertical range of the storm. This may be due to a deficiency of coordinate systems in hurricane convective systems, or it may be necessary to replace power-law forms for $K_1$ and $K_2$ by polynomials: this requires further study.
Figure 6. (a) and (b). Profiles of $\mu(\rho)$ (in convenient arbitrary units), for values of $F_1$ from 6 to 48. In (a) $m = 3.0$, $n = 1.8$; in (b) $m = 20$, $n = 1.5$. In both (a) and (b) the ratio $R_1/R_2 = 20$. 
Figure 7. Typical calculated model vertical profiles of $u$, $v$ and $\omega$ in convenient arbitrary units. (a) $\overline{K}_1 = 45$, $\overline{K}_2 = 1.5$, $m = 2.0$, $n = 1.2$ (---: $\bar{K}_1 = 30$, $\bar{K}_2 = 0.3$, $m = 2.0$, $n = 1.2$). (b) $\bar{K}_1 = 30$, $\bar{K}_2 = 0.3$, $m = 4.0$, $n = 1.2$. (c) $\bar{K}_1 = 40$, $\bar{K}_2 = 2.0$, $m = 4.0$, $n = 1.2$. (d) $\bar{K}_1 = 25$, $\bar{K}_2 = 2.5$, $m = 2.0$, $n = 2.1$.

(d) The detailed structure of a particular model

A study of the effect, on the velocity distribution, of varying the basic set of parameters and comparing with available data, sets the range of possible numerical values for these parameters. Figures 8, 9, 10, 11, 12 show computed distributions of the fields of tangential velocity, radial velocity, vertical velocity, potential temperature anomaly and diabatic heating, respectively, for a typical model storm. The numerical values chosen were $\overline{K}_1 = 40$, $\overline{K}_2 = 4$ (corresponding to maximum values of $K_1 = 4 \times 10^3 \text{m}^2\text{s}^{-1}$, $K_2 = 4 \text{mb}^2\text{s}^{-1}$), $m = 3.5$, $n = 1.8$, $p_1 = 900 \text{mb}$, $C = 1.0$, $F = 0.6$ and $\alpha = 3.0$; the storm has an ‘eye’ radius of 19.5 km and the maximum horizontal wind of 50-6 m s$^{-1}$ occurs at 47-4 km. The tangential velocity field is very similar to that obtained by Sundqvist; the other components are in reasonable agreement with some other published results of numerical modelling (e.g. Peng and Kuo 1975). Although the maximum in the $\omega$ field is nearer to $r_0$ than to $r_{\text{max}}$, this will not affect the computation of the model eddy viscosity spatial distribution in the $\rho$ coordinate, and is unlikely to affect significantly sea surface temperature to velocity distribution relationships.
The overall shape of the calculated potential temperature anomalies is reasonably characteristic of observational data, the maximum value being rather lower than that observed in storms of comparable strength but occupying a larger volume. There are few published data regarding observed distributions of fields of diabatic heating, but the fields produced by our model are not inconsistent with those of Krishnamurti (1962) and Peng and Kuo (1975), the maximum being at about 600 mb (below the level of the maximum temperature anomaly).

The geopotential function is calculated as indicated in section 2, the arbitrary constant involved being chosen such that the 1000 mb surface is at zero height at large radius. The model sea surface pressure is then calculated by linear interpolation between geopotential
Figure 10. Computed distribution of vertical velocity (mb s$^{-1}$). (See also caption to Fig. 8.)

Figure 11. Computed distribution of potential temperature (K) anomalies. (See also caption to Fig. 8.)

Figure 12. Computed distribution of diabatic heating ($4.2 \times 10^{-4}$ W kg$^{-1}$). (See also caption to Fig. 8.)
heights of the 1000, 950 and 900 mb surfaces. Using the particular parameter values given
above, the model sea surface pressure reaches a minimum value of 958 mb at the eye, the
variation with radius being shown in Figure 13. The calculated air temperature at the sea
surface increases slowly from 24·8°C at \( r_0 \) to 26·2°C at 700 km, the associated potential
temperature decreasing from 27·1°C to 26·2°C. Equation (30) was then used to calculate
the specific humidity distribution: surface air was assumed to be saturated at the point \( r_B \)
in Fig. 1, taken to be the edge of the condensation region, where the heating function at
the surface drops to a value of 1% of its maximum; in our particular model this occurred
at a radius of 32 km, the calculated variation of surface specific humidity being given by
Figure 13. The model sea surface temperature was calculated from the integral condition,
Eq. (31), to be 25°C, for a drag coefficient of \( 10^{-3} \) and an integration outer boundary for
the moisture balance equation taken at 460 km (where the inflow drops below 1·5 m s\(^{-1}\)).

The variation with height in the computed horizontal eddy diffusivity experienced by
the model is shown in Fig. 14. It is seen to decrease with height, \( 10^5 m^2 s^{-1} \) being attained
at 910 mb, which is within 50 mb of the surface in the innermost region. This shows that in
numerical modelling, a (horizontal) eddy coefficient which varies with height is essential.
The estimates of Anthes (1974) of \( 4 \times 10^4 m^2 s^{-1} \) and of Riehl and Malkus (1961) of \( 10^7 m^2 s^{-1} \)
for the mature stages are not inconsistent with our surface values. The fact that in numerical modelling considerably smaller values for this coefficient are found to be necessary, highlights the important problem which needs to be solved in all numerical models of unravelling the effect of gridpoint errors, the effect of finite resolution schemes, from the physical effects of sub-grid-scale dynamics. In our model the numerical complexities are removed as we are dealing with an analytic continuous solution.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure15}
\caption{Variation of $F$ with sea surface temperature (°C).}
\end{figure}

\(e\) The effects of variation of parameters on the model storm

A second general point of interest emerged when comparing the characteristics of different model systems obtained by varying the parameters controlling the radial variation, but keeping the total kinetic energy of the region, bounded by $r_b$ and an outer cylindrical surface of radius 700 km, to a prescribed value. The parameters controlling the vertical variation, $\bar{K}_1$, $\bar{K}_2$, $m$, $n$, $p$, and $C$, were kept constant and the model was computed for different values of the 'radial variation parameter, $F$', from 0.3 to 1.0 at intervals of 0.05; for a given $C_D$, the associated sea surface temperature was also calculated. Graphs of sea surface temperature against $F$, for storms with the same value of $a$ and the same evaporation system, were found to have turning values with minima in sea temperature. A typical result is shown in Fig. 15. We note that although the evaporation system is to some extent arbitrary, the existence of a minimum is not, in that all models studied are found to have a turning value. For a given value of $C_D$, the storm which requires the lowest sea temperature has a structure which is the most favourable for producing evaporation. The value of $F$ expresses its

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure16}
\caption{Shaded region indicates values of $m$ and $n$ (associated with a kinetic energy of $5 \times 10^{17}$ J) leading to typical observed storm characteristics.}
\end{figure}
effect in controlling the radius of the maximum wind speed: thus if the storm is in a state
of minimum sea temperature, any movement of its radius of maximum wind requires a
higher sea temperature to maintain the moisture balance. This suggests that such a storm
is then in a preferred state for a given kinetic energy and a given value of the parameter \(a\).

Calculations were also carried out to study the effects of varying the indices \(m\) and \(n\); details are given in Evans (1978). For models with kinetic energy \(5 \times 10^{17}\) J in the region
enclosed by the 700 km radius, the region of values of \(m\) and \(n\) which lead to typical storm
characteristics is shown in Fig. 16.

4. Conclusions

(i) An analytic form of solution has been obtained for the dynamical equations of a
hurricane in its quasi-stationary, axisymmetric state, valid for general functional dependences
of the eddy viscosity coefficients on height above the sea surface. Numerical exponents
defining specific power-law forms are then found for the latter by matching calculated
tangential velocity profiles to typical observed profiles. The sensitivity of calculated inflow
and outflow characteristics on choice of parameter values is studied. In particular, it is
found that the structure of tangential velocity distribution depends more on the ratio of
horizontal to vertical eddy flux coefficients than on the magnitude of these.

(ii) Using an empirical relationship between equivalent potential temperature and
pressure at the sea surface, the distribution of humidity can be calculated in the surface
region of the model outside the condensation (convective) storm core. By then applying a
domain, integral condition concerning the water vapour budget, a general result is found
concerning preferred radial distributions of tangential velocity for a given storm kinetic
energy.

(iii) Because of the comparative simplicity of the necessary computation, the model
can serve as a useful basis for testing quickly the effects of varying the numerical values
given to various physical parameters. The mathematical analysis is capable of improvement
by taking \(K_1\) to be a function of \(r\) with consequent effects on the radial distribution of the \(u\),
\(\omega\) and \(Q\) fields. Due to the nature of the separation-of-variables method this, however, will
not alter the functional forms found for \(K_1(p)\) and \(K_2(p)\).

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