Baroclinic instability governed by the modified quasi-geostrophic equations

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SUMMARY

Results are presented for the baroclinic instability of zonal flows on a β-plane according to the modified quasi-geostrophic equations (which retain a non-Doppler term in the rigid horizontal boundary condition). Numerical techniques, with up to 180 interior levels, are used. In the absence of a β-effect the instability of the longer waves is fairly sensitive to the non-Doppler term, the more so if static compressibility is neglected. A realistic β-effect markedly reduces this sensitivity: the weakly unstable long waves are essentially internal mode phenomena and are consequently little affected by the non-Doppler term. It is confirmed that the stable external mode Rossby waves, where they exist, are radically altered when the term is included.

1. INTRODUCTION

The purpose of this paper is to report the results of a numerical investigation into the baroclinic instability of fluid motion governed by the modified quasi-geostrophic equations in β-plane form. These equations, in which geometric height is the vertical coordinate, are not restricted to systems having BH ≪ 1, and in this respect alone they differ from the standard β-plane quasi-geostrophic equations (Pedlosky 1964). Here \( B = (1/\theta_0)(d\theta_0/dz) \) (\( \theta_0 \) = potential temperature, \( z \) = altitude) is the reference state static stability, and \( H \) is a scale height, either of the motion or of the reference state density field.

For brevity we shall refer to the standard and modified quasi-geostrophic equations as the 'S' and 'M' sets respectively. As detailed by White (1977), the M set thermal wind, continuity and vorticity equations contain terms not present in the S set forms, but the potential vorticity equation, when written in terms of the (horizontal) stream function, \( \psi \), is unchanged. The only difference in the evolution of \( \psi \) according to the two sets arises because the M set horizontal boundary condition contains the extra term \(-B(\partial \psi / \partial t)\). White (1978) has studied the parallel with the quasi-geostrophic equations in log-pressure coordinates, in which a formally similar extra term appears when the boundary condition on specified pressure surfaces is \( w = 0 \) rather than \( \omega = 0 \): when the basic state is isothermal, and under certain other conditions, an exact isomorphism exists.

Because of its behaviour under zonal coordinate transformation the term \(-B(\partial \psi / \partial t)\) is conveniently referred to as the 'non-Doppler' term. A physical interpretation of the non-Doppler property of \(-B(\partial \psi / \partial t)\) was given by White (1977) (henceforward indicated as
W). It takes account of the difference in the directions of apparent vertical as measured by observers in frames rotating with different steady angular velocities about the earth's rotation axis. It compensates precisely for the centrifugal, non-Galilean, effect whereby a boundary which appears horizontal to one observer appears to the other observer to be inclined.

The non-Doppler term was retained in baroclinic instability analyses by Charney (1947) and Kuo (1952), and commented upon briefly by Burger (1962). Geisler (1974), Geisler and Dickinson (1975) and Geisler and Garcia (1977), in various linearized analyses, retained the corresponding term in the log-pressure formulation. Assuming a realistic $\beta$-effect, Geisler and Garcia stated that the effect of the non-Doppler term on baroclinic instability was small, though they did not give quantitative information. Using pressure as vertical coordinate, and also assuming a realistic $\beta$-effect, Wiin-Nielsen (1971a) had previously found that an equivalent non-Doppler term caused generally small changes in the instability properties of a simple baroclinic flow. W, on the other hand, solved an $f$-plane baroclinic instability problem in the S and M formulations, neglecting static compressibility, and found appreciable differences in stability properties, especially in the long waves. Blumen (1978) obtained approximate solutions to a better-posed $f$-plane problem in which static compressibility was retained. He also found considerable differences between the S and M results.

After outlining in section 2 the mathematical and numerical aspects of the problem, we present $f$-plane and $\beta$-plane results in sections 3 and 4, an important aim being to reconcile the results of Geisler and Garcia (1977) and W. Our main concern is the behaviour of unstable waves, but the properties of the stable external Rossby mode are discussed briefly in the concluding section 5. Conditions for neutral stability in the M formulation have been studied by Geisler and Dickinson (1975) and Blumen (1978).

2. EQUATIONS AND NUMERICAL TECHNIQUES

The linearized potential vorticity equation satisfied by small perturbations $\psi'$ on a zonal flow $\bar{u}(z)$ on a $\beta$-plane is

$$\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left\{ \nabla^2 \psi' + \frac{f_0^2}{\rho_0} \frac{\partial}{\partial z} \left( \frac{\rho_0}{N_0^2} \frac{\partial \psi'}{\partial z} \right) \right\} + \frac{\partial \psi'}{\partial x} \left( \beta - \frac{f_0^2}{\rho_0} \frac{\partial}{\partial z} \left( \frac{\rho_0}{N_0^2} \frac{\partial \bar{u}}{\partial z} \right) \right) = 0,$$

whilst at rigid horizontal boundaries

$$\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial \psi'}{\partial z} - \frac{\partial \psi'}{\partial x} \frac{\partial \bar{u}}{\partial z} - B \frac{\partial \psi'}{\partial t} = 0.$$  (2)

Here the notation is that used in W. Eq. (2) is the M set boundary condition; the S set condition is obtained from it by neglecting the non-Doppler term $-B \partial \psi'/\partial t$. Wave-like solutions

$$\psi_{ki} = \text{Re} \{ F(z)e^{ik(x-ct)} \} \sin ly$$

must have

$$(U-C)(d^2 F/dZ^2 - \kappa dF/dZ - p^2 F) + F(\gamma - d^2 U/dZ^2 + \kappa dU/dZ) = 0$$  (3)

with

$$(U-C)dF/dZ - F dU/dZ + C Bz F = 0,$$

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at rigid horizontal boundaries. Here $Z = z/z_\ast$, $U = u/u_\ast$, $C = c/u_\ast = C_r + i C_i$, $\kappa = -(B/\rho_0) \frac{d}{dZ} (\rho_0/B)$, $p^2 = (g B z^2 j_0^2) (k^2 + i^2)$, and $\gamma = g B z^2 \beta j_0^2 u_\ast$. These non-dimensional quantities are similar to those defined by Green (1960), but the choice of the height.
and velocity scales $z_*$, $u_*$ depends on the particular problem in hand (see sections 3 and 4).

An upper lid boundary condition will be applied. Thus Eq. (4) is to be satisfied at some height $Z_L = z_L/z_*$ as well as at the ground ($Z = 0$). The resulting eigenvalue problem can be solved in its continuous form by analytical techniques if $U(Z)$, $\rho_0(Z)$ and $B(Z)$ are fairly simple functions, but numerical methods are more versatile and will be employed in this study. $F(Z)$ is specified at $N$ interior levels $Z_n = Z_L(n - 1/2)/N$ ($n = 1, 2, \ldots, N$) and at fictitious levels $Z_0 = -Z_L/2N$, $Z_{N + 1} = Z_L(1 + 1/2N)$. Use of second-order difference approximations for the $Z$ derivatives appearing in Eqs. (3) and (4) gives $(N + 2)$ homogeneous equations for the $(N + 2)$ values $F(Z_n) = F_n$ ($n = 0, 1, 2, \ldots, N + 1$). Solutions exist only for certain values of $C$, and to each eigenvalue a form $\{F_n\}$ belongs. Our approach is to use a matrix method to find all solutions for $N = 10$, and to obtain much more accurate results for the solutions of interest by applying a complex interpolation technique with $N = 20, 60, 100$ and finally $180$. The matrix method, which was used by Gadian (1978), employs the $QR$ algorithm (Wilkinson 1965). The interpolation method is the familiar 'shooting method' (Green 1960; Stone 1970).

The baroclinically unstable solutions described in sections 3 and 4 correspond to $N = 180$ and are accurate to well within $1\%$. For insignificant logistic reasons the external Rossby mode solutions discussed in section 5 were obtained with $N = 100$.

3. $f$-PLANE PROBLEMS ($\beta = 0$)

We shall examine the case of constant vertical shear and static stability when static compressibility is retained ($\kappa > 0$). It is convenient to define $z_* = z_L$ and $u_* = u(z_L) - u(0) = \Delta u$. Eqs. (3) and (4) become

\begin{equation}
(U_B + Z - C)(d^2F/dZ^2 - \kappa dF/dZ - p^2F) + \kappa F = 0 \quad . \quad . \quad (5)
\end{equation}

\begin{equation}
(U_B + Z - C) dF/dZ - F + CBz_* F = 0 \quad \text{at} \quad Z = 0, 1. \quad . \quad . \quad (6)
\end{equation}

Here $U_B = u(0)/\Delta u$ and $\kappa = -(1/\rho_0)(d\rho_0/dZ)/z_L/H_\nu$.

Because of the non-Doppler term in Eq. (6), $M$ set stability properties depend on the (non-dimensionalized) surface flow $U_B$. Fig. 1 shows $pC_1$ and $(C_T - U_B)$ as functions of $p$ for $U_B = -\frac{1}{2}, 0, +\frac{1}{2}$ when $\kappa = 1$ and $Bz_* = 0.1$. Also shown are the $S$ set curves (which are of course independent of $U_B$). The quantity $pC_1$ is a true growth rate measure only in the limit $l \to 0$, but it is convenient to refer to $pC_1$ as 'the growth rate'; $(C_T - U_B)$ is a measure of the phase speed relative to the fluid.

The wavenumber $p_{\text{max}}$ of maximum instability is $\approx 1.5$ in all four cases. In this section we shall refer to waves having $p > p_{\text{max}}$ as 'short waves' and those having $p < p_{\text{max}}$ as 'long waves'. Growth rates and relative phase speeds in the short waves are little affected by the inclusion of the non-Doppler term in Eq. (6). Long waves are, however, quite considerably affected, though the variation with $U_B$ is somewhat less than it is when static compressibility is neglected ($\kappa = 0$; see $W$, Fig. 1). The overall behaviour reflects the variation with wavenumber of the vertical structure functions (not shown): the vertical scale of the waves generally decreases as the horizontal scale decreases and thus the non-Doppler term in Eq. (6) is less important in the short-wave part of the spectrum than in the long-wave part. Long-wave instability in the $M$ formulation increases as $U_B$ increases. Similar behaviour occurs in the $S$ set if the boundaries are tilted towards the isentropes whilst keeping the basic zonal flow unchanged (Hide 1969; Blumsack and Geirars; Mason 1975). Consistent with $W$'s physical interpretation of the non-Doppler term, its inclusion in the $M$ set evidently corresponds closely to the adoption of a vertical coordinate direction which is
Figure 1. Non-dimensional growth rates (left) and phase speeds (right), as functions of non-dimensional wavenumber \( p \), of the unstable modes in four \( f \)-plane baroclinic stability analyses (see text for precise definitions of the non-dimensionalizations). Basic flow is \( U = (Z+\text{constant}) \) with uniform static stability. The continuous curve gives the results obtained using the S set. The three broken curves give the results obtained using the M set for the basic flows indicated alongside each curve. In all four cases parameter values are \( \gamma = 0, \kappa = 1, B_{sz} = 0.1 \) and the upper lid is at \( Z = 1 \). Curves interpolated from values obtained for \( p \) at intervals of 0.1 using the shooting method with 180 interior levels.

that of the apparent vertical measured in a frame moving with the height-averaged zonal flow.

Note that a range of unstable long waves have steering levels outside the fluid when \( U_b = 0 \) or \( \frac{1}{2} \). Similar behaviour occurred in the case \( \kappa = 0 \) (see W).

Our numerical methods could detect no growing modes below \( p = 0.4 \) when \( U_b = -\frac{1}{2} \) nor below \( p = 0.15 \) when \( U_b = 0 \). Blumen (1978) investigated analytically an \( f \)-plane problem in which \( \kappa = 1 \) and the zonal flow was such that the mean state potential vorticity vanished. He established the existence of long-wave cut-offs in the M formulation. We consider it very probable that such cut-offs exist in the present linear shear problem. In this respect the behaviour would be similar to that when \( \kappa = 0 \). However, W found that the stable very long waves had very large phase speeds when \( \kappa = 0 \); this aspect of the long-wave behaviour was shown by Blumen to be a consequence of neglecting static compressibility, and thus to be physically irrelevant. On heuristic grounds it seems inappropriate to neglect static compressibility in the M formulation, since dynamic compressibility is still present. Indeed, our general impression is that such a step has far fewer adverse repercussions than might be expected. (It should be noted here that Wiin-Nielsen (1971a) established a formal isomorphism between (i) stability problems posed in the standard height coordinate formulation with \( w = 0 \) at rigid upper and lower boundaries and static compressibility neglected; and (ii) problems posed in pressure coordinates with \( \omega = 0 \) at upper and lower pressure surfaces. Isomorphism under formal interchange of \( Z \) and \( p_* \), where \( p_* \propto (1000 \text{ mb} - \text{ pressure}) \), occurs if \( U(Z) = U(p_*) \) and \( N^2(Z) = \sigma(p_*) \), where \( \sigma \) is the relevant static stability measure. It can be shown that the isomorphism does not extend to the related non-Doppler problems; unlike the situation in the M set height coordinate problem with static compressibility neglected, the pressure coordinate non-Doppler coefficient is not a constant multiple of the static stability measure \( \sigma \) which appears in the relevant potential vorticity equation.)
The most important result of this section is that long-wave stability properties are appreciably changed by the inclusion of the non-Doppler boundary condition term, their instability being greater the more westerly the mean flow. In both respects the sensitivity of the long waves is somewhat less than in the \( \kappa = 0 \) problem. We next examine what happens when a realistic \( \beta \)-effect is included.

4. \( \beta \)-PLANE PROBLEMS

A fairly wide range of \( \beta \)-plane problems has been investigated; only two typical cases, both including static compressibility, will be described here. It is convenient to take \( z_* = \) density scale height, \( H_\rho \), and \( u_* = u(z_*) - u(0) = \Delta u \). Thus \( \Delta u \) is the change in basic zonal flow between the lower boundary and an altitude \( H_\rho \); the upper boundary need not be at \( z_* = H_\rho \).

![Diagram](image)

Figure 2. As Fig. 1, but for three \( \beta \)-plane baroclinic stability analyses with \( \gamma = 1 \). (The M set results for the basic flow \( U = Z \) are omitted for clarity.)

Fig. 2 shows \( pC_1 \) and \( (C_1 - U_B) \) as functions of \( \rho \) for the S set (full line) and the M set with \( U_B = -\frac{1}{2}z_\kappa \) (broken line), \( +\frac{1}{2}z_\kappa \) (dashed line), in the constant static stability problem with \( U = U_B + Z \), \( \gamma = \kappa = Z_L = 1 \), and \( Bz_* = 0.1 \). (The format of Fig. 2 is thus the same as that of Fig. 1 except that the M set curves for \( U_B = 0 \) are omitted for clarity.) The S set problem is one of those studied by Green (1960). In the region of maximum baroclinic instability \( (\rho \simeq 1.3) \) the effect of the non-Doppler term is qualitatively the same as in the \( f \)-plane case (Fig. 1), but with a noticeably smaller variation of maximum growth rates – an effect which is consistent with the higher value of \( \rho_{\text{max}} \) in the case \( \gamma = 1 \). In the vicinity of \( \rho = 1.5 \) the sensitivity of M set growth rates to variations of \( U_B \) is comparable with that found when \( \gamma = 0 \).

The weakly unstable long waves \( (\rho < 1.3) \) are insensitive to the non-Doppler term – as regards vertical structure (not shown), growth rates and phase speeds. So far as we can tell (the smallest value of \( \rho \) studied was 0.1) there is no long-wave cut-off. The behaviour of the long waves in the M set is thus fundamentally changed by the addition of a realistic \( \beta \)-effect. This reflects the fact that S set unstable long waves are internal mode phenomena with more rapid variations of amplitude and phase with height than those of the fastest growing
baroclinic waves; consequently the unstable long waves might be expected to be insensitive to the non-Doppler term in Eq. (6).

An interesting detail of the M set long-wave behaviour is that growth rates decrease with increase of $U_B$ – opposite to the behaviour in the region of major instability and in the $f$-plane case. At present we can give no explanation for this result.

Fig. 3, whose basic format is the same as for Fig. 2, gives the stability data for the case

$$U - U_B = \tanh Z / \tanh 1 = f(Z); B = B_0 \left[ 1 + 2e^s/(e^s + 1) \right], s = 3(Z - 1) \cdot$$

with the upper lid at $Z_L = 4$ (i.e. 4 density scale heights), $\gamma_0 = \gamma(B_0) = 1$, $\kappa_0 = \kappa(B_0) = 1$, $B_0 z_s = 0.1$. The inset to Fig. 3 shows the profiles of $f(Z)$ and $B$: they correspond to maximum shear in the ‘troposphere’, $0 \leq Z \leq 1$, and a rapid increase in static stability near the ‘tropopause’, $Z = 1$. In the region of major instability the behaviour is much the same as in

![Graph](image)

Figure 3. As Fig. 2, but for a basic flow profile $U - U_n = \tanh Z / \tanh 1 = f(Z)$ and static stability $B(Z) = B_0 \left[ 1 + 2 \exp \left( 3(Z - 1) / (1 + \exp[3(Z - 1)]) \right) \right]$ with $\rho = \rho(B_0)$, $\gamma_0 = \gamma(B_0) = 1$, $\kappa_0 = \kappa(B_0) = 1$, $B_0 z_s = 0.1$ and the upper lid at $Z = 4$, $f(Z)$ and $B(Z)$ are shown on the inset to the growth rate diagram.

Fig. 2, there being a perceptible but slight dependence of M set properties on $U_B$. There are now two subsidiary growth rate maxima in the long-wave spectrum and growth rates here are generally considerably larger than in the uniform stratification problem; but the variation between S and M results is slight. No long-wave cut-off could be found.

Qualitatively similar differences between S and M behaviour thus appear in both problems. They have been found, too, in all other cases in which a realistic $\beta$-effect is imposed.

5. Conclusion

From the results described in sections 3 and 4, the apparent discrepancy between Geisler and Garcia’s (1977) findings and those of W can be explained. The neglect of static compressibility in $f$-plane problems tends to exaggerate the sensitivity of the long waves to the non-Doppler term in the horizontal boundary conditions. More important, typical $\beta$-effects drastically reduce growth rates in just the wavenumber region where $f$-plane
solutions are sensitive to the non-Doppler term. The remaining unstable long waves are internal mode phenomena with a small enough vertical scale to render the non-Doppler terms insignificant. Around the wavelength of maximum instability the non-Doppler effect is small.

Approximations of four types are made in deriving the S set equations from the hydrostatic primitive equations on a sphere: (a) spherical geometry is replaced by Cartesian; (b) the latitude variation of the Coriolis parameter is subjected to the β-plane approximation; (c) certain ageostrophic effects are neglected; (d) terms formally small when BH ≪ 1 are neglected. See Pedlosky (1964). From the studies by Stone (1966), Simons (1972), Simmons and Hoskins (1976), Frederiksen (1978) and others, the importance of approximations (a), (b) and (c) in baroclinic instability problems can be gauged. For the fastest-growing waves on flows similar to the observed zonal average it seems that (a) and (b) are generally more important than (c). Our study suggests that (d) is an uncritical approximation, perhaps even less critical than (c) for the fastest-growing waves. Such conclusions about the unimportance of (d) are however not universally valid. For example, Wini-Nielsen (1971b), Geisler (1974) and White (1978) have noted that the retrogression phase speeds of long (neutral) external Rossby modes are very much altered by inclusion of the non-Doppler term. This has been verified in the present study for a variety of upper lid heights and stability profiles. Fig. 4 shows the S and M phase speeds C, as functions of p in the constant shear, constant static stability problem with \( \gamma = \kappa = Z_L = 1 \) and \( Bz = 0\cdot1 \). In both formulations the external Rossby mode appears in the same wavenumber region as the long (internal mode) baroclinic waves (Green 1960; Geisler and Dickinson 1975), but the S set retrogression phase speeds are very much larger than those in the M set. It is to be expected that the external Rossby modes should be sensitive to the non-Doppler term, for in addition to high phase speeds these modes have only a slow variation of amplitude with height. These two effects combine to make the contribution of the non-Doppler term very important in Eq. (6).
White (1978) also noted that a progressive mode, which has no counterpart in the S set, can exist in a uniform westerly flow bounded by an upper lid, according to the M formulation. In recent studies this mode has been traced in slightly baroclinic flows by applying the shooting method (section 2). As the (height-independent) shear is increased from zero, the phase speed of the mode, at any given wavenumber, decreases until a critical level occurs at the upper lid. Thereafter the mode has not been traced, and it does not appear to be associated with an instability.

REFERENCES