Theory of the sunpillar

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(Received 9 March 1979; revised 11 June 1979)

SUMMARY

The radiance distribution predicted by classical physical models of the sunpillar is calculated. It is shown that the loci of constant tip angle calculated by Stuchtey differ from loci of constant radiance only as a result of variation of the reflection coefficient with arc distance from the source. At least at low source elevations, crystals oscillating about a horizontal equilibrium position are more efficient as pillar producers than those rotating about horizontal C₆ axes, but there is no conclusive evidence that the latter never produce pillars. It is shown that the variation of the cross-sectional area of the incident beam has important effects, such as production of a minimum radiance at the source. Neglect of this factor by recent workers casts doubt on some of their conclusions.

No evidence is found to suggest that the classical models need replacement; tentative suggestions are made that diffraction, transfer effects, and modes other than external reflection in crystals rotating about horizontal C₆ axes may enhance the formation of pillars at higher source elevations.

1. Introduction

The sunpillar is one of the halo phenomena (chionisms), which are produced by refraction and/or reflection of light by the ice crystals that compose atmospheric cirrus clouds. The phenomenon takes the form of a vertical column of light through the source, and is seen only when the latter is at a low elevation.

The colour of the pillar is observed to be identical to that of the source, and the generally accepted explanation is in terms of external reflection from plane crystal faces. As to the faces involved, there are two possibilities. The first is that the hexagonal faces of plate crystals, oscillating about their equilibrium position with the six-fold symmetry axes of the crystal lattice (henceforth simply 'C₆ axes': they are identical to the normals to the hexagonal cross-section) vertical, are involved. The second, mentioned by Minnaert (1952), is that the rectangular faces of columnar crystals, rotating about horizontal C₆ axes, are responsible.

These classical models have come in for some criticism. It has often been remarked that the first, at least, seems incapable of accounting for the fact that the sunpillar is observed more above the sun than below. The difficulty of producing pillars at higher source elevations by this model is also often referred to. None of the authors who note this give specific references to such observations, which makes this point almost impossible to investigate. The second model, attributed above to Minnaert, seems to have been introduced in response to these difficulties, but does not appear to have been generally regarded as successful.

A difficulty of rather a different kind has been put to the author (Bignell, private communication 1972). It was suggested that neither of the classical models is capable of accounting for the narrowness of certain sunpillars, which are hardly greater in width than the solar disc, and have relatively sharp edges.

It seemed to the author that all the difficulties in deciding whether the two models were capable of representing the observed features of sunpillars, arose from the fact that their consequences had never been analysed in detail, and, in particular, that the radiance distributions they predict had not been determined. The determination of the radiance distribution was therefore set as the aim of the investigation described in the present paper.

At the time of carrying out this research, the author was unaware of similar work by Greenler et al. (1972). The most advanced quantitative theory of the sunpillar to hand was
that of Stuchtey (1919), who has calculated the loci of illuminated points produced by crystals having the same tip angle (Fig. 1). These have usually been regarded as curves of constant radiance, which seems reasonable for a distribution of crystals sharply peaked at zero tip angle, but less clearly true for a more slowly varying distribution.

2. MATHEMATICAL THEORY

To obtain a formula for sunpillar radiance it is most convenient to proceed from the general formula for halo sterisent (flux per unit solid angle per unit scattering volume), Eq. (4.6) of White (1978):

\[
(JN/4\pi^2) \sum_i c_i T_i W_i |\delta(\eta, \rho)| |\delta(\tilde{\mu}, \alpha)| .
\]

(1)

where \( J \) is the point source flux density, \( N \) the number of crystals per unit volume, and \( \tilde{\mu}, \alpha \) are respectively the sine of the polar distance and the azimuthal angle of the point of observation in an arbitrary system of Eulerian angles. The quantities \( \eta, \rho \) are the two parameters required to specify the orientation of a body with two degrees of rotational freedom, so that area in \( (\eta, \rho) \) space is equal to \((1/4\pi) \times \) probability.

Even in the case of our first model, there are effectively two degrees of rotational freedom. Although the crystal may have a third rotation about its tilted \( C_6 \) axis (in addition to rotation about a vertical axis and tipping), this does not affect the sunpillar.

In general, there is more than one value of \( (\eta, \rho) \) corresponding to a given value of \( (\tilde{\mu}, \alpha) \), and we denote the branches by the integer subscripts \( i \). The quantity \( c \) is the normal cross-sectional area at incidence of the beam that undergoes the required mode of scattering (sequence of refractions and reflections in the ice crystal) – in the present case, just a single external reflection at one of the faces we have designated. The symbol \( T \) denotes the transmission coefficient and \( W \) is the unity-normalized distribution of crystals. The quantities \( c, T, W \) are single-valued functions of \( (\eta, \rho) \), and hence the branches \( c_i, T_i, W_i \) of these quantities as functions of \( \tilde{\mu}, \alpha \) are in one-to-one correspondence with those of \( (\eta, \rho) \).

The physical interpretation of the Jacobian in Eq. (1) is that it represents the ratio of an area of \( (\eta, \rho) \) space to the corresponding area of \( (\tilde{\mu}, \alpha) \) space (which is carried into the celestial sphere by an equal area projection).

To convert the sterisent expression (1) into radiance (radiant power per unit solid
angle per unit area, for which the term 'intensity' was formerly used), we multiply by the path length \( p \) through the cloud in the direction of observation. An equivalent procedure is to divide by the extinction cross-section, and multiply by the optical depth through the cloud in the direction of observation. This is strictly valid only for small optical depth, where only single scattering is important. However, the scattering phase function is found by division of Eq. (1) by the extinction cross-section, and normalization to the albedo for this mode. We can generalize to take account of a spectrum of particle size and shape by distribution-weighted integration over every relevant parameter.

For definiteness, we shall take \( \mu \) as the sine of elevation of the point of observation, and \( \alpha \) as the azimuth difference of this point and a point source at elevation \( \zeta \). Further, \( \eta \) is taken as the azimuth difference of the source and the horizontal axis about which the crystal is free to rotate, and the parameter of this rotation is \( \rho \). Generally we shall have \( W = W(\rho) \) at most.

Let a unit vector in the direction of the source be

\[
\begin{bmatrix}
\cos \gamma \sin \beta, \\
\cos \gamma \cos \beta, \\
\sin \gamma
\end{bmatrix}
\]  

in a frame fixed in the crystal, and in this frame let the unit vector in the direction of the virtual image be

\[
\{\cos \gamma' \sin(2\delta' - \beta'), \cos \gamma' \cos(2\delta' - \beta'), \sin \gamma\}'.
\]

We shall interpret the \( z \) axis of this system as the horizontal axis of rotation of the crystal, and thus for both the proposed sunpillar models

\[ \gamma' = \gamma. \]

Such a law holds when all the faces at which refraction/reflection takes place form parts of the faces of a right polygonal prism whose axis is the \( z \) axis, the total number of internal reflections at polygonal faces is even, and entry and exit occur either both at polygonal faces, or both at rectangular faces (class 1 modes). If the latter, we may interpret \( 2\delta' \) as the angle between the entry and exit faces, and \( \beta \) and \( \beta' \) as the angles of incidence and emergence, respectively (Fig. 2), of the projections of rays into the principal section (the \( x-y \) plane). If an internal reflection at a polygonal face is added, Eq. (4) is replaced by

\[ \gamma' = -\gamma. \]

It is shown by White (1976) that for these cases

\[ \partial(\mu, \alpha)/\partial(\eta, \rho) = -\cos^2 \psi + \cos^2 \zeta \mp 2 \cos \zeta \cos \psi \cos 2\theta \{1 + (\partial \beta/\partial \delta')_\tau\}, \]

where the upper sign corresponds to Eq. (4), and the lower to Eq. (5). For both our sunpillar

![Figure 2](image)  

Illustrating the definitions of \( \beta \), the angle of incidence at the entry face, \( \beta' \), the angle of emergence at the exit face, and \( \delta' \), the half angle between these faces. This figure is drawn in projection into the principal section.
models $\beta = \beta', 2\delta' = \pi, c = a\sin\frac{1}{2}\theta$, where $\theta$ is the phase angle and $a$ the total area of the two hexagonal faces in the first model, and of the six rectangular faces in the second. This takes account of all faces that can contribute. Thus, by a standard property of Jacobians, the sterinsent is

$$\left( JA/8\pi^2 \right) W(\rho) T(\theta)(\cos^2\psi + \cos^2\zeta - 2\cos\psi \cos\zeta \cos\alpha)^{-1}\sin\frac{1}{4}\theta,$$

where $A$ is the total area of hexagonal, or rectangular, faces per unit volume, according to which model is used. The introduction of $A$ takes account of a spectrum of particle size and shape.

Let us return now to the forms of the loci of constant tip angle plotted by Stuchtay (1919). Let $\phi$ be the angle between the plane containing the directions of the source and the virtual image, and the plane of the source vertical. We have left the definition of $\rho$ sufficiently ambiguous for us now to interpret it as the angle made by the outward normal to the reflecting face with the upward vertical, which is identical in magnitude to the tip angle, and we obtain for the equation of these curves $\cos\rho = \cos\phi \cos\zeta \cos\frac{1}{2}\theta + \sin\zeta \sin\frac{1}{2}\theta$, or, since $\sin\psi = \sin\zeta \cos\theta - \cos\phi \sin\theta \cos\zeta$, $2 \cos\rho \sin\frac{1}{4}\theta = \sin\zeta - \sin\psi$.

Returning our attention to Eq. (6), we see by a little manipulation that

$$\left( JA/16\pi^2 \right) W(\rho) T(\theta) |\cos\rho|.$$

We see that it is only the variation of $T$, in this case simply a reflection coefficient, that causes any divergence of the loci of constant radiance from the Stuchtay curves.

We may rationalize (7) intuitively in terms of the well-known result (van de Hulst 1957) that external reflection from any randomly oriented convex particles produces a distribution of scattered radiation that differs from the isotropic only by the presence, as a factor, of a reflection coefficient $T(\theta)$. The particles we are concerned with are not randomly oriented: they have a distribution proportional to $W(\rho)|\cos\rho|$, hence the form of Eq. (7). The numerical factor also checks out by this method, using Sec. 8.41 of van de Hulst.

3. The Distribution

In the case of our second model, employing crystals with a horizontal $C_4$ axis, it seems likely that there will be, at most, slightly greater stability with a pair of opposite rectangular faces horizontal, and we can take $W(\rho) \equiv 1$ to a good approximation.

For our first model, this is a more difficult question. Two physical mechanisms can account for the oscillation. Very small crystals oscillate as a result of Brownian motion. Large crystals oscillate because the Reynolds number is high, and a turbulent wake is generated behind each crystal as it falls. Unfortunately, insufficient seems to be known of the aerodynamics of ice particles to enable the distribution to be deduced in either case (even for Brownian motion, a solution of the aerodynamic problem is necessary).

We therefore resorted to statistical hypotheses, and considered distributions proportional to $\exp(-k\rho^2)$. We may note two cases in which this is precisely the correct form. It is so for Brownian motion when the restoring force is such as would produce simple harmonic oscillations (Einstein 1956), at least if the particles are all the same. The second hypothesis leading to this form is an exponentially decaying distribution of crystals with respect to the total energy of simple harmonic oscillations, provided $a$ does not vary with the energy. The variation of the pillar with $k$ is studied in this paper: in any particular instance of either of these two special cases, knowledge of the dynamics is necessary to determine its value.
4. Numerical results and discussion

In computation of numerical values, the quantity $T$, the reflection coefficient for unpolarized light, was readily found by Fresnel’s laws of reflection. The correction for finite extent of the source disc is made numerically, by integration. In Figs. 3, 4, 5 are plotted respectively the sunpillars for $k = 0, 10, 100$, for zero source elevation, in the form of radiance contour maps. These maps, and Figs. 6, 7 are drawn on a simple cylindrical projection: away from the horizon this slightly exaggerates the horizontal as compared with the vertical scale.

We see that the classical theories do indeed predict quite sharp edges of the pillar near the source: only farther away does it become more diffuse: this is in good agreement with observation. There does not appear to be a great deal of variation in the sharp-edged region of the pillar with $k$, but the diffuse region becomes less extensive as $k$ increases. For small $k$, the diffuse region is an ‘outer’ rather than an ‘upper’ region, and the curves of constant radiance there closely match those for a point source.

It is evident, then, that the pillar with sharp edges and of the same width as the source arises as a consequence of the finite extent of the source. It is also essential, as we see from Figs. 6, 7, that the source be at a low elevation. These are for an elevation of $5^\circ$, for which, as we see, a pillar is still formed, but it no longer has sharp edges. Increase of $k$ from zero gives a stronger pillar.

The reason for this behaviour seems likely to be as follows. A point source at the horizon is coincident with the undersource. For a point source, radiance is singular at the undersource. With the source and undersource coincident at the horizon, the radiance rises continuously to a singularity as this point is approached from above or below, but for approach from the sides radiance is uniformly bounded until the point is actually reached. This may be associated with the production of the sharp edges of the pillar for a source of finite size. Indeed, with an elevated point source, radiance rises continuously to a singularity at the undersource for all directions of approach, and a pillar with sharp edges is not produced for a finite source sufficiently elevated.

An interesting feature of our plots is the presence of a minimum radiance of the pillar at the source. There is a shallow minimum even with the pillar at the horizon, and for $5^\circ$ source elevation the radiance drops to a small fraction of its value above and below the source. We can understand why this happens if we recognize that near a point source, the Stuchtley curves tend to curves of constant radiance. Hence the width of the point source sunpillar is zero here, and the averaging process involved in correction for the finite size of the source disc produces a nearly zero radiance as compared with other points. The matter is less clear for a source at the horizon, since although the width of the pillar is zero at the source, the radiance is singular there. In fact, as we have seen, the net result of these two opposing effects is that a minimum is still produced, though it is inevitably much shallower than would be the case with an elevated source.

This is true, at least, in all but a vertical strip two or three minutes of arc in width extending above the source centre for some $4^\circ$, where the radiance seems to rise again to very high values. It is thought that this feature is spurious, probably a consequence of the use of what is in effect a generalized trapezoidal rule for integration near a singularity. It is likely to be the same cause that produces rather erratic values near the source: affected areas are left blank in the plots presented.

With a source at considerably greater elevations than $5^\circ$, we can get a maximum radiance at the source due to weakening of the pillar structure as compared with the radial decay of radiance away from the source produced by the reflection coefficient $T(\theta)$, provided $k$ is sufficiently small.
Figure 3. Sunpillar for zero source elevation, and $k = 0$. Hence this is a pillar produced by reflection by crystals rotating about horizontal $C_d$ axes. Loci of constant radiance are at interval $25J4p/132\pi^2$. The most flared loci have radiance unity on this scale. Height of figure $10^\circ$, width 2°.

Figure 4. As Fig. 3, with $k = 10$, and units twice those in Fig. 3. This pillar is produced by plate crystals oscillating about their equilibrium position. Height of figure $64^\circ$, width 2°.

Figure 5. As Fig. 3, with $k = 100$, and units eighty times those in Fig. 3. This is again produced by oscillating plate crystals, but with a distribution more sharply peaked at zero tip than that for $k = 10$. Height of figure $10^\circ$, width 2°.

Figure 6. As Fig. 3, for a source at 5° elevation. Height of each section $10^\circ$, width 2°. Section at left above horizon, that at right below. Interval increased tenfold after the first ten contours from zero: fifteen contours shown.

Figure 7. As Fig. 6, with $k = 10$, and units 8/5 those in Fig. 3. Height of each section $94^\circ$, width 2°.

**Note**

Figures 3–7, as reproduced, are to be viewed from the right, and the terms 'left', 'right', 'height', 'width', etc., refer to this sense of viewing.
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In the case of crystals free to rotate about horizontal C\textsubscript{6} axes, there is another mechanism which might increase the sunpillar radiance at the position of the source, and that is through class 1 modes other than simple external reflection. Radiance plots for this case indicate that the sharp edges of the pillar are retained for sufficiently small source elevations. No such mechanism can operate in the case in which plate crystals oscillate about their equilibrium position, and the local minimum pillar radiance at the source must be observed, according to geometric optics.

Do real sunpillars show this minimum? It is not easy to make observations near the source, but no instance has been found in which the minimum is definitely absent. The balance of probability is that sunpillars result largely from external reflection. They are seen in falling snow, in a way that other haloes are not. As remarked by Humphreys (1964), inclusions in snow crystals probably prevent them producing haloes involving transmission.

One feature of many real sunpillars that our theory does not seem to represent is that they are observed mainly above the sun. Minnaert (1952) refers to this difficulty of classical sunpillar theory, and remarks on the possibility that it may be due to the increased thickness of cloud through which one looks in a more nearly horizontal direction. However, he was unable to decide whether this would be an advantage or a disadvantage. In fact, the question is capable of quantitative resolution (White 1973, 1978). Initially there is an increase of radiance with cloud thickness, subsequently a decrease. Note that the formulae developed by White relate strictly only to the case of randomly oriented particles. In the present case, analysis shows that this mechanism may be slightly less effective if crystals only of the class producing the pillar are present, but the presence of other scattering/absorbing matter can compensate for this.

A further point which must be taken into consideration is that the sun is often so low when pillar observations are made that obstructions render the sky below the sun invisible anyway. On an occasion when conditions were favourable (Ripley and Saugier 1971), the pillar below the sun was brighter than above: indeed, the latter was not observed.

No attempt to deal with the transfer problem has been thought worth while in the present paper, in view of the introduction of additional parameters, but one qualitative result may be noted. Primary scattering from those reflecting faces that are fortuitously horizontal produces an image of the source reflected in the plane of the horizon. Secondary scattering of this radiation results in a reflection of the pillar in the horizon. Thus multiple scattering may have some ability to enhance the brightness of the pillar near and above the source, and at higher source elevations.

As k increases, we see from the plots that the pillar becomes brighter and less flared for a given optical depth (assuming an extinction cross-section proportional to a, a fair approximation for thin plates or long columns) for the relevant source elevations. The optical depth is, roughly speaking, a measure of the visibility of the cloud by its own scattering, and the obscuration of bodies behind it. But for oriented particles, the extinction cross-section is anisotropic, and in the limit $k \to \infty$ is given by $a \sin \zeta$, within our ‘fair approximation’. Thus the pillar is much brighter for a cloud of given ‘visibility’ than would be the case for a halo phenomenon produced by more nearly randomly oriented particles, since $\zeta$ is small. Crystals rotating about horizontal C\textsubscript{6} axes have $k = 0$ and are thus less efficient as pillar producers than plates oscillating about a horizontal equilibrium position. There is certainly insufficient evidence to suggest that the former never give pillars: the simultaneous observability of other haloes, either tangent arcs or parhelia, may give some clue, but is unlikely to be conclusive.

We must note other recent work on theoretical modelling of the sunpillar. Greenler et al. (1972) have developed a numerical model, using a distribution $W(\rho) \propto H(|\rho_o| - |\rho|)$, where $H$ denotes the usual Heaviside unit function, for oscillating plate crystals, and $W(\rho) \equiv 1$,
as in the present paper, for rotating columnar crystals. In the latter case, they find a maximum sunpillar radiance at the source, even at quite large source elevations, and claim that these crystals are capable of producing short pillars through a source at high elevations. They take no account of the factor $c$ which, as we have shown here, has the effect of producing a minimum radiance at the source. Thus their conclusion would appear not to stand. It could be claimed, though they do not do so, that near the source their model approximates the effect of modes other than external reflection in the columnar crystals, but even then the effect of externally reflected light can hardly be ignored.

A theory of sunpillar radiance has also recently been developed by G. J. Thompson and A. B. Fraser, though only one radiance plot (Thompson 1978) appears to have been published. Not only have these workers, like Greenler et al., neglected the factor $c$, but they have also made an error in the formal solution of the problem. This means that Thompson's radiance plot is unreliable.

5. DIFFRACTION EFFECTS

So far we have considered only geometric optics. It was put to the author at the commencement of this research that diffraction by plate crystals with the $C_6$ axes vertical (no oscillation) was capable of giving a sunpillar with perfectly sharp edges, and of the same width as the source, at least when the latter is at the horizon. To see that this is not the case, consider the diffraction by that part of the crystal shown solid in Fig. 8. It is that light which is scattered in the same manner as if this 'block' of the crystal extended to infinity in both directions which is capable of giving the desired diffraction effects. The method of solution of the general problem of diffraction by such an 'extended block' is well known to be that of seeking solutions of Maxwell's equations proportional to $e^{ikz}$ where $z$ is a coordinate along the extended block. Such solutions lead to a law of the form of Eq. (4), when expression (2) is a unit vector in the direction of the source, and (3) is a unit vector in the direction reciprocal to that of scattering. The $x$ and $y$ axes need not be specified for the present purpose.

Thus the radiation scattered by crystals with the $z$ axes so defined, directed towards a

![Figure 8](image)

Figure 8. Light diffracted by the 'block' of the crystal, shown by solid lines, may be capable of producing sunpillar effects. The manner of extension of the block to infinity referred to in the text is indicated by the dash-dot lines.
given point on the horizon, is to be found on a small circle centred at that point and passing through the source. This is exactly the same as the locus on which we should find the geometric optics scattered radiation due to a crystal with the $z$ axis so directed, but with an extra degree of rotational freedom about this axis. Thus we have disproved the proposition that diffraction from crystals with one degree of rotational freedom is capable of giving a sunpillar equal in width to the source. For the proposition to have been true, we should have required the diffracted radiation to be found on a vertical great circle through the source for all azimuths of the $z$ axis. There is no reason to expect the pillar produced by diffraction from crystals with one degree of rotational freedom to have substantially sharper edges than that produced by reflection by crystals with two degrees of rotational freedom.

A similar argument may be constructed for columnar crystals with horizontal $C_3$ axes, which we take as the $z$ axes. Indeed, here it is clearer that the diffraction will be represented by a solution proportional to $e^{i\mu}$, with only minor 'end effects', whether or not the crystal has rotational freedom about the $z$ axis, and Eq. (4) is obeyed.

In the case of oscillating plate crystals, the pillar produced by diffraction will be more diffuse than that due to reflection. Diffraction does offer a process whereby plate crystals can produce a pillar with maximum intensity at the source when the latter is substantially above the horizon. In view of the preceding remark, it may be best to invoke the absence of oscillation. A question which cannot be regarded as resolved is whether a diffraction model can represent the absence of colour from sunpillars.

**Note added in proof**

Although internal reflections may contribute to the pillar produced by plate crystals oscillating about the horizontal position, the minimum radiance at the source is maintained in this case.

**References**


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