Energetics of diabatic mesoscale circulation:  
A numerical study

By G. A. DALU  
Istituto di Fisica dell'Atmosfera  
CNR, Roma

and  
J. S. A. GREEN  
Atmospheric Physics  
Imperial College, London

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SUMMARY

Mesoscale circulations, induced by mesoscale available potential energy diabatically generated in the boundary layer, are studied through the numerical integration of primitive equations. The energetics of a sea-breeze simulation is studied in some detail. After 12 hours of simulation the kinetic energy is about a quarter of the mesoscale available potential energy diabatically produced. We comment on the energy lost by the upstream difference scheme, and on some aspects of the unavailability of available potential energy in such a system that together describe this level of inefficiency.

1. INTRODUCTION

Green and Dalu (1980) explored how convective available potential energy CAPE and mesoscale available potential energy MAPE are diabatically generated in the boundary layer. Here we see what actually happens in a conventional numerical model of a sea-breeze circulation, intended to simulate the coastal zone from Rome to the Appenines. Energy is lost by the numerical scheme, and only a small fraction of the energy available on the mesoscale is actually converted into kinetic energy. We examine the accuracy with which the numerical scheme portrays the energy transformations, observing that the criterion for one-step accuracy is not the same as that for integration out to a given time. We also examine physical reasons for the inability of the circulation to use the potential energy supplied to it. This we do by means of a chain of four efficiencies, two numerical and two physical, which connect the model kinetic energy to the idealized input of potential energy.

2. EQUATIONS OF MOTION AND ENERGY INTEGRALS

The x-axis is perpendicular to the land–sea boundary, y-axis parallel, z-axis vertically up. Motion is assumed independent of the y-coordinate.

In the Boussinesq equations with hydrostatic pressure and constant density, appropriate for a shallow layer (Charney 1973), $\phi$ is the log-potential temperature and $\delta p$ is the deviation of the pressure from that in a standard atmosphere. The entropy source $D\phi/Dt = Q$ drives the motion and is assumed to be a known function (see Dalu 1978).

\[
\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} + w \frac{\partial \phi}{\partial z} = \frac{D\phi}{Dt} = Q \quad .
\]
(1)

\[
\frac{\partial}{\partial z} \left( \frac{\delta p}{\rho} \right) = g \phi \quad .
\]
(2)

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} -fv + \frac{\partial}{\partial x} \left( \frac{\delta p}{\rho} \right) = 0 \quad .
\]
(3)

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + w \frac{\partial v}{\partial z} + fu = 0 \quad .
\]
(4)
\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad . \quad . \quad . \quad (5)
\]

Rigid horizontal boundaries are assumed at \( z = 0 \) and \( z = H \) which imposes constraints on the motion that allow the pressure field to be deduced. Integrating Eq. (5) from \( z = 0 \) to \( z = H \) shows that

\[
\int_0^H \frac{\partial u}{\partial x} dz = 0 \quad . \quad . \quad . \quad (6)
\]

at all times.

Thus integrating \( \partial \phi / \partial x \) of Eq. (3) from \( z = 0 \) to \( z = H \) gives a diagnostic equation for the pressure

\[
\frac{\partial^2}{\partial x^2} \int_0^H \frac{\delta p}{\rho} dz = \int_0^H \frac{\partial}{\partial x} \left( u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} - f v \right) dz \quad . \quad . \quad . \quad (7)
\]

This equation is easier to use in centred difference form than the first integral with respect to \( x \). The set (1) to (7) can be integrated in time, as in Dalu (1978).

When the energy equation

\[
\frac{\partial}{\partial t} \left\{ \frac{1}{2}(u^2 + v^2) \right\} + \left( u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z} \right) \left( \frac{1}{2}(u^2 + v^2) + \frac{\delta p}{\rho} \right) - g w \phi = 0 \quad . \quad . \quad . \quad (8)
\]

is integrated over a volume bounded by rigid walls, the work done by the pressure and the advection of kinetic energy vanish. In each case their role is to redistribute energy, not to create it. There remains an equation for mechanical energy:

\[
\frac{d}{dt} \left\{ \frac{1}{2}(u^2 + v^2) \right\} = g \overline{w \phi} \quad . \quad . \quad . \quad (9)
\]

where

\[
\mathcal{S} = \int_{z=0}^{z=H} dz \int_{x=0}^{x=L} S(x,z,t) dx
\]

where \( d/dt \) denotes the ordinary rate of change with time and the overbar will be used throughout to denote the integral with respect to volume in Eq. (9).

Equation (9) expresses the conversion of potential energy into kinetic energy. It represents a mechanical effect in the sense that only the dynamical Equations (2)–(5) have been used so far, not the thermal Eq. (1).

Using this thermal equation allows the rate of release of potential energy to be written.

\[
g \overline{w \phi} = g \phi \frac{Dz}{Dt} = g \frac{D}{Dt}(z \phi) - g \overline{z Q} = \frac{d}{dt}(g \overline{z \phi}) - g \overline{z Q} \quad . \quad . \quad . \quad (10)
\]

The r.h.s. of Eq. (10) represents the difference between the diabatically produced potential energy (second term on right) and the potential energy remaining in the system; notice that \( -g \overline{z \phi} \) is the potential energy, as in Green (1970). When the flow is adiabatic (10) is equivalent to Lorenz's form: Lorenz (1955).
3. Numerical scheme

Details of the scheme are as in Dalu (1978). It is similar to those used by Neumann and Marher (1971), Pielke (1974) and Pielke and Marher (1978). Prognostic equations are integrated forwards in time using upstream differences and linear interpolation.

Small-scale convection is treated, as in Dalu (1978), by supposing that energy supplied at the surface during the new time-step is uniformly redistributed through an isentropic boundary layer up to some finite height determined by the buoyancy and the ambient field of temperature. Entrainment with a constant coefficient is allowed. The numerical equivalent of the balance-equation (7) is used to calculate the height-integral of \( \delta p/\rho \) using velocities of the \textit{old} time-step. Pressure is calculated everywhere using the field of entropy diabatically perturbed. This pressure and mass continuity are used to produce velocity at the new time-step. Finally the entropy is advected by the new velocity field. The scheme does not necessarily satisfy conservation integrals, particularly Eqs. (9) and (10) of the continuous set.

Using this numerical scheme we have integrated the equations for an idealized heat source typical of a sea-breeze situation. This is as described in Dalu (1978) and is basically that 0·1 of the solar flux is converted into sensible energy but only on the 'land' part of the system. There the flux of energy at the surface is \( 0·1S \cos 42^{\circ} \sin (2\pi t/24) \), \( 0 \leq t \leq 12 \) hr. The entrainment factor (\( r \) of Green and Dalu 1980) is held constant at the value \( r = 0·2 \). Realistic flow fields, like those shown in Fig. 1, result.

4. Numerical efficiency

We choose to examine separately the two advection processes represented by Equations (1) and (3)+(4). Figure 2(a) shows the ratio
calculated from the model. Over the time when substantial energy has been supplied (12 to 18 h, say) the numerical scheme consistently loses some 20 to 30% of the mechanical energy that it has already represented numerically.

To obtain Eq. (10) from Eq. (8) the terms representing the advection of $\frac{1}{2}v^2$ and $\delta p/\rho$ in the numerical scheme must vanish when integrated over the whole system. The numerical value of both these integrals is found to be very small.

Figure 2(b) shows the ratio suggested by Eq. (10)

$$F_2 = g \int_0^t \omega \varphi \, dt / \left( g \left[ \frac{\partial \varphi}{\partial z} \right] - g \int_0^t zQ \, dt \right)$$

where now

$$\frac{1}{2}v^2 = F_1 F_2 \left( g \left[ \frac{\partial \varphi}{\partial z} \right] - g \int_0^t zQ \, dt \right)$$

Figure 2. Numerical efficiency of integration scheme. (a) $F_1$ as defined by Eq. (11): mechanical conversion of potential into kinetic energy. (b) $F_2$ as defined by Eq. (12): conversion of thermal potential into mechanical potential energy.
Again the numerical scheme has an efficiency of only 70 to 80% in the energetic part of the day. In both of the steps represented by \( F_1 \) and \( F_2 \) energy is lost on its way from thermal to kinetic. In the energetic part of the day, both efficiencies become smaller, i.e. the integration gets worse, as the time step is decreased, which suggests that spatial rather than temporal truncation is responsible for the inaccuracy.

Figure (2) shows that the ratio \( F_1 \) is generally smaller than \( F_2 \). This is consistent with the different spectral distributions of velocity and temperature. Since heating is discontinuous in space, the temperature field contains high frequency components that are rapidly dissipated by the numerical scheme, as shown by the smallness of \( F_2 \) particularly in the early stages when advection has not yet had a chance to smooth the field of temperature. In contrast, acceleration is redistributed through the system by the pressure force. The field of velocity is correspondingly smoother, of larger spatial scale, hence dissipated less.

How the numerical scheme treats the propagation of waves can be described through the integration of the 'shallow water' equations:

\[
\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = 0 \quad (14),
\]
\[
\frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + \frac{c^2}{g} \frac{\partial u}{\partial x} = 0 \quad . \quad (15)
\]

where the depth \( h \) should be identified with the temperature in our set. The integration scheme uses \( h \) (i.e. \( \phi \)) at the current time to forecast \( u \), and the new updated value of \( u \) to calculate \( h \). This implicit character of the scheme causes wave propagation to be stable, and not heavily damped when the C.F.L. condition \( c \Delta t \leq \Delta x \) is satisfied. Thus the solution can be written \( h = a \exp(i \kappa x) \).

The amplitude \( a \) depends on the three parameters \( \mu = U \Delta t / \Delta x, \quad m = c \Delta t / \Delta x \) and \( \theta = k \Delta x \) and is given by

\[
[a^2 + \mu (1 - \exp(-i \theta))]^2 + a (2m \sin \frac{\theta}{2})^2 = 0 \quad . \quad (16)
\]

The limit \( \theta^2 \ll 1 \), gives

\[
|a|^{\Delta t} = \exp - \frac{1}{2} (1 - \mu \pm m) \theta^2 \quad U \Delta x \quad . \quad (17)
\]

for the amplitude. We notice that dependency on the length of the time-step has nearly disappeared, leaving the advection distance \( U \Delta t / \Delta x \) as the dominant term. Stability is ensured if \( \mu + m < 1 \) and good energy conservation demands \( \theta^2 \ll 1 \) at about 60 points per wave-length. Dissipation is a quadratic function of wave length: dissipation for the mode with 30 points per wave-length is 4 times that for 60 points.

In comparing the analytic and numerical values for the energy, we take \( U = 3 \text{ m s}^{-1}, \quad c = 5 \text{ m s}^{-1} \) (i.e. vertical wave-length of 3 km in the conditions of Fig. 1) whence for the lowest mode \( \theta \approx 0 \cdot 1, \quad \mu = 0 \cdot 3, \quad m = 0 \cdot 5 \) for a time-step of 10 min. The value taken for \( c \) is consistent with the fact that the computation became unstable for a time-step of 12 min.

Assuming energy split equally between the waves propagating upstream and those propagating downstream, we find that the energy in the lowest mode is decreased by the factor 0.84 after 12 h. The next mode at \( \theta = 0 \cdot 4 \) retains 0.56. These figures are consistent with the spectral distribution of energy and the values of \( F_1 \) and \( F_2 \) of Fig. 2. We note that the finite difference scheme leading to Eq. (17) has made a Galilean transformation in the bracketed term, \( \mu \) being replaced by \( \mu \pm m \), but not in the other term where we find \( U \) and not \( U \pm c \). The physical system transforms \( U \) into \( U \pm c \), as does the phase speed represented by the solution for the complex \( a \) in Eq. (16).

If Eqs. (14) and (15) are treated similarly instead of being staggered in time, then \( U \) is replaced by \( U \pm c \) throughout and the resulting scheme is numerically unstable for \( c > U \).
5. PHYSICAL EFFICIENCY

Green and Dalu (1980) identified the Mesoscale Available Potential Energy:

\[ \text{MAPE} = \frac{g}{6} \left( \frac{\partial \phi}{\partial z} \right)^{-1} \frac{D(L-D)}{L} \left\{ \int_0^t Q \, dt \right\} \left\{ \frac{r^3 + (1 + r)^3}{(r + \frac{1}{r})^2} \right\} \]  \hspace{1cm} (18)

This is the potential energy that would be available if all the energy supplied went into creating an isentropic boundary layer (capped by a suitable inversion) then all that energy was released in the subsequent motion.

We now define the ratio

\[ E_1 = g \left[ \frac{\int zQ \, dt}{\text{MAPE}} \right] \]  \hspace{1cm} (19)

so

\[ \frac{1}{2} \bar{v}^2 = F_1 F_2 E_1 \text{MAPE}. \]

If the numerical method had retained all the energy (i.e. if \( F_1 = F_2 = 1 \)) then \( E_1 \) would measure the total kinetic energy that the actual mesoscale motion managed to obtain compared to the idealized (not necessarily maximum) represented by MAPE. We therefore regard \( E_1 \) as a measure of the physical efficiency.

Figure 3(a) shows that the sea breeze is very inefficient at using the energy supplied to it on the right side. It is possible that this potential energy is used by some other mesoscale motion. Processes omitted from the model are those representing along-coast variations. These could take the form of a lobed breeze unequally distributed along the coast and might resemble baroclinic waves of very short (100 km) wave-length. Neglect of the onshore component of the wind is a serious qualification but places where the mean wind blows parallel to the coast (perhaps the north African coast) might be examined for synoptic evidence of such a phenomenon.

We can speculate on possible reasons for the inefficiency. It is clear that it is not the attainment of thermal-wind balance that inhibits the release of potential energy. Alongshore velocity differences of \( \pm 3 \text{ m s}^{-1} \) are obtained consistent with \( f = 10^{-8} \text{ s}^{-1} \) and transverse displacements of 60 km. Temperature differences of a few K are spread over a zone some 50 km side so

\[ \frac{g}{f} \frac{\partial \phi}{\partial x} \left( f \frac{\partial v}{\partial z} \right) \sim 6. \]  \hspace{1cm} (20)

and the temperature field is not in balance with the thermal wind.

One reason for the physical inefficiency represented by \( E_1 \) is that the system has not had time to rearrange itself to a state of minimum potential energy. The response time of the mesoscale motions is given by the distance travelled by the cold air divided by its velocity:

\[ D(2 \text{ MAPE}/LH)^{-1} = \frac{1}{2} D/v \sim 75 \text{ km}/(\text{say } 2 \text{ m s}^{-1}) \sim 8 \text{ h} \]

which is indeed comparable with the growth time of the boundary layer.

In the Rome–Appenines calculation it is the presence of the inland–wall that determines \( D \), and it is its presence that inhibits the release of potential energy. The potential energy so far unconverted into kinetic energy at a given time can be expressed as:

\[ g(\zeta - z) \phi \]  \hspace{1cm} (21)

where \( \xi(x,z,t) \) is the height that each particle would be found at, if the total potential energy of the system were minimized consistent with mass continuity. The value of \( \xi \) can be found merely by listing the particles in order of increasing potential temperature; \( \zeta - z \) is a virtual displacement.

Up to this time the energy converted is, according to Eq. (10)
Figure 3. Physical efficiency. (a) $E_1$ defined by Eq. (19): the ratio of the potential energy released by the model to that defined by the idealized processes represented by MAPE. (b) efficiency of feedback between mesoscale advection and the development of the convective boundary layer: $E_2$ defined by Eq. (24).

\[ g \left( \bar{z}\phi - \int_0^t \bar{z}Q \, dt \right) \]  \hspace{1cm} (22)

so the total energy input is

\[ \{ (21) + (22) \} = g \left( \bar{z}\phi - \int_0^t \bar{z}Q \, dt \right) \]  \hspace{1cm} (23)

The final factor in the chain is

\[ E_2 = \frac{g \left( \bar{z}\phi - \int_0^t \bar{z}Q \, dz \right)}{\text{MAPE}} \]  \hspace{1cm} (24)

plotted in Fig. 3(b). This factor is unity in the absence of motion and represents the feedback between the mesoscale motion and the convection.

Figure 1 shows that the convective boundary layer well inland has been advected
upwards by the mesoscale motion. Subsequent supply of thermal energy gives slightly deeper and more vigorous convection and more MAPE than the simple static case of Green and Dalu (1980). In contrast near the coast Fig. 1 shows that the mesoscale advection of cool air confines the convection to a shallow layer with less production of MAPE than the simple case.

Thus the mesoscale motion feeds back on the convection – to their mutual advantage in the updraught region and disadvantage in the downdraught region. The most interesting feature of $E_2$ is the accurate compensation of this feedback.

6. CONCLUSION

We have explored one way in which the energetics of a simple system can be examined. Energy loss in a conventional forward-in-time upstream-in-space integration scheme represented by $F_1 F_2$ cannot be ignored. Only a small fraction $E_1$ of the available energy is used. Feedback between the mesoscale and the convection, represented by $E_2$ is negligible when integrated over the whole system.

It is clear that better numerical methods are needed in order to clarify some problems. The sea-breeze front is smoothed out and integration through the night hours too heavily damped in these calculations to be useful. In spite of the poor showing of the 'upstream' finite difference scheme, the circulation patterns are plausible and the calculations raise interesting problems about the mechanism for release of potential energy on the mesoscale.

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