On stochastic perturbation and long-term climate behaviour

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Summary

A very simple energy-balance equation for the mean surface temperature of the planet (Budyko-type model) is perturbed by a stochastic term, simulating the numerous ‘weather’ processes not considered in the deterministic feedback term. In the first instance, a symmetrical deterministic process with three equilibrium states is considered. Using standard methods for determination of the probability density distribution, the characteristic time which the climate system spends in the neighbourhood of one stable equilibrium before shifting to the other is expressed in terms of the characteristic time of the feedback process and the time constant of the perturbing process. Realistic corresponding values are found. An Appendix discusses the possibility of using the analogy between stochastically perturbed Newtonian and quantum-dynamical systems in research on climate change.

I. Introduction

Although the earth’s climatic history is poorly documented, it seems to be established that in the last million years there have been about eight major changes. Mason (1976, see his Fig. 1) describes these changes as passage from a cold to a warm epoch (or vice versa) in a transition of duration orders of magnitude less than the intervening epochs of minor variation or quasi-steady conditions. However, the power spectrum of the data (as exhibited, e.g., in Fig. 7 of Mason’s paper) shows marked peaks at periods of 20 000 and 40 000 years, smaller than the main peak centred at 100 000 years. These periods correspond to the periodicities noted by Milankovitch (1930) in the earth’s orbital parameters, namely changes in the eccentricity (period 96 000 years) obliquity of the axis (40 000 years), and the precession of perihelion (21 000 years). It is, therefore, obviously relevant to examine climatic models to test whether the observed behaviour must be related to external causes, such as orbital changes as in the Milankovitch hypothesis or variations of the solar constant; or whether alternatively, they could arise from the extreme complexity of the interactions within the system, as Lorenz (1968) hypothesized.

Schneider and Dickinson (1974) have reviewed the approach to the study of climate through a hierarchy of models, in which the simpler classes are used to suggest the relative relevance of physical processes, the most important of which can then be treated more explicitly at the next level of complexity. Amongst the simpler classes in the hierarchy is the energy balance model of the type proposed by Budyko (1969) and Sellers (1969). This class of model showed such interesting physical possibilities (e.g. a completely ice-covered earth), that its properties have since been investigated by many authors. The present paper continues this line of enquiry by investigating quantitatively the effect of stochastic perturbations associated with internal causes, following Hasselman’s (1976) suggestion that the effect of the short period weather disturbances could be parametrized in this way. Following Hasselman, many authors have treated this argument, notably Lemke (1977), Robock (1978) and Fraedrich (1978, 1979). Detection of the relevant variations in climatological averages requires consideration of the multiplicity of steady states, which in turn implies consideration of the long-term behaviour of the system. Although it is implicit in the cited papers that finite amplitude stochastic disturbances could drive the system outside the stability domain of a steady state, this possibility has not been quantitatively investigated, either because of
linearization of the equation around the steady state (by Lemke and Fraedrich, for example), or because of the shortness of the simulated period (Robock, for example, considers durations of several hundred years only). In the linearized models the perturbed equation simulates a diffusion process that assigns its maximum likelihood asymptotically around the stable steady states (see section 5 of this paper). Linearized equations will not detect changes in time-averaged statistics associated with periods in which the process is in the neighbourhood of a steady state other than the one near which it started (i.e. behaviour which is 'almost intransitive' in Lorenz’s (1968, 1970) language). Furthermore, such transitions are rare, and their investigation requires durations much longer than those within which deterministic behaviour is relevant – e.g. the relaxation times introduced by stabilizing feed-backs.

The analysis to be put forward in this paper avoids linearization and is not limited to short periods. It shows that a new time constant is introduced by stochastic perturbation, and that the value of this constant is relevant to the understanding of long-term climatic changes and to testing climate models with respect to 'stochastic sensitivity'. In section 2 most of the standard mathematical material to be employed is reviewed. In section 3 a hypothetical energy-balance model of zero geometrical dimensions, formulated in terms of a potential function with a structure fitting the phenomenological description of climate change of Mason (1976), is tested against stochastic perturbations of amplitude appropriate to weather fluctuations. In section 4, in order to study the stochastic sensitivity of a model with a more explicit physical basis, the concept is applied to that of Fraedrich (1979). Finally, section 5 investigates the effect of stochastic perturbation on the bifurcation set for a model with a ‘catastrophe’ pathology (see e.g. Fraedrich 1979).

2. STOCHASTIC EQUATIONS

The major concern of this paper is to analyse the behaviour of the solutions of equations generated by stochastic perturbation of systems of deterministic ordinary differential equations. In this section the basic concept and the methods to be used are outlined. The reader familiar with the technique and terminology may omit this material, retaining only the final results (Eqs. (2.16) and (2.19)).

Let us consider the system of differential equations (or dynamical system):

$$\frac{d}{dt}\{x(t)\} = f(x) \quad \cdots \quad \cdots \quad (2.1)$$

where $x(t) = (x_1, x_2, \ldots, x_n)$ is an $n$-dimensional vector and $f$ is an $n$-component vectorial field.

We can regard the time-dependent solution of this equation as the 'orbit' or 'trajectory' of a point in phase space, the $n$-dimensional space having as reference the Cartesian framework generated by the components of the $x$-vector, and for many purposes it is sufficient to know the behaviour of the trajectory in the neighbourhood of the steady solutions obtained by setting

$$f(x) = 0 \quad \cdots \quad \cdots \quad (2.2)$$

If $x_s$ is a steady solution (i.e. $dx_s/dt = 0$), its stability properties may be determined by studying the sign of the eigenvalues of the Jacobian matrix

$$J = \left( \frac{\partial f_i}{\partial x_j} \right) \quad i, j = 1, 2, \ldots, n \quad \cdots \quad \cdots \quad (2.3)$$
computed on \( x_0 \). If all the eigenvalues have real part < 0, any trajectory starting close to \( x_0 \) will not diverge from another trajectory starting close to \( x_0 \), and furthermore the trajectories will approach the steady state \( x_s \) asymptotically, i.e.

\[
\lim_{t \to \infty} x(t) \to x_s .
\]  

(2.4)

On the other hand, if any eigenvalue has real part > 0, the steady state is unstable and trajectories starting close to it will diverge and will leave its neighbourhood. The set of all the initial conditions for which Eq. (2.4) is verified is termed the attraction domain (or basin or simply set) of \( x_0 \).

Now consider the effect generated by a stochastic perturbation on a trajectory starting from some initial deterministic datum. Denote the stochastic process by \( W(t) \) and its increment between times \( t \) and \( t + \Delta t \) by \( \Delta W \). If \( P \) is the probability density of the process \( W(t) \), \( t > 0 \), the expectation value \( E \) is defined by

\[
E(W) = \int_0^\infty W P \, dt .
\]  

(2.5)

From the class of all processes, we select the Gaussian process \( G(t) \) for which any set \( \{G(t); 0 \leq t \leq T\} \) is Gaussian, and we define a Wiener process as one for which all increments \( \Delta W \) are mutually independent and constitute a Gaussian process. For \( W(0) = 0 \), we have \( E\{W(t + \Delta t) - W(t)\} = 0 \), \( E\{W(t + \Delta t) - W(t)^2 = \Delta t \) and the distribution of \( W(t + \Delta t) - W(t) \) coincides with the distribution of \( \Delta W \) and is Gaussian with zero mean and variance \( \Delta t \).

Consider the simple deterministic difference equation

\[
x(t + \Delta t) - x(t) = f(x(t)) \Delta t .
\]  

(2.6)

and perturb \( x \) by a Wiener process, such that the variance of \( x \) in unit time is \( \varepsilon \), with dimensions of \( x^2 t^{-1} \). Then the expected change of \( x \) in time \( \Delta t \) due to the process \( W(t) \) is \( \varepsilon \Delta t \Delta W \). The equation becomes

\[
x(t + \Delta t) - x(t) = f(x(t)) \Delta t + \varepsilon \Delta W
\]  

(2.7)

and its solution will be a stochastic process \( x(t, x_0) \) where \( x(0) = x_0 \).

Noting that if \( W(t) \) is not differentiable, division by \( \Delta t \) is not allowed, then we find that as \( \Delta t \to 0 \), Eq. (2.7) becomes an "Ito equation"

\[
dx = f(x) \, dt + \varepsilon \, dW
\]  

(2.8)

Equation (2.8) is defined on the real numbers (\( \mathbb{R}^n \)) and we require the probability that a process starting in the interval \([a, b]\) of \( \mathbb{R}^n \) will be found outside this interval at time \( t \). Let \( P(x, t; x_0) \) be the density probability of \( x \) at time \( t \) for \( x(0) = x_0 \). \( P \) is determined by the Fokker–Planck equation (see Hasselman 1976).

\[
\frac{\partial}{\partial t} P = \sum_{i=1}^n \frac{\varepsilon}{2} \frac{\partial^2}{\partial x_i^2} P + \sum_{i=1}^n f_i(x) \frac{\partial}{\partial x_i} P
\]  

(2.9)

If \( f(x) = -\nabla V(x) \), \( V = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right) \) for some \( V \) then a steady solution of Eq. (2.9) \((\partial P/\partial t \equiv 0)\) is

\[
P = N \exp\{-2V(x)/\varepsilon\}
\]  

(2.10)

(where \( N \) is an appropriate normalization factor) and every element of \( \mathbb{R}^n \) will eventually
have been occupied: if the process starts in \([a, b]\), sooner or later it will leave \([a, b]\). Further, the time spent in \([a, b]\) before exit will be different for each realization of the process: the exit time is a random variable and we are interested in its expectation value. To determine this, in the case of a single stochastic equation, we use a theorem and a corollary due to Gihman-Skorohod (1972, p. 106).

Theorem: Let \(\tau_x(a, b)\) be the time that a sample trajectory starting from any \(x\) in interval \([a, b]\) will leave the interval. Then if \(m(x) = E\{\tau_x(a, b)\}\); \(m(x)\) is given by the solution of the equation:

\[
\frac{e}{2} \frac{d^2}{dx^2} \{m(x)\} + f(x) \frac{d}{dx} \{m(x)\} = -1; \quad m(a) = m(b) = 0 .
\] (2.12)

This theorem is based on intensive use of the Ito formula and proof and details can be found in Gihman-Skorohod (1972).

Corollary: Set

\[
\phi(y) = \exp\left\{ -\int_a^y \frac{2}{e} f(\zeta) \, d\zeta \right\}.
\]

then \(m(x) = E\{\tau_x(a, b)\} = - \int_a^x 2\phi(y) \int_a^y \frac{dz}{e \phi(z)} \, dy + \int_a^b 2\phi(y) \int_a^y \frac{dz}{e \phi(z)} \, dy \int_a^x \phi(z) \, dz / \int_a^b \phi(z) \, dz \) (2.13)

Our objective is to compute an \(E(\tau)\) in the case for which \(f(x) = -d\{V(x)\}/dx\) and particularly \(E(\tau_c)\) for the process starting from \(c\) in the class of \(V(x)\) whose behaviour in the interval \([a, b]\) has the form illustrated in Fig. 1. We have:

\[
\phi(x) = \exp\left\{ -\frac{2}{e} \int_a^x f(z) \, dz \right\} = \exp\left[\frac{2}{e} \{V(x) - V(a)\}\right] .
\] (2.14)

Figure 1. Potential \(V(x)\) in the interval \([a, b]\) (illustrating Eq. (2.15))
Substitution into Eq. (2.13), algebraic manipulation and use of the condition that as \(b\) approaches \(c\) the exit time from \([a, c]\) approaches zero leads to
\[
E\{\tau, (a, b)\} \approx \frac{2}{\varepsilon} \int_a^b \exp \left\{ -\frac{2}{\varepsilon} V(x) \right\} dx \int_a^c \exp \left\{ \frac{2}{\varepsilon} V(x) \right\} dx.
\] (2.15)

We approximate Eq. (2.15) by using
\[
\int_a^b \exp\{\kappa g(x)\} \, dx \approx \left\{ -\pi / \kappa g''(a) \right\}^{\frac{1}{2}} \exp\{\kappa g(a)\}
\]
which holds for
\[
\frac{d}{dx} g(a) = 0, \quad \frac{d^2}{dx^2} g(a) < 0, \quad \kappa \gg |g(a) - g(b)|
\]
so that for \(\varepsilon\) small, Eq. (2.15) becomes
\[
E\{\tau, (a, b)\} \approx \frac{\pi}{\{V''(a), V''(c)\}^{\frac{1}{2}}} \exp \left\{ \frac{2}{\varepsilon} \frac{V(a) - V(c)}{\lambda} \right\}.
\] (2.16)

Note that the formula does not depend on \(b\), because, given the fast growth of \(V(x)\) for \(x > c\), there is no significant contribution to the integral. \((V(x) \text{ is required to grow as some power } > 2 \text{ of } x \text{ because in the neighbourhood of } c, V(x) \sim 0(x^2).)\)

The last step, concluding this review of known results, is to evaluate from which point, \(a\) or \(b\), the process will leave the interval \([a, b]\). It is intuitive, considering the actual shape of \(V\), to believe that the process will necessarily leave from \(a\), but for a more quantitative approach let us introduce the probability, \(p_x(x; b)\), that starting from some point \(x_0\) in \([a, b]\) the process \(x\) will arrive at \(a\) before \(b\). The probability that the process will arrive at \(b\) before \(a\) is \(p_y(x; a)\). Then \(p_y(x; b)\) is given by the following corollary:

Corollary (Gihman-Skorohod 1972, p. 111): Set
\[
\phi(y) = \exp\left[ -\int_a^y \frac{2f(z)}{\varepsilon} \, dz \right]
\]
then
\[
p_x(x, b) = \int_x^b \exp\left\{ \frac{2}{\varepsilon} V(y) \right\} dy / \int_a^b \exp\left\{ \frac{2}{\varepsilon} V(y) \right\} dy
\] (2.17a)
and
\[
p_y(x, b) = \int_a^x \exp\left\{ \frac{2}{\varepsilon} V(y) \right\} dy / \int_a^b \exp\left\{ \frac{2}{\varepsilon} V(y) \right\} dy
\] (2.17b)
so that \(p_x(x, b) > p_y(x, b)\) so long as
\[
\int_x^b \exp\{2V(x)/\varepsilon\} \, dx > \int_a^x \exp\{2V(x)/\varepsilon\} \, dx.
\] (2.18)

Considering \(x = c\) and \(V\) as depicted in Fig. 1, it follows:
\[
\int_x^b \exp\{2V(x)/\varepsilon\} \, dx > \int_a^c \exp\{2V(x)/\varepsilon\} \, dx.
\] (2.19)

If \(V(x) = -x^2 + O(x^n)\), it appears clear that as soon as \(V(b) \geq V(a)\), the process will leave from \(a\) (it is trivially true in the case \(0(x^n) = 0\) from symmetry considerations). Another interesting application of Eqs. (2.17a) and (2.17b) is that if we start in \([a, b]\) with \(V(a) > V(b)\) then the trajectory will arrive at \(b\) before \(a\).

Finally, let us remark that the assumption \(\varepsilon\) constant can be relaxed. In such a case, however, the final results are no longer useful since
\[ \phi(y) = \exp \left[ - \int_a^y 2f(z)/c(z) \, dz \right] \]
and the exit time must be evaluated by a different procedure.


In this section we inquire whether a hypothetical and formally specified 'energy balance model' can reproduce the climatic phenomena outlined in section 1 of this paper and documented by Mason (1976). Broadly, these are that the climate changes, in relatively short periods, between two states, colder and warmer, which are occupied for longer periods of approximately equal duration between transitions. To simulate this we postulate a function governing temperature, the deterministic component of which

(a) has two stable steady states and one unstable steady state, and
(b) is antisymmetric about the unstable steady state.

For mathematical convenience we further assume that the function has polynomial form. We work in terms of a 'non-dimensional temperature', \( T \), which may, for example, be related to an actual temperature \( K \) by

\[ T = (K - K_{\text{standard}})/K_{\text{standard}} \]

or some similar device. The stochastically perturbed energy balance equation becomes an equation for the time development of this non-dimensional quantity with the form

\[ dT = F(T) \, dt + \varepsilon^T \, dW \]  \hspace{1cm} (3.1)

where \( F \) (dimension \( t^{-1} \)) is the deterministic relaxation related to the actual balance between incoming and outgoing radiation, \( dW \) is the Wiener process; \((dW)^2 \) has dimension \( t \), \( \varepsilon \) has dimension \( t^{-1} \), and \([E(\varepsilon^T \, dW)]\) is non-dimensional. Furthermore, we require that \( F(T) \) has a potential \( V(T) \) such that

\[ V(T) = -\int F(T) \, dT \]  \hspace{1cm} (3.2)

so that \( dV/dT = 0 \) defines the steady states, and the sign of \( d^2V/dT^2 \) their stability.

Further appeal to the climatic record shows that

(c) in the last million years the changes in actual representative mean temperatures have been confined to a range of order 10 K.

For the restricted purpose of this section, the three climatic 'observations' (a), (b), and (c) can be related to a potential function of the quartic form displayed in Fig. 2, where \( \alpha \) and \( \beta \) correspond to the observed limiting temperatures, \( T_{01} \) and \( T_{03} \) are the stable steady states, and \( T_{02} = \frac{1}{2}(T_{01} + T_{03}) \) is the unstable steady state. We require \( V(\alpha), V(\beta) \geq V(T_{02}) \) and arbitrarily scale \( V(T) \). Furthermore, we set \( T_{02} \equiv 0 \) by taking the standard reference temperature to be the actual temperature at the unstable steady state. We then have

\[ F(T) = -aT(T^2 - 1) \]  \hspace{1cm} (3.3)

as the deterministic component of the evolution of \( T \). We stress that Eq. (3.3) is an approximation to the balance equation in an interval of \( T \), so that it would be misleading to consider the term \( aT^3 \) as related to emission and \( aT \) related to absorption of radiation. Linearizing Eq. (3.3) around a point distant a small increment \( \delta T \) from a stable steady state, we find the deterministic trajectory behaves as
with a relaxation time $\beta = \frac{1}{2} a$, which may be used to relate the constant $a$ to the relaxation time of other deterministic energy balance models with generating functions $F(T)$ constructed from physical considerations of radiative and convective energy transfer. A further consideration relating our phenomenologically based formulation of $F(T)$ to other physically based formulations is to be found in the work of North et al. (1979) on the generality of models in which $F(T)$ has a potential $V(T)$.

The linearized form of the stochastically perturbed form of Eqs. (3.1) and (3.3) is

$$dT = -2aT dt + \varepsilon dW$$  \hspace{1cm} (3.5)

Equation 3.5 is known in the physics literature as an Ornstein-Uhlenbeck process (see Wax 1954). Feller (1966, p. 324) has shown that the density probability distribution for Eq. (3.5) is

$$q(t, T(0), T) = \left[ \frac{2a}{\pi \varepsilon (1 - \exp(-4at))} \right]^\frac{1}{2} \exp \left[ \frac{-2a{T - T(0) \exp(-4at)}}{\varepsilon (1 - \exp(-4at))^2} \right]$$  \hspace{1cm} (3.6)

The variance of the process is given by:

$$\sigma^2 = \varepsilon (1 - \exp(-4at)) / 4a$$  \hspace{1cm} (3.7)

approaching the asymptotic value:

$$\sigma^2 = \varepsilon / 4a$$  \hspace{1cm} (3.8)

in agreement with Hasselman's results.

An inspection of Eq. (3.6) shows that there is a finite probability of finding the process governed by Eq. (3.5) in the neighbourhood of the other steady state, so that a sample path of Eq. (3.5) which has left the attraction domain will reach the other steady state with probability 1. Furthermore, Eq. (3.8) shows that the sample path will reach the new stable steady-state in finite time. However, for long intervals Eq. (3.5) is not a good approximation and the entire non-linear equation must be retained. Then it appears evident that a mean time $\tau$ will exist, such that the solution will leave the neighbourhood of $T_{01}$, crossing the boundary close to the unstable steady state, and approaching $T_{03}$. The time $\tau$ marks an
abrupt change in the statistics of the process, because after leaving the attraction domain the representative point will fall down towards the other steady state, and will oscillate around it for another time space $\tau$.

In the last part of this section we compute $\tau$. The expected exit time from a domain $[0,L]$, $L$ arbitrarily large, around the steady state $T_0$ is:

$$E(\tau_j(0,L)) = \frac{2}{\varepsilon} \int_0^{+1} \exp\left\{2V(T)/\varepsilon\right\} dT \times \int_0^{L} \exp\left\{-2V(T)/\varepsilon\right\} dT \quad \text{(3.9)}$$

$$= 2^{1/2} \pi \beta \exp(4\beta\varepsilon)^{-1} \quad \text{(3.10)}$$

which is independent of any translation of the potential $V(T)$ on the $T$ axis.

Equation (3.10) is the expression we require relating the three major time constants of our simple climate model – the persistence time $\tau$ of the major stable regimes, the characteristic time $\beta$ of the stabilizing ‘feedback’ mechanisms, and the characteristic time $\varepsilon^{-1}$ of the stochastic ‘weather’ perturbations. Table 1 illustrates the extreme sensitivity of $\tau$ to the choice of $\varepsilon^{-1}$ and $\beta$ with the domains for which $10^5$ yr $> \tau > 10^3$ yr indicated. The rationale for the choice of $\varepsilon^{-1}$ in this table is: 25 d – approximately the value assumed by Fraedrich (1978) based on the duration of ‘blocking patterns’; 10 d – the mean lifetime of a water molecule in the atmosphere, characterizing energy conversions involving condensation and evaporation; 5 d – characteristic of the lifetime of cyclonic storms in temperate latitudes; 2.5 d – a minimum estimate of the ‘rundown’ time of the atmosphere, the characteristic time for viscous dissipation of atmospheric kinetic energy. We regard the range of possible values of $\varepsilon^{-1}$ as reasonably well defined, but for the values of $\beta$ which have been used by other investigators (e.g. 8 yr by Fraedrich (1979)), this range of $\varepsilon$ gives a $10^6$-fold range of $\tau$. However we regard, the climatic record as establishing $\tau$ for the last glacial epoch as of order $10^4$ yr: Table 2 shows corresponding values of $\varepsilon^{-1}$ and $\beta$. (It indicates that Fraedrich’s (1978, 1979) combination of $\varepsilon^{-1} = 25$ d, $\beta = 8$ yr is not compatible with our solution.)

**TABLE 2. CHARACTERISTIC TIMES OF STOCHASTIC ($\varepsilon^{-1}$) AND DETERMINISTIC ($\beta$) PROCESSES FOR AN EXIT TIME OF $10^5$ YR**

<table>
<thead>
<tr>
<th>$\tau$ (yr)</th>
<th>$\varepsilon^{-1}$ (d)</th>
<th>$\beta$ (yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^5$</td>
<td>2.5</td>
<td>8.0</td>
</tr>
<tr>
<td>$10^4$</td>
<td>5.0</td>
<td>3.6</td>
</tr>
<tr>
<td>$10^3$</td>
<td>10.0</td>
<td>1.7</td>
</tr>
<tr>
<td>$10^2$</td>
<td>25.0</td>
<td>0.6</td>
</tr>
</tbody>
</table>

There is no trace in the climatic record of the totally glaciated earth which is indicated as a possibility by the Budyko and Sellers models. The anti-symmetrical deterministic function of Eq. (3.3) is not the most appropriate for investigation of this state; absence from the record requires a very short exit time $\tau$, or alternatively requires that the potential barrier $\Delta V$ towards the ice-covered earth solution be so high that to reach such a steady-state would require $\tau$ greater than the age of the earth.

**4. LONG-TERM EFFECTS IN A ‘REALISTIC’ MODEL**

In this section we consider an energy balance model with a more realistic deterministic term than that studied in section 3, and examine the effect of stochastic perturbations on some properties of its solutions. More realistic models are formulated in terms of para-
<table>
<thead>
<tr>
<th>$d$</th>
<th>$\varepsilon^{-1}$</th>
<th>yr</th>
<th>1</th>
<th>1-5</th>
<th>2</th>
<th>2-5</th>
<th>$\gamma$</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>30</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>$6.85 \times 10^{-2}$</td>
<td></td>
<td>$3.1 \times 10^{16}$</td>
<td>$2.5 \times 10^{11}$</td>
<td>$7.5 \times 10^7$</td>
<td>$2.4 \times 10^7$</td>
<td>$3.3 \times 10^4$</td>
<td>$1.7 \times 10^3$</td>
<td>$7.6 \times 10^2$</td>
<td>$4.5 \times 10^2$</td>
<td>$6.4 \times 10^2$</td>
<td></td>
</tr>
<tr>
<td>5.0</td>
<td>$1.37 \times 10^{-2}$</td>
<td></td>
<td>$3.7 \times 10^6$</td>
<td>$1.2 \times 10^6$</td>
<td>$8.2 \times 10^4$</td>
<td>$1.6 \times 10^4$</td>
<td>$8.5 \times 10^3$</td>
<td>$2.8 \times 10^2$</td>
<td>$2.2 \times 10^2$</td>
<td>$2.4 \times 10^2$</td>
<td>$5.3 \times 10^2$</td>
<td></td>
</tr>
<tr>
<td>10.0</td>
<td>$2.74 \times 10^{-2}$</td>
<td></td>
<td>$4.1 \times 10^4$</td>
<td>$2.9 \times 10^4$</td>
<td>$8.5 \times 10^2$</td>
<td>$4.3 \times 10^2$</td>
<td>$1.4 \times 10^2$</td>
<td>$1.1 \times 10^2$</td>
<td>$1.7 \times 10^2$</td>
<td>$1.8 \times 10^2$</td>
<td>$4.9 \times 10^2$</td>
<td></td>
</tr>
<tr>
<td>25.0</td>
<td>$6.85 \times 10^{-2}$</td>
<td></td>
<td>$1.7 \times 10^3$</td>
<td>$7.6 \times 10^1$</td>
<td>$5.5 \times 10^1$</td>
<td>$4.8 \times 10^1$</td>
<td>$4.6 \times 10^1$</td>
<td>$6.4 \times 10^1$</td>
<td>$8.5 \times 10^1$</td>
<td>$1.5 \times 10^2$</td>
<td>$4.6 \times 10^2$</td>
<td></td>
</tr>
</tbody>
</table>
meters related to the physical processes of radiation, convection, and storage of energy, and many investigators have studied change in structure of the solution as one or more of the parameters is varied — for example, two or more steady states may coalesce at a critical value of a parameter. With the added stochastic term the question is to understand the behaviour of the equation.

\[
dT = f(\mu, \ldots; T) dt + \epsilon^t dW
\]  

(4.1)
as one parameter \( \mu \) is changed. In particular, we will examine the exit time \( \tau(\mu) (A, B) \) where \( A, B \) is some interval of \( T \).

As a particular example, we will consider a zero-dimensional model due to Fraedrich (1979) and first examine \( \tau(\mu) (A, B) \) for the case where \( \mu \) is a parameter related to albedo. In this model (using Fraedrich’s notation) the deterministic equation for a vertically integrated global temperature \( T \) is

\[
\frac{dT}{dt} = a(-T^4 + b_\mu T^2 - d_\mu) 
\]  

(4.2)
with

\[
a = (e_{SA}\sigma/c) \\
b_\mu = (\frac{1}{2} \mu I_0/e_{SA}\sigma) b_2 \\
d_\mu = -(\frac{1}{2} \mu I_0/e_{SA}\sigma)(1 - a_2)
\]

where \( I_0 \) is the solar constant, \( \sigma \) the Stefan-Boltzmann constant, \( c \) the thermal capacity of a well-mixed ocean layer of depth 30 m covering 70.8% of the earth’s surface, \( e_{SA} \) the effective emissivity, \( a_2 (>1) \) and \( b_2 \) parameters chosen to adjust the albedo magnitude (\( \alpha \)) and slope of the albedo—temperature relation (which Fraedrich takes to be \( \alpha = c_1 + c_2 T^2 \)) to acceptable values at the present value of \( T \) (288.6 K). The parameter \( \mu \), a multiplier of \( I_0 \), allows simulation of changes in the solar constant or the earth’s orbital parameters (Von Woerkom (1953)).

The steady-state equation

\[
T^4 - b_\mu T^2 + d_\mu = 0
\]  

(4.3)

\[
T^\pm = \left\{ \frac{1}{2} b_\mu \pm (\frac{1}{2} b_\mu^2 - d_\mu)^{\frac{1}{4}} \right\}^{\frac{1}{4}}
\]  

(4.4)
has two physically meaningful solutions

with \( T^+ \) stable and \( T^- \) unstable (see Fraedrich 1979). With Fraedrich’s choice of the constants, there is a critical value of \( \mu, \mu_c = 0.63 \) for which \( (\frac{1}{2} b_\mu^2 - d_\mu) = 0 \) and \( T^+ = T^- \), i.e. the two steady states coalesce.

We now add the stochastic term \( \epsilon^t dW \) to Fraedrich’s Eq. (4.2), using his value of \( 2.7 \times 10^{-2} K^2 \ \text{yr}^{-1} \) for \( \epsilon \), and investigate the behaviour of the exit time \( \tau_T(\mu) (T_-, T_+) \), \( T_\pm \) large, as \( \mu \rightarrow \mu_c \), using the approximate formula developed in section 2. The ‘potential’ \( V(\mu, \ldots; T) \) is

\[
V(\mu, \ldots, T) = a(\frac{1}{2} T^5 - \frac{1}{2} b_\mu T^3 + d_\mu T)
\]  

(4.5)
with a maximum at the unstable steady state \( T^- \) and a minimum at the stable state \( T^+ \). Figure 3 shows \( V(T) \) for a typical \( \mu > \mu_c \) and illustrates the fact that after the exit time \( \tau_T + (\mu) (T^-, T_+) \) has elapsed, the solution will fall irreversibly to the ‘ice-covered earth’ state. Using the formulae of section 2, we have computed this exit time for a range of \( \mu \) between 0.632 and 0.670. Selected results are listed in Table 3. This illustrates that so long as
STOCHASTIC PERTURBATION AND LONG-TERM CLIMATE BEHAVIOUR

\[ v(T) \]

Figure 3. Potential \( V(T) \) for the Fraedrich model of section 4.

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( V(T^+) - V(T^-) )</th>
<th>( \tau (\text{yr}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.632 E+00</td>
<td>-0.5703 E+01</td>
<td>0.1684 E+21</td>
</tr>
<tr>
<td>0.638 E+00</td>
<td>-0.9335 E+01</td>
<td>0.6844 E+32</td>
</tr>
<tr>
<td>0.644 E+00</td>
<td>-0.1357 E+02</td>
<td>0.2650 E+46</td>
</tr>
<tr>
<td>0.650 E+00</td>
<td>-0.1838 E+02</td>
<td>0.6806 E+61</td>
</tr>
<tr>
<td>0.656 E+00</td>
<td>-0.2371 E+02</td>
<td>0.9123 E+78</td>
</tr>
<tr>
<td>0.662 E+00</td>
<td>-0.2956 E+02</td>
<td>0.5399 E+97</td>
</tr>
<tr>
<td>0.668 E+00</td>
<td>-0.3590 E+02</td>
<td>0.1248 E+118</td>
</tr>
</tbody>
</table>

\( \mu \) exceeds the critical value 0.630, even by a very small amount, stochastic perturbation associated with internal mechanisms will not drive the system defined in the model to the ice-covered earth state within an interval comparable with the age of the earth.

The situation is, however, very different if we change the parameter \( a \) of the Fraedrich model, physically the ratio \( e_{SA}/c \) of emissivity to thermal inertia. Table 4 illustrates this. In these computations the thermal inertia has been increased by factors from 2 to 6 with \( e_{SA} \) unchanged. With factors of 5 and 6, i.e. mixed layer depths of 150 m and 180 m, the exit time to the ice-covered earth condition is less than \( 10^5 \) yr over the whole range of \( \mu \) investigated. This suggests that the value of \( \tau_{T^+}(\mu) (T^-, T_0) \) could be used as a scale analysis tool in any globally averaged energy balance model. This question is being pursued.

5. OUTLINE OF AN ALTERNATIVE APPROACH

The preceding results are particular cases of the general facts that we attempt to outline in this section.
There are some examples in which almost intransitive behaviour has been obtained when considering only the deterministic behaviour of the solution. Such a case was studied by Fraedrich (1978, 1979) for a Budyko-type model. The approach was based on catastrophe theory as developed by Thom (1975). Essentially, the phenomenon considered is the coalescence of some steady solutions as the parameters entering the equation (including in the climatic application the time constants of the feed-backs) reach certain critical values.

We consider a minor variation of the model presented in the preceding section, characterized by the deterministic drift $F(T) = -T^3 + aT + d$. The set of values of the parameters $a$ and $d$ for which ‘catastrophe’ or ‘bifurcation’ occurs, is given by $D = d^2/4 - a^3/27 > 0$.

In this case the equation

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$V(T^+) - V(T^-)$</th>
<th>$\tau$ (yr)</th>
</tr>
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<tbody>
<tr>
<td><strong>4a</strong></td>
<td>0.632</td>
<td>E+00</td>
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<td></td>
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</tr>
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<td>E+00</td>
<td>-0.9190</td>
</tr>
<tr>
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<td>0.656</td>
<td>E+00</td>
<td>-1.1185</td>
</tr>
<tr>
<td></td>
<td>0.662</td>
<td>E+00</td>
<td>-1.4178</td>
</tr>
<tr>
<td></td>
<td>0.668</td>
<td>E+00</td>
<td>-1.795</td>
</tr>
<tr>
<td><strong>4b</strong></td>
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<td>E+00</td>
<td>-0.9506</td>
</tr>
<tr>
<td></td>
<td>0.638</td>
<td>E+00</td>
<td>-0.1555</td>
</tr>
<tr>
<td></td>
<td>0.644</td>
<td>E+00</td>
<td>-0.2262</td>
</tr>
<tr>
<td></td>
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<td>-0.3063</td>
</tr>
<tr>
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<td>0.656</td>
<td>E+00</td>
<td>-0.3952</td>
</tr>
<tr>
<td></td>
<td>0.662</td>
<td>E+00</td>
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</tr>
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<td>E+00</td>
<td>-0.7658</td>
</tr>
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<td>0.656</td>
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<td>-0.9881</td>
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<td>0.662</td>
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<td>-0.1231</td>
</tr>
<tr>
<td></td>
<td>0.668</td>
<td>E+00</td>
<td>-0.1495</td>
</tr>
<tr>
<td><strong>4d</strong></td>
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<td>E+00</td>
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<tr>
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</tr>
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</tr>
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<td></td>
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<td>-0.4986</td>
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</tbody>
</table>
has three roots, whilst if \( D \leq 0 \), it has only one real root (see Zeman 1976). Here we are interested in the effect that stochastic noise will have on the structure of the bifurcation set. With this aim we introduce the Fokker–Planck equation (see Hasselman 1976) associated with the stochastic variant of Eq. (5.1) with a deterministic drift.

In this case the Fokker–Planck equation is:

\[
\frac{\partial P}{\partial t} = -\partial G/\partial T
\]

where

\[
G = FP - \varepsilon \partial P/\partial T
\]

and \( P \) is the probability density \( P(T,t) \) for the process. Let us consider \( F(T) = -d[V(T)]/dT \) then a steady solution of Eq. (4.2) is:

\[
P_{ST}(T) = N \exp\{-V(T)/\varepsilon\}
\]

where \( N \) is an appropriate normalization. From Eq. (5.4) it appears that the steady-state probability density changes in correspondence with the bifurcation set. In fact, the density, Eq. (5.4), attains its extrema within the bifurcation set. Therefore, within the bifurcation set the probability, Eq. (5.4), has two maxima and a minimum. The main difference between the deterministic and stochastically perturbed process is that in the deterministic behaviour the solution of the differential equation leaves the attraction domain only in the boundary of the bifurcation set, while in the stochastic case the shift can take place at any point within the bifurcation set at a mean time given by Eq. (2.16). Furthermore, if \( P_{ST}(T) \) is also the asymptotic probability distribution \( (P(T,t) \to P_{ST}(T) \text{ as } t \to \infty) \) and the process described by the stochastic perturbed version of Eq. (5.1) is ergodic

\[
\left( \lim_{t \to \infty} \int_0^t \xi dt = \int_\mathbb{R} \xi dT \right) \text{ everywhere, } \xi \text{ being any statistic of the process},
\]

an interesting application of Eq. (5.4) is possible. The ergodic property tells us that the probability of finding the solution in the domain around some deterministic equilibrium is:

\[
P = \int_{T_1}^{T_2} P_{ST}dT = N \int_{T_1}^{T_2} \exp\{-V(T)/\varepsilon\} dT
\]

where \( N \) is an appropriate normalizing factor.

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APPENDIX

In this appendix we refer to a formal analogy between the problem of finding the energy levels of a quantum mechanical system and the solution of a stochastic perturbation equation. It seems likely that, reversing the argument, the powerful techniques already developed in quantum mechanics (for example, JWKB, see Froman and Froman 1965) can help to investigate many aspects of climatic changes initiated by stochastic processes.

Consider a classical system with a potential $V(X)$ and impose a stochastic perturbation. With $b(X) = -\operatorname{grad} V(X)$, the stochastic equation is

$$dX = b(X, h/m) dt + (h/m)^{3/2} dW$$

(A1)

where $W$ is a Wiener process (compare Eq. (2.5)). We seek a steady probability equation for the process Eq. (A.1). Consider the situation when $b$ is defined by the equation

$$\frac{1}{2} b^2 + \frac{1}{2}(h/m) d b/dX = (V - E)/m$$

(A2)

(see Nelson 1966 for the derivation of Eq. (A.2), which is his Eq. 37). The connection with Schroedinger's equation is straightforward if we note that Eq. (A.2) is a Riccati equation which can be linearized using

$$b(X) = \frac{1}{2}(h/m) d\ln(\psi^2)/dX$$

(A3)

(compare Eq. (5.4), which is $b(T) = \varepsilon d(\ln P)/dT$), when the equation for $\psi$ becomes

$$\frac{1}{2}(h^2/m) d^2\psi/dX^2 + (E - V)\psi = 0$$

The wave function $\psi$ may be interpreted as the logarithm of the probability density distribution associated with the stochastic diffusive equation (A.1), and, for example, the well-known tunnelling effect in quantum mechanics can be interpreted as the exit of the classical trajectory from a domain of stability. Discussion of the connection between quantum mechanics and stochastic dynamics can be found in Nelson (1966, 1968).

REFERENCES


