The accuracy of Gadd's modified Lax–Wendroff algorithm for advection

551.509.313

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(Received 10 April 1980; revised 10 November 1980)

Gadd (1978) has reviewed the Lax–Wendroff approximation to the advection equation

\[ \frac{\partial \theta}{\partial t} = -u \frac{\partial \theta}{\partial x} \]  \hspace{1cm} (1)

and described a modification that is simple, but gives significantly reduced phase speed errors and compares favourably with fourth-order accurate schemes. In this note we show that Gadd’s algorithm is second-order accurate, but close to an approximation that is third-order accurate and has fourth-order accurate phase speeds. The analysis is based on the dispersion relation for the finite difference equation. This highlights the well-known fact that high orders of accuracy in the calculation of individual terms do not guarantee accurate solutions. A leap-frog, or time centred, approximation to the two-dimensional equation that has a fourth-order accurate dispersion relation is also presented. It is interesting because it is not a naive combination of two one-dimensional versions, and it has similar stability properties to Gadd’s algorithm.

The dispersion relation is usually obtained by substituting solutions of the form \( \exp(i(\sigma t - kx)) \), where, for simplicity, it is supposed that there are only two independent variables \( x \) and \( t \), and deriving

\[ \sigma = \sigma(k) \]  \hspace{1cm} (2)

For Eq. (1)

\[ \sigma = uk \]  \hspace{1cm} (3)

When functions of the form \( \exp(i(\sigma t - kx)) \) are substituted in the finite difference equation \( \sigma \) will only appear as the product \( \sigma \delta t \) and \( k \) as the product \( k \delta x \). The dispersion relation takes the form

\[ \sigma \delta t = \sigma \delta t(k \delta x) \]  \hspace{1cm} (4)

where, as in the continuous case, the function \( \sigma \) depends on other variables, e.g. \( u \), but now also \( \delta x \) and \( \delta t \). The approximation is defined to be \( n \)th order accurate if

\[ \sigma = \sigma + O((k \delta x)^{n+1}) \]  \hspace{1cm} (5)
The expansion is in the non-dimensional variable \( k \delta x \) which is small for waves \( \exp ikx \) that are well resolved (i.e. have many grid points within a wavelength). Since the derivation of Eq. (4) involves all terms in the original equation, whatever they may be, it is clear that this approach combines the effect of time and space \((\tau \text{ and } x)\) truncation.

It is possible to achieve higher orders of accuracy in the dispersion relation using only second order approximations for the individual terms of the equation. Gadd approximates Eq. (1) in two steps

\[
\theta_{j+\frac{1}{2}}^{n+1} = \left( \theta_j \right)_{n+1}^{n} - \frac{1}{2} \mu (\delta_x \theta)_j^{n+1} \tag{6}
\]

\[
\theta_{j+1}^{n+1} = \theta_j^{n} - \mu \left\{ (1 + a) (\delta_x \theta)^{n+1} - a (\delta_x \theta)^{n+1} \right\} \tag{7}
\]

where the notation follows Gadd, i.e. \( n \) labels the time level \((t = t_0 + n \delta t)\), \( j \) labels the spatial grid point \((x = x_0 + j \delta x)\), and \( \mu = u \delta t / \delta x \). Gadd showed that this approximation is stable for \( \mu < 1 \) if

\[
a < \frac{1}{4} (1 - \mu^2) \tag{8}
\]

and arbitrarily (or with a view to reducing the dissipation) chose the limiting case. Substituting \( \exp(\tilde{\sigma} \delta t - k x) \) in Eqs. 6 and 7 gives

\[
\exp(\tilde{\sigma} \delta t) = 1 - 2 \mu \sin^2 \alpha (1 + 4 a \sin^2 \alpha / 3) + 2i \mu \sin \alpha \cos \alpha (1 + 4 a \sin^2 \alpha / 3) \tag{9}
\]

where \( \alpha = k \delta x / 2 \). (Equation (9) is Eq. (11) of Gadd (1978).) The series expansion of \( \tilde{\sigma} \) for small \( \alpha \) gives

\[
\tilde{\sigma} \delta t = 2 \mu \alpha - 4 \mu^2 (1 - \mu^2 - 2a) / 3 + 2i \mu^2 \alpha^2 (1 - \mu^2 - 4a / 3) + O(\alpha^4) \tag{10}
\]

The exact relation, Eq. 3, can be rewritten

\[
\sigma \delta t = 2 \mu \alpha \tag{11}
\]

so \( \tilde{\sigma} = \sigma + O(\alpha^4) \) only if

\[
a = \frac{1}{4} (1 - \mu^2) \tag{12}
\]

For this choice of \( a \) there is an error in the \( \alpha^4 \) term, and the scheme is third-order accurate and fourth-order dissipative. Tables 1 and 2 give the damping per time step \( \exp(-1 \text{Im} \tilde{\sigma} \delta t) \) and the relative phase speeds \( \text{Re} \tilde{\sigma} / \sigma \) as functions of \( \mu \) and wavelength and are directly comparable with Gadd’s Tables 1 and 2. It can be seen that Eq. (12) gives a significant improvement in the phase speeds but, as expected, it is more dissipative than Gadd’s choice. Figures 1 and 2 show the result of advecting a step function through 100 time steps using the two schemes. It is worth commenting that Eq. (12) gives fourth-order accuracy in the phase speed, and that the effect of the dissipation of either scheme is not always undesirable.

The dispersion relation can be used to derive a fourth-order accurate approximation to the two-dimensional advection equation

\[
\frac{\partial \theta}{\partial t} = -u \frac{\partial \theta}{\partial x} - v \frac{\partial \theta}{\partial y} \tag{13}
\]

The new approximation is
### TABLE 2. Relative phase speed as a function of the non-dimensional advecting velocity ($\mu$) and the wavelength in grid lengths ($L$) for $a = \frac{1}{4} (1 - \mu^2)$

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<th>7</th>
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Figure 1. A step function after advection through 100 time steps for 5 values of the non-dimensional advecting velocity $\mu = u \delta t / \delta x$. Gadd's modification of the Lax–Wendroff scheme was used with $a = \frac{1}{4}(1 - \mu^2)$.

Figure 2. A step function after advection through 100 time steps for 5 values of the non-dimensional advecting velocity $\mu = u \delta t / \delta x$. Gadd's modification of the Lax–Wendroff scheme was used, but with $a = \frac{1}{4}(1 - \mu^2)$ rather than the value proposed by Gadd.
\[ \theta_{i,j}^{n+1} - \theta_{i,j}^{n-1} = -2\mu ((1 - r^2 - a) \delta \theta - a \delta_{4,0}^x \delta \theta - v^2 \delta_{2,x} \delta_{2,y} \delta \theta) \\
- 2v((1 - \mu^2 - b) \delta_{2,y} \theta - b \delta_{4,0}^x \theta + \mu^2 \delta_{2,x} \delta_{2,y} \theta) \]  
where \( v = v \frac{\partial t}{\partial y}, a = \frac{1}{6}(1 - \mu^2) \) and \( b = \frac{1}{6}(1 - \mu^2) \). Note that the naive combination of the fourth order one-dimensional approximation to the two terms on the right of Eq. (13) would give

\[ \theta_{i,j}^{n+1} - \theta_{i,j}^{n-1} = -2\mu ((1 + a) \delta \theta - a \delta_{4,0}^x \theta - 2v((1 + b) \delta_{2,y} \theta - b \delta_{4,0}^x \theta) \]  

This is not Eq. (14), and it follows that high accuracy is not necessarily preserved when multidimensional approximations are formed by combining one-dimensional approximations.

A step function has been advected through 100 time steps using this scheme in one dimension. The results shown in Fig. 3 are rough and appear less satisfactory than those of Fig. 2, but a small amount of fourth-order dissipation gives results very similar to Fig. 2. Centred schemes are often preferred because they are not dissipative, but meteorological models are rarely free of any form of dissipation, and it is not clear where the advantage lies in comparing these two schemes.

Equation (14) is stable for

\[ |\mu| + |v| < 1 - \varepsilon \]  

where \( \varepsilon = 0.009 \). \( \varepsilon = 0 \) if \( \mu \) and \( v \) are both less than 0.856. \( \partial \theta / \partial x \) and \( \partial \theta / \partial y \) are calculated with fourth-order accuracy if \( v^2 = \mu^2 = 0 \) and \( a = b = \frac{1}{6} \), but the advantage of this accuracy is lost if time truncation is significant (\( \mu \) or \( v \) not small), and these choices give a scheme that is only stable for \( |\mu| - |v| < \frac{1}{4} \).

**Acknowledgment**

I am grateful to Dr Gadd for several conversations during the preparation of this paper. At a late stage, Dr Gadd received a letter from P. L. Roe pointing out that the choice \( a = \frac{1}{6}(1 - \mu^2) \) gives increased accuracy for the phase speed.

**Reference**