On the scattering of sound in a turbulent flow

By R. E. ROBSON and R. POTTS

Physics Department, James Cook University, Townsville, Australia

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SUMMARY

The formula for the cross-section for scattering of sound in a turbulent flow, upon which SODAR meteorology relies, is derived from a simplified, physically appealing argument.

1. INTRODUCTION

The acoustic sounder (SODAR) has become an established diagnostic tool in meteorology and although much has been written concerning interpretation of the returns, there is not always agreement as to what meteorological phenomena are actually taking place (Brown and Hall 1978). A basic equation in SODAR meteorology is the formula for the cross-section for the scattering of sound by turbulence, but the standard derivation of this (Tatarski 1961, 1971) involves application of a series of approximations to the equations of hydrodynamics in order to obtain a wave equation, which is subsequently solved using the Green’s function technique and the Born approximation. The ordering of the approximations can be confusing and their justification is not always clear. A fundamental understanding of the various terms contributing to the cross-section is therefore difficult to obtain from existing theories, whereas it is clear that such basic concepts are necessary (but of course not sufficient) for correct interpretation of sounder returns. In the present paper we attempt to illuminate these concepts through a simplified argument, which relies more upon physical intuition than mathematical rigour.

2. THEORY

The speed of sound in a fluid depends upon the temperature $T$ and flow velocity $v$, both of which fluctuate in a turbulent flow. Consider first of all a plane acoustic wave of amplitude $p_0$ incident at an angle $\alpha$ to the normal of a boundary between two regions of a fluid (Fig. 1). Mass density and speed of sound are denoted by $\rho_j$ and $c_j$ respectively ($j = 1, 2$).

![Figure 1. Scattering of sound from an interface between two distinct media.](image)
Elementary texts on acoustics (see, for example, Kinsler and Frey 1962) show that the ratio of the amplitudes \( p_o \) and \( p_s \) of incident and reflected waves respectively is given by

\[
\frac{p_s}{p_o} = \frac{\rho_2 c_2 \cos \alpha \rho_1 c_1 \left[ 1 - \left( \frac{c_2}{c_1} \sin \alpha \right)^2 \right]}{\rho_2 c_2 \cos \alpha + \rho_1 c_1 \left[ 1 - \left( \frac{c_2}{c_1} \sin \alpha \right)^2 \right]} \tag{1}
\]

If the fluid behaves as an ideal gas, then \( c = \sqrt{(\gamma \rho / \rho)} \), where \( \gamma \) is the ratio of specific heats and \( \pi \) is the hydrostatic pressure of the fluid, which must be the same on both sides of the interface. Assuming the discontinuity to be weak, that is, \( c_1 \approx c_2 \approx c \), then to first order in \( c' = c_1 - c_2 \) we have

\[
\frac{p_s}{p_o} = \frac{c' \cos \theta}{2c \cos \theta / \alpha} \tag{2}
\]

where \( \theta = \pi - 2\alpha \) is the scattering angle. This result can be extended to the situation where the two regions are moving with velocities \( v_1 \) and \( v_2 \) respectively, by accounting for doppler shifts in the usual way. Thus, to first order in both \( c' \) and \( v' = v_1 - v_2 \) it can be shown that

\[
\frac{p_s}{p_o} = \left( \frac{c' - \mathbf{k} \cdot \mathbf{v}}{2c / \omega} \right) \left( \frac{\cos \theta}{\sin^2(\theta/2)} \right) \tag{3}
\]

where \( \omega \) and \( \mathbf{k} \) denote the angular frequency and wave vector respectively.

Equation (2) is really only appropriate to a specular reflection from a sharp interface between two layers. Large-scale layering does occur in the atmosphere (for example, in temperature inversions) but reflection in this way is generally much less important than scattering from the associated large number of small-scale turbulent eddies (Gilman et al. 1946). We consider scattering from an ensemble of eddies, which are partitioned from the bulk of the medium following the somewhat idealized picture in Fig. 1, where one side now represents the bulk, undisturbed medium and the other the fluctuation. Thus we now take \( c', v' \) to represent deviations from the corresponding mean values \( c, v \) and assume that apart from the angular dependence, Eq. (2) still applies, at least in some average sense.

We assume that a plane wave of wave number \( \mathbf{k} \), maximum amplitude \( A_o \), incident at position \( \mathbf{r} \) will be scattered into an outgoing spherical wave centred on \( \mathbf{r} \), the amplitude of which is proportional to the amplitude \( p_o = A_o \exp(i\mathbf{k} \cdot \mathbf{r}) \) of the incident wave and (apart from angular dependence) to the fractional reflected amplitude given by (2) as well as to the size of the scattering volume. The contribution to the pressure amplitude at the detector, located at position \( \mathbf{R} \) (see Fig. 2) for a wave scattered at angle \( \theta \) by eddies in a volume \( d^3r \) centred on \( \mathbf{r} \) is thus given by:

\[
k^2 A_o f(\theta) \left( \frac{c'}{c} \frac{\mathbf{k} \cdot \mathbf{v}}{\omega} \right) e^{i\mathbf{k} \cdot \mathbf{R}} \frac{e^{i|\mathbf{R} - \mathbf{r}|}}{|\mathbf{R} - \mathbf{r}|} d^3r \tag{3}
\]

where \( f(\theta) \) is an angular-dependent ‘structure factor’ (corresponding to \( \cos \theta / \sin^2(\theta/2) \) in the preceding discussion) and the factor \( k^2 \) is required by dimensional considerations. Normally the detector is far removed from the scattering region, enabling us to take \( \mathbf{R} \gg \mathbf{r} \). The amplitude at the detector due to scattering from the entire region volume \( V \) is thus approximately:

\[
p_s = k^2 A_o f(\theta) \left( \frac{c'}{c} \frac{\mathbf{k} \cdot \mathbf{v}}{\omega} \right) \frac{e^{i|\mathbf{k}|D}}{R} \int_V \exp(-i\Delta\mathbf{k} \cdot \mathbf{r}) d^3r \tag{4}
\]

where \( \Delta\mathbf{k} = \mathbf{k}_s - \mathbf{k} \) and \( \mathbf{k}_s \) is the scattered wave vector, \( |\mathbf{k}_s| = |\mathbf{k}| \).
The intensity of the scattered wave is (Kinsler and Frey 1962)

\[ \frac{|p_i|^2}{2\rho c} \hat{R} \]

This, like \( c' \) and \( \nu' \), is a randomly fluctuating quantity, whereas we are interested in the ensemble average, which we denote by \( \langle \ldots \rangle \). Thus, the intensity recorded by detector is

\[
I_s = \hat{R} \langle |p_i|^2 \rangle / 2\rho c \\
= \hat{R} \frac{k^4 |A_0|^2 |f(\theta)|^2}{2\rho c R^2} \int_V d^3 r_1 \int_V d^3 r_2 e^{-i \Delta k \cdot (r_1 - r_2)} \times \\
\times \left( \left( \frac{c'}{c} - \frac{\mathbf{k} \cdot \mathbf{v}}{\omega} \right)_{r_1} \left( \frac{c'}{c} - \frac{\mathbf{k} \cdot \mathbf{v}}{\omega} \right)_{r_2} \right)
\]  

(5)

In applying (5) to the atmosphere, we assume ideal gas behaviour and also for simplicity neglect humidity fluctuations. Thus \( c \propto T^{4} \) and hence temperature fluctuations \( T' \) about a mean value \( T \) produce fluctuations in \( c \) according to the expression:

\[
\frac{c'}{c} = \frac{T'}{2T}
\]  

(6)

Equation (5) involves correlations between fluctuating quantities at position \( r_1 \), and \( r_2 \). If the turbulence is homogeneous, that is, if the properties of the fluid are invariant under arbitrary translations of the coordinate frame, then the correlation functions depend only upon \( \mathbf{r} = r_1 - r_2 \). Moreover, if the flow is incompressible, temperature and velocity fluctuate independently at two different points. Thus, we can write

\[
\langle v_i(r_1) v_j(r_2) \rangle = B_{ij}(r_1 - r_2) \quad (i, j = 1, 2, 3, )
\]  

(7a)

\[
\langle T'(r_1) T'(r_2) \rangle = B_T(r_1 - r_2)
\]  

(7b)

and moreover

\[
\langle T'(r_1) v_j(r_2) \rangle = 0 = \langle T'(r_2) v_j(r_1) \rangle
\]  

(7c)
Substituting Eqs. (6) and (7) into (5) then gives
\[ I_s = \frac{\hat{R}^4 k^4 |A_0|^2 |f(\theta)|^2}{2\rho c R^2} \int \int d^3 r_1 d^3 r_2 e^{-\Delta k \cdot (r_1 - r_2)} \left\{ \frac{B_t(r_1 - r_2)}{4T^2} + \frac{k_i k_j}{\omega^2} B_{ij}(r_1 - r_2) \right\}, \]  
(8)
where summation over repeated indices is implied. One volume integral can be done immediately, since the integrand depends only upon \( r = r_1 - r_2 \). Moreover, by introducing Fourier transforms
\[ \Phi(\kappa) \equiv (2\pi)^{-3} \int d^3 r \, B(r) e^{-i\kappa \cdot r}, \]  
(9)
for both temperature and velocity correlation functions, the expression (8) takes the following simplified form:
\[ I_s = \frac{\hat{R}^4 4\pi^3 V k^4}{\rho c R^2} |A_0|^2 |f(\theta)|^2 \left\{ \frac{\Phi_T(\Delta k)}{4T^2} + \frac{k_i k_j}{\omega^2} \Phi_{ij}(\Delta k) \right\}. \]  
(10)
where the spectral functions \( \Phi_T \) and \( \Phi_{ij} \) are defined as in (9), with \( \kappa = \Delta k \).

The acoustic power associated with scattering into solid angle \( d\Omega \) centred on direction \( \theta \) is \( I_s R^2 d\Omega \). On the other hand, the intensity of the incident wave is
\[ I_0 = \frac{|A_0|^2}{2\rho c}, \]
from which it follows that the differential cross-section per unit volume of the scattering region is
\[ \frac{d\sigma}{d\Omega} = 8\pi^3 k^4 |f(\theta)|^2 \left( \frac{\Phi_T(\Delta k)}{4T^2} + \frac{k_i k_j}{\omega^2} \Phi_{ij}(\Delta k) \right). \]  
(11)

If the turbulence is assumed to be isotropic (or at least isotropic for a range of wave numbers \( \kappa \)), then from symmetry considerations it can be shown that (Tatarski 1961)
\[ \Phi_T(\kappa) = \Phi(\kappa) \]  
(12a)
and
\[ \Phi_{ij}(\kappa) = \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) E(\kappa) \]  
(12b)
where \( E(\kappa) \) is the spectral density of turbulent energy:
\[ \frac{1}{2} \left\langle (\nu')^2 \right\rangle = \int E(\kappa) 4\pi k^2 \, dk \]  
(13)
The assumption of isotropic turbulence for the atmosphere is usually valid in the inertial subrange, for which \( \lambda \sim 0.1 \, \text{m} \), where \( \lambda = 2\pi/k \). It is crucial for determining the final form of the cross-section formula, upon which acoustic sounder interpretations are based. In particular, the second term in curly brackets in (11), accounting for wind fluctuations warrants close examination: Substituting for \( \Phi_{ij} \) from (12b), there results a factor
\[ k_i k_j \left( \delta_{ij} - \frac{(\Delta k_i)(\Delta k_j)}{|\Delta k|^2} \right) = k^2 - \frac{(k \cdot \Delta k)^2}{|\Delta k|^2} \]
which, when \( \Delta k = k_s - k \), \( \cos \theta = k \cdot k_s / k^2 \) is substituted, yields simply
\[ k^2 \cos^2(\theta/2). \]  
(14)
Explicit expressions for \( \Phi_T \) and \( E \) could, if desired, be substituted according to the so-called
'2/3 law' appropriate to the inertial subrange of the turbulent spectrum. (For a lucid discussion of these and other matters related to turbulent structure, see Chapter 2 of Tatarski (1961).)

If Eqs. (12) are employed in Eq. (11), with \( \kappa = |\Delta k| = 2k \sin \frac{1}{2} \theta \), we find, making use of (14),

\[
\frac{d\sigma}{d\Omega} = 8\pi^2 k_0^4 |f(\theta)|^2 \left\{ \frac{\Phi_1(2k \sin \frac{1}{2} \theta)}{4T_1} + \frac{\cos^2(\frac{1}{2} \theta) + \cos(\frac{1}{2} \theta)}{c_2} E(2k \sin \frac{1}{2} \theta) \right\}
\]

(15)

where we have substituted \( \omega = kc \).

The more elaborate theories produce a similar expression for \( d\sigma/d\Omega \) but with \( f(\theta) \) specified as \( \cos \theta/2\pi \) (Monin 1962, Tatarski 1971). The simplified model introduced earlier predicts an angular dependence of \( \cos \theta/\sin^2(\frac{1}{2} \theta) \) for \( f(\theta) \), which is at least qualitatively in accordance with the experimental result (Kallistratova 1961) of zero scattered intensity at \( \theta = \pi/2 \). In this respect, our model is better than earlier theories (Obukhov 1941, Tatarski 1961) where \( f(\theta) \) had the constant value \( (2\pi)^{-1} \).

The origins of the two terms in curly brackets in (11) or (15) can be traced back to and understood in terms of the simplified model introduced at the start. The appearance of the \( \cos^2(\frac{1}{2} \theta) \) factor associated with the second term in (15) is a direct result of the isotropic turbulence assumption, and has the significance that turbulent fluctuations in flow velocity do not contribute to backscattering (\( \theta = \pi \)), a fact which is important in interpretation of returns from monostatic acoustic sounders. If the turbulence is anisotropic, such as immediately following breaking of Kelvin–Helmholtz waves, when a well-formed turbulent cascade had not had time to develop, then backscatter from wind fluctuations is in general possible, since there will be no \( \cos^2(\frac{1}{2} \theta) \) term; there may in addition be contributions from temperature–velocity cross correlations, for which the relationship (7c) no longer applies under these circumstances. In short, the most important features of the scattering cross-section derive from the form (2) of the reflection coefficient in simple reflection at the boundary of two dissimilar media, plus symmetry considerations; the usual theories tend to obscure this observation.

Finally, we note that Eq. (15) is similar in form to expressions for cross-sections for other quite different experiments in physics, notably scattering of sound in a fluctuating ocean (Flatté 1979), and of light from fluids and X-rays from solids (Goodstein 1975). The spectral density of the appropriate correlation functions evaluated at wave number \( 2k \sin \frac{1}{2} \theta \) generally appears as in (15) as does the structure factor \( f(\theta) \) which is determined by the nature of the individual scatterers. In general then, backscattering \( \theta = \pi \) effectively probes the structure of a medium on a scale of half the wavelength of the incident radiation. In this connection we should also mention other methods of remote sensing in meteorology (Rogers and Vali 1978) and oceanography (Ward and Dexter 1976).

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