Comparison of some different approximations in the statistical theory of relative dispersion

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SUMMARY

The effect on the theory of relative dispersion of some different approximations to the two-particle Lagrangian correlation function is examined. Two of these, one due to Taylor, the other to Smith and Hay, are treated in detail.

Through numerical solution of the dispersion equations, the influence of the initial cluster size, a number of simple variations of the spatial separation argument of the two-particle correlation function, the number of particles in the cluster and the ratio of Lagrangian to Eulerian integral scales are examined. With the exception of the initial cluster size, which is important in the early stage of growth, these parameters are relatively unimportant, particularly compared to the overall difference between the two approximations.

In qualitative agreement with Batchelor’s inertial range theory, both the Taylor and Smith-Hay approximations show linear growth at small time with an accelerated growth region at intermediate time. However, between these regions, the Smith-Hay solution shows a region of less-than-linear growth for which there appears to be no observational or theoretical support. This regime is more pronounced, and the difference between the two approximations greater, for initially small clusters.

Comparison with suitably documented observations, while not entirely definitive, shows a degree of consistency and suggests the Taylor approximation to be the more appropriate.

1. INTRODUCTION

In the past twenty years or so, the problem of specifying the dispersion of pollutants in the atmosphere has received a great deal of attention from experimental, empirical and theoretical points of view (Pasquill 1974; Slade 1968; Turner 1970). Most of this has centred on the time-mean distribution of material from a continuous source.

The companion problem of the separation of a pair (or cluster) of particles, which involves dispersion relative to a moving origin (the centre-of-mass of the cluster) rather than absolute or total dispersion as in the continuous source case, has received less attention. However, it is hardly less important since, besides the obvious direct application to the spread of clouds of material, it is also relevant in specifying the mean-square concentration field and hence concentration fluctuations. In particular, the mean-square particle separation is a critical parameter in the two-particle probability density function from which the mean-square concentration field may in principle be calculated for an arbitrary source configuration (Lamb 1980; Durbin 1980). On a more practical if more heuristic level, Gifford’s (1960) fluctuating plume model requires as input a measure of both total and relative dispersion.

The importance of two-particle statistics in the relative dispersion problem was first recognized by both Brier (1950) and Batchelor (1950) who independently demonstrated the involvement of the correlation between the velocities of two different particles separated in both space and time. This two-particle Lagrangian correlation function is fundamental to the relative dispersion problem, but has apparently not been studied in detail either experimentally or theoretically. For example, in developing his inertial range theory of relative dispersion, Batchelor (1950) relied on dimensional analysis to deduce (to within a dimensionless constant of order unity) the rate of expansion at ‘small’ and ‘intermediate’ times, but invoked no knowledge of this correlation function other than the trivial small-time limit. Various attempts experimentally to verify Batchelor’s (1950) theory (Gifford 1957a, b; 1977), also throw no light on the nature of this function, or its effect on relative dispersion. However, various approximate forms of the two-particle Lagrangian correlation function have been proposed in this context (Brier 1950; Batchelor 1952; Smith and Hay 1961). It is the purpose of this paper to compare these approximations and
in particular to assess their effect on prediction of the mean-square separation of a cluster of particles. The wider relevance to prediction of concentration fluctuations has already been mentioned. The theory developed here follows closely that of Smith and Hay (1961) but greater attention is given to the finer points and assumptions involved in ensemble averaging and averaging over particles (or pairs of particles) in the cluster. In particular, differences between the two-particle and many-particle cases are examined closely.

2. Theory

(a) Relative dispersion equation

We consider the general case of a cluster of \( N \) particles, and will be particularly concerned with the cases of \( N = 2 \) and \( N = \infty \). Following Batchelor (1950), the equation for the mean-square separation of an arbitrary pair of particles within the cluster is

\[
\frac{d}{dt} \langle l_2^2(t) \rangle = 2 \int_0^t \langle (u_i(t) - u_i(\tau)) \cdot (u_j(t) - u_j(\tau)) \rangle d\tau. \tag{1}
\]

A list of notation is contained in the Appendix. Note that tensor algebra is not used.

Now Eq. (1) contains two types of velocity product. The first, of the form \( u_i(t) \cdot u_i(\tau) \) refers to the same particle at two different times and thus represents a Lagrangian auto-covariance. The second, of the form \( u_i(t) \cdot u_j(\tau) \) involves one particle at time \( t \) and a second at time \( \tau \) and thus is a two-particle Lagrangian covariance. Rewriting Eq. (1) in terms of the corresponding velocity correlation functions for stationary, homogeneous, isotropic turbulence we have

\[
\frac{d}{dt} \langle l_2^2(t) \rangle = 4 \langle u^2 \rangle \int_0^t \{ R_E(t - \tau) - R_{2L}(t, \tau) \} d\tau, \tag{2}
\]

which can then be averaged over all pairs in the cluster to give formally,

\[
\frac{d}{dt} \langle l_2^2 \rangle = 6 \int_0^t \frac{d}{dt} \sigma^2(t) = 4 \langle u^2 \rangle \int_0^t \{ R_E(t - \tau) - R_{2L}(t, \tau) \} d\tau. \tag{3}
\]

Without loss of generality, we assume in Eqs. (2) and (3) and for the rest of this paper that \( t > \tau \).

An assumption implicit in the analysis given here is that the effects of molecular diffusion may be ignored, so that the theory is restricted to scales large compared to the Kolmogorov scale, \( (v^3/\epsilon)^{1/4} \) (Batchelor 1950; Durbin 1980).

(b) Approximations for \( R_{2L}(t, \tau) \)

Smith and Hay (1961), although presenting their analysis somewhat differently, effectively assumed

\[
R_{2L}(t, \tau) = \langle R_E[|r_i(t) - r_j(t) + U(t - \tau)/\beta|] \rangle, \tag{4}
\]

where the Lagrangian separation in time has been incorporated into the argument of the trace Eulerian-space correlation function, \( R_E \), as an Eulerian separation in space using the Hay-Pasquill (Hay and Pasquill 1959) and Taylor (Pasquill 1974, p. 9 and p. 42) hypotheses. We defer specification of the ratio of Lagrangian to Eulerian integral scales, \( \beta \), until Section 3. Averaging over the ensemble distribution of particle separations is discussed in detail below.

A related assumption, not considered by Smith and Hay (1961) is

\[
R_{2L}(t, \tau) = \langle R_E[|r_i(\tau) - r_j(\tau) + U(t - \tau)/\beta|] \rangle. \tag{5}
\]

Neither of these forms takes account of the fact that the separation of the particles changes
over the period $t - \tau$. In the spirit of Eq. (4) and Eq. (5), a version which incorporates the actual separation $r_i(t) - r_j(\tau)$ is

$$R_{2L}(t, \tau) = \langle \{r_i(t) - r_j(\tau) - U(t - \tau) + U(t - \tau)\beta_i\} \rangle. \quad (6)$$

The extra term $U(t - \tau)$ is subtracted to avoid including the separation in time twice—once in a Lagrangian sense and once in an Eulerian sense.

In an unpublished note in 1935 (referred to by Batchelor 1952), G. I. Taylor proposed a set of approximations which, ignoring covariances between different velocity components, is equivalent in the present notation to

$$R_{2L}(t, \tau) = \left\langle \{u_i(t)u_j(t)\} \right\rangle / \langle u^2 \rangle^2 = \langle R_E[\{r_i(t) - r_j(t)\}] \rangle R_L(t - \tau) \quad (7)$$

or

$$R_{2L}(t, \tau) = \left\langle \{u_i(\tau)u_j(\tau)\} \right\rangle / \langle u^2 \rangle^2 = \langle R_E[\{r_i(\tau) - r_j(\tau)\}] \rangle R_L(t - \tau), \quad (8)$$

where the two-particle covariances at fixed time, $\langle u_i(t)u_j(t) \rangle$ and $\langle u_i(\tau)u_j(\tau) \rangle$ have been written here as an average of the Eulerian space covariance over the ensemble distribution of particle separations at times $t$, $\tau$ respectively. Thus the two-particle correlation function reduces to a simple product of an averaged Eulerian space correlation and a Lagrangian autocorrelation. The only difference between Eq. (7) and Eq. (8) is the time at which the Eulerian function is evaluated. Again, both ignore any change in the separation over the interval $(t - \tau)$.

Another approximation in the same spirit is

$$R_{2L}(t, \tau) = \langle R_E[\{r_i(t) - r_j(\tau) - U(t - \tau)\}] \rangle R_L(t - \tau) \quad (9)$$

which involves the actual separation $r_i(t) - r_j(\tau)$ in the argument of the Eulerian space component. The reason for the term $-U(t - \tau)$ is as before.

For completeness, two approximations used by Brier (1950) are also mentioned here. In the first part of his paper he used, in the present notation,

$$R_{2L}(t, \tau) = \langle R_E[\{r_i(0) - r_j(0)\}] \rangle R_L(t) R_L(\tau). \quad (10)$$

In the later part of his paper, he assumed the second Taylor approximation, Eq. (8), in addition to Eq. (10). These two equations enable the Lagrangian correlation functions to be written in terms of Eulerian space correlations. In particular

$$R_L(t - \tau) = \{\langle R_E[\{r_i(t - \tau) - r_j(t - \tau)\}] \rangle / \langle R_E[\{r_i(0) - r_j(0)\}] \rangle\}^{15} \quad (11)$$

Both these assumptions made by Brier (1950) suffer from obvious physical defects. In the first instance, Eq. (10) taken alone results in a two-particle correlation function which decreases with time of travel, regardless of the separation of the particles at times other than $t = 0$. In the second case Eq. (11) is a Lagrangian correlation function (for a single particle) which depends on the expected separation of a pair of particles. Neither of Brier’s approximations is pursued further here, although Storebø and Bjorvatten (1977) have used Eq. (10) to calculate the centre-of-mass correlation function in their treatment of the motion of multiple clusters.

Thus the main thrust of the present paper is to compare the effect on the theory of relative dispersion of using the Taylor Eqs. (7)-(9) and Smith-Hay Eqs. (4)-(6) approximations for the two-particle Lagrangian correlation function.

(c) Explicit consideration of the averaging processes

Equations (2) and (3) are quite general and essentially free from approximation other than the general conditions imposed on the nature of the turbulence. However, to cast them into a usable form requires explicit consideration of the averaging over the pairs in the cluster and the averaging over the distribution of particle separations as well as introduction of specific forms for the correlation functions.
Different particles in the cluster are identifiable only because of the influence their initial positions have on their subsequent expected positions. Thus averaging over all pairs of the cluster reduces to averaging over the distribution of initial separations. For a large number of particles, we assume a continuous Gaussian distribution with zero mean and variance $2 \sigma^2(0)$. Thus, for any function $f$ of the initial separation

$$\bar{f(l_{ij}(0))} = \frac{1}{8\pi^{3/2}\sigma(0)^3} \int f \exp\{-l_{ij}^2(0)/4\sigma^2(0)\} \, dl_{ij}(0). \quad (12)$$

The distribution of separations appropriate to Eqs. (6) and (9) is considered first, since results for Eqs. (4), (5), (7) and (8) arise as special cases. Again a Gaussian distribution is assumed so that only the mean and variance need to be specified and in Cartesian coordinates the three components may be considered separately. For the x-component (with similar equations for the y- and z-components)

$$\langle x_i(t) - x_j(t) \rangle = x_i(0) - x_j(0) \quad (13)$$

and

$$\langle \{x_i(t) - x_j(t)\}^2 \rangle = \langle x_i^2(t) \rangle + \langle x_j^2(t) \rangle - 2\langle x_i(t)x_j(t) \rangle.$$  

While the mean depends very simply on the initial separation, the variance does not. Clearly, the problem lies in the correlation between $x_i(t)$ and $x_j(t)$, which depends in some way on $\mathbf{R}_{ij}(t - \tau)$ itself, and is in general a complicated function of the initial separation. Thus, even if we could write down a formal expression for the distribution of $\{r_i(t) - r_j(t)\}$, the subsequent process of averaging over the pairs of the cluster through Eq. (12) would be difficult if not impossible. Further progress requires additional assumptions.

As a starting point, assume that the translation of the centre of mass of the cluster over the period $(t - \tau)$ is a result of the mean motion only. Then

$$x_i(t) - x_j(t) = x_i(0) - x_j(0) + U(t - \tau), \quad (14)$$

so that the distribution of $x_i(t) - x_j(t)$ may be approximated by the distribution of $\{x_i(t) - x_j(t)\}$ since $U(t - \tau)$ is fixed. The first and second moments of the latter distribution are

$$\langle x_i(t) - x_j(t) \rangle = x_i(0) - x_j(0)$$

and

$$\langle \{x_i(t) - x_j(t)\}^2 \rangle = \langle x_i^2(t) \rangle + \langle x_j^2(t) \rangle - 2\{\langle x_i(t)x_j(t) \rangle \} = x_i^2(0) - x_j^2(0) + 2\langle x_i(t)x_j(t) \rangle \quad (15)$$

We note here that for Eqs. (4) and (5) or (7) and (8), for which the distribution of the separation at a given time is of interest, Eq. (14) with $t = \tau$ is exact and Eq. (15) is equivalent to Eq. (13). However, those versions ignore all motion over the period $(t - \tau)$, and thus already embody the assumption neglecting the eddy motion of the centre of mass.

While it appears we have not progressed very far, since the variance of $x_i(t)$ and the correlation between $x_i(t)$ and $x_j(t)$ (or for that matter, $x_i(t)$ and $x_i(t)$) are still unknown functions of the initial positions, in some special cases Eq. (15) can be reduced to a more manageable form.

Assume that the second order central moments can be replaced by the cluster-averaged values.

$$\langle \{x_i(t) - x_j(t)\}^2 \rangle - \langle x_i^2(t) \rangle = \langle x_i^2(t) \rangle - x_i^2(0) + \langle x_i^2(t) \rangle - x_i^2(0) + 2\langle x_i(t)x_i(t) \rangle - x_i^2(0)/(N - 1) \quad (16)$$

so that Eq. (15) can be approximated by

$$\langle x_i(t) - x_j(t) \rangle = x_i(0) - x_j(0)$$

and

$$\langle \{x_i(t) - x_j(t)\}^2 \rangle - \langle x_i(t) - x_j(t) \rangle^2 \quad (17)$$

$$= \{(N - 1)/N\} \{\sigma_i^2(t) + \sigma_j^2(t) - 2\sigma_i^2(0)\} + \{2/(N - 1)\} \{\langle x_i(t)x_i(t) \rangle - x_i^2(0)\}.$$
Under what circumstances is this last assumption valid? It is certainly so for \( N = 2 \), since then averaging over the cluster is trivial and Eqs. (15) and (17) are equivalent. The correlation between \( x'(t) \) and \( x'(\tau) \) is still unknown, but clearly vanishes as \( (t - \tau) \) tends to infinity and goes to unity as \( (t - \tau) \) tends to zero, giving the lower and upper limits

\[
\frac{1}{2} \{ \sigma^2(t) + \sigma^2(\tau) - 2\sigma^2(0) \} \leq \text{var} \{ x_i'(t) - x_j'(\tau) \} \leq \frac{1}{4} \{ (\sigma(t) + \sigma(\tau))^2 - 4\sigma^2(0) \}. \tag{18}
\]

If the alternative parameterizations Eqs. (4) or (5) are used instead of Eqs. (6) or (7) or (8) instead of (9)), (17) retains the same form but with \( \tau = t \) (or vice versa) and the upper limit of Eq. (18) becomes an equality. We thus suspect in general that the upper limit is close to the correct result, but defer final judgement until Section 3.

For other values of \( N \), the assumption is valid for small \( t \) since then the distributions corresponding to both Eqs. (15) and (17) approach a Dirac \( \delta \)-distribution centred on \( x_i'(0) - x_j'(0) \). It is also valid when \( \langle x_i'^2(t) \rangle \) is much greater than \( x_i'^2(0) \), that is for \( t \) suitably large, and is in fact only in doubt for that portion of the growth where \( \langle x_i'^2(t) \rangle \) is of order \( x_i'^2(0) \). For initially small clusters this region represents an insignificant part of the growth, but in other cases may be the region of interest. It is not clear in general how to determine precisely when the approximation fails or how badly it does so.

For large \( N \), from Eq. (17),

\[
\langle [x_i'(t) - x_j'(\tau)]^2 \rangle - \langle [x_i'(t) - x_j'(\tau)] \rangle^2 = \sigma^2(t) + \sigma^2(\tau) - 2\sigma^2(0). \tag{19}
\]

Finally, since the variables \( x_i' - x_j' - \langle x_i'(0) - x_j'(0) \rangle \) and \( x_i'(0) - x_j'(0) \) to the same approximation are uncorrelated, their distributions can be combined simply by adding variances so that, from Eqs. (19) and (12),

\[
\langle R_E \rangle = \frac{1}{8\pi^3 \tilde{\sigma}^3} \int_0^\infty R_E \exp \left\{ -\left| r_i'(t) - r_j'(\tau) \right|^2 / (4\tilde{\sigma}^2) \right\} \, dr \tag{20}
\]

for \( N \) large, where \( \tilde{\sigma}^2 = \frac{1}{2} \{ \sigma^2(t) + \sigma^2(\tau) \} \).

From Eq. (18), the corresponding two-particle result is

\[
\langle R_E \rangle = \frac{1}{8\pi^3 \tilde{\sigma}^3} \int_0^\infty R_E \exp \left\{ -\left| r_i'(t) - r_j'(\tau) - s'(0) \right|^2 / (4\tilde{\sigma}^2) \right\} \, dr \tag{21}
\]

for \( N = 2 \), where

\[
\frac{1}{2} \{ \sigma^2(t) + \sigma^2(\tau) - 2\sigma^2(0) \} \leq \tilde{\sigma}^2 \leq \frac{1}{4} \{ (\sigma(t) + \sigma(\tau))^2 - 4\sigma^2(0) \}.
\]

When the spatial separation argument of \( R_E \) is evaluated at a fixed time, as in Eqs. (4) and (5) or (7) and (8), results identical in form to Eqs. (20) and (21) are obtained, but with \( t = \tau \) on the right-hand side.

\[(d)\] **Incorporation of specific forms for \( R_{21}(t, \tau) \)**

**Smith-Hay approximation**

For \( N \) large, substituting Eq. (6) into Eq. (20), writing \( \{ U(t - \tau)/\beta + r_i'(t) - r_j'(\tau) \} \) equal to \( r \) and integrating over the sphere \( r = \text{constant} \), gives

\[
\overline{R_{21}(t, \tau)} = \frac{1}{2\pi^{1/2} \tilde{\sigma}} \int_0^\infty s^{1/2} R_E(s) \left\{ \exp \left\{ -\left( (r - s') / 2\tilde{\sigma} \right)^2 \right\} - \exp \left\{ -\left( (r + s') / 2\tilde{\sigma} \right)^2 \right\} \right\} \, ds, \tag{22}
\]

where \( s' \) is \( \{ U(t - \tau)/\beta \} \). For an exponential correlation function

\[
R_E(r) = \exp(-r/l_E), \tag{23}
\]

\[
\overline{R_{21}(t, \tau)} = (e^{\tilde{\sigma}^2/2\tilde{\chi}}) \left( e^{\tilde{\chi}'(2\tilde{\eta}^2 + \tilde{\chi}')} \text{erfc}(\tilde{\eta} + \tilde{\chi}'/2\tilde{\eta}) - e^{-\tilde{\chi}'}(2\tilde{\eta}^2 - \tilde{\chi}') \text{erfc}(\tilde{\eta} - \tilde{\chi}'/2\tilde{\eta}) \right), \tag{24}
\]

using the non-dimensional variables \( \tilde{\eta} \) equal to \( \tilde{\sigma}/l_E \) and \( \tilde{\chi}' \) equal to \( s'/l_E \).
Substituting from Eq. (24) into Eq. (3), transforming to space variables using the Taylor hypothesis and rearranging, the dispersion relation is

$$\frac{d\bar{\eta}^2}{d\chi} = 2\beta^2 i^2 \int_0^\chi \left( e^{-\chi'} - \frac{e^{\chi'}}{2\chi'} \right) \left[ e^{\chi'}(2\bar{\eta}^2 + \chi') \text{erfc}\{\bar{\eta} + (\chi'/2\bar{\eta})\} - e^{-\chi'}(2\bar{\eta}^2 - \chi') \text{erfc}\{\bar{\eta} - (\chi'/2\bar{\eta})\} \right] d\chi' \quad \ldots \quad (25)$$

for \( N \) large, where \( \chi = x/\beta l_E \) and \( 2\bar{\sigma}^2 = \sigma^2(\chi) + \sigma^2(\chi - \chi') \).

For the two-particle case, Eq. (21) yields an equation identical in form to Eq. (24) but with \( \chi' \) equal to \( |x + l_i(0)/l_E| \) replacing \( \chi' \) and \( \bar{\sigma} \) as in Eq. (21). The two-particle dispersion equation is then

$$\frac{d\bar{\eta}^2}{d\chi} = 2\beta^2 l^2 \int_0^\chi \left( e^{-\chi'} - \frac{e^{\chi'}}{2\chi'} \right) \left[ e^{\chi'}(2\bar{\eta}^2 + \chi') \text{erfc}\{\bar{\eta} + (\chi'/2\bar{\eta})\} - e^{-\chi'}(2\bar{\eta}^2 - \chi') \text{erfc}\{\bar{\eta} - (\chi'/2\bar{\eta})\} \right] d\chi' \quad \ldots \quad (26)$$

for \( N = 2 \), with

$$\frac{1}{2}(\sigma^2(\chi) + \sigma^2(\chi - \chi') - 2\sigma^2(0)) \leq \bar{\sigma}^2 \leq \frac{1}{8}[(\sigma(\chi) + \sigma(\chi - \chi'))^2 - 4\sigma^2(0)] \quad \ldots \quad (27)$$

For the alternative versions Eqs. (4) and (5) of the Smith-Hay formulation, equations identical in form to Eqs. (24) – (27) result but with \( \sigma^2(\chi) \) replacing \( \sigma^2(\chi - \chi') \) for Eq. (4) and vice versa for Eq. (5).

**Taylor approximation**

Averaging Eq. (9) using Eq. (20) with an exponential space correlation function gives

$$\overline{R_{2L}(t, \tau)} = R_L(t - \tau)\left( e^{\bar{\eta}^2(1 + 2\bar{\eta}^2)} \text{erfc}(\bar{\eta}) - 2\pi^{-\frac{1}{2}} \bar{\eta} \right), \quad \ldots \quad (28)$$

for \( N \) infinite, so that after some algebra, the dispersion relation in space variables is

$$\frac{d\bar{\eta}^2}{d\chi} = 2\beta^2 l^2 \int_0^\chi e^{-\chi'} \left( 1 + 2\pi^{-\frac{1}{2}} \bar{\eta} - e^{\bar{\eta}^2(1 + 2\bar{\eta}^2)} \text{erfc}(\bar{\eta}) \right) d\chi' \quad \ldots \quad (29)$$

Note that in the case of the Taylor approximation we could more correctly have retained unchanged the Lagrangian part of the correlation function and used time as the independent variable. However, to facilitate comparison with the Smith-Hay equations, we have used the Hay-Pasquill and Taylor hypotheses in deriving Eq. (29).

For the two-particle case, the analysis is similar to the derivation of Eq. (24), the final results being

$$R_{2L}(t, \tau) = R_L(t - \tau)\left( e^{\bar{\eta}^2(2\lambda_{ij}(0))} \text{erfc}\{\bar{\eta} + \lambda_{ij}(0)/2\bar{\eta}\} - \exp\{-\lambda_{ij}(0)\} \left( 2\bar{\eta}^2 - \lambda_{ij}(0) \right) \text{erfc}\{\bar{\eta} - \lambda_{ij}(0)/2\bar{\eta}\} \right), \quad \ldots \quad (30)$$

and

$$\frac{d\bar{\eta}^2}{d\chi} = 2\beta^2 l^2 \int_0^\chi e^{-\chi'} \left( 1 - e^{\bar{\eta}^2(2\lambda_{ij}(0))} \text{erfc}\{\bar{\eta} + \lambda_{ij}(0)/2\bar{\eta}\} - \exp\{-\lambda_{ij}(0)\} \left( 2\bar{\eta}^2 - \lambda_{ij}(0) \right) \text{erfc}\{\bar{\eta} - \lambda_{ij}(0)/2\bar{\eta}\} \right) d\chi' \quad \ldots \quad (31)$$

for \( N = 2 \), where \( \lambda_{ij}(0) = l_{ij}(0)/l_E \). For each of the three versions of the Taylor formulation, the specification of \( \bar{\sigma} \) and \( \bar{\eta} \) is the same as the corresponding Smith-Hay version.

3. Evaluation and Discussion of Relative Dispersion Curves

The various dispersion relations have been evaluated numerically by approximating the
derivative, $d\eta^2/d\chi$, by a forward finite difference and the integral on the right by a trapezoid-rule sum. The step size required for suitable precision is limited by the finite difference approximation and depends on $\chi$. A precision of order 1% is achieved with a step size varying from $10^{-3}$ for $\chi$ of order $10^{-2}$, to $10^{-1}$ for $\chi$ of order 10.

It is convenient at this stage to discuss the value adopted for the ratio of integral scales, $\beta$. According to Pasquill (1974, p. 89 and p. 359) the product $\beta i$ is a constant which for lateral correlation functions takes the value 0.44 (albeit with considerable uncertainty). However, since here we are concerned with the Eulerian trace length-scale which is one third larger, it is appropriate to use

$$\beta i = \text{constant} = 0.33$$

(32)

With such a choice the use of non-dimensional variables as in Section 2 results in a universal set of curves depending only on the initial cluster size. Smith and Hay (1961) on the other hand pragmatically take $\beta$ to be constant. It is important to note here that these differing assumptions concerning $\beta$ do not affect the shape of the resulting dispersion curves, merely the scaling and thus the relationship between solutions under different conditions.

Figure 1. Non-dimensional relative dispersion curves (increase over the initial size) for initial cluster sizes $\eta(0) = 10^{-3}, 10^{-2}$ and $10^{-1}$. Other relevant parameters are $N = \infty$, $\sigma^2 = \frac{1}{2}(\sigma_X^2 + \sigma^2_X - \chi^2)$ and $\beta i = 0.33$.

Also shown are the total dispersion ($\eta_t$) curve and lines of slope $\frac{1}{2}$ and 1.
(a) Variation of initial cluster size

Figure 1 shows non-dimensional cluster relative dispersion curves \((N \text{ large})\) for initial cluster sizes \(\eta(0) = 10^{-3}, 10^{-2}\) and \(10^{-1}\) and for \(\sigma^{2}\) equal to \(\frac{1}{2}\{\sigma^{2}(\chi) + \sigma^{2}(\chi - \chi')\}\). Note that the increase over the initial size, \(\{\eta^{2}(\chi - \eta^{2}(0))\}^{1/2}\), is plotted. Also shown for comparison are the total dispersion curve, which for an exponential correlation function is specified by the relation

\[\eta^{2}(\chi) - \eta^{2}(0) = 2B^{2}i^{2}(e^{-\chi} + \chi - 1),\]

and lines representing slopes of 1 and \(\frac{1}{2}\).

We see that the Taylor curves in particular are linear for small \(\chi\). Taking the limit as \(\chi\) tends to zero of Eq. (29) gives

\[d\eta^{2}/d\chi = 2B^{2}i^{2}\left(1 + 2\pi^{-\frac{1}{2}}\eta(0) - \left[\exp(\eta^{2}(0))\right] \{1 + 2\eta^{2}(0)\} \text{erfc}(\eta(0))\right)\chi,\]

which for small \(\eta(0)\) reduces to

\[d\eta^{2}/d\chi = (8\pi^{-\frac{1}{2}}B^{2}i^{2}\eta(0))\chi.\]

It can readily be confirmed that the numerical results for small \(\chi\) are in close agreement with the linear relation Eq. (34) while for \(\eta(0)\) less than \(10^{-2}\) the limit Eq. (35) is a good approximation. As long as Eq. (35) is valid, the curves show a square-root dependence on the initial cluster size \(\eta(0)\). Elsewhere the dependence on \(\eta(0)\) is weaker.

On the other hand, for the Smith-Hay relation (25) the limit Eq. (34) applies only so long as \(\chi\) is much less than \(2\eta^{2}\). Indeed, for \(2\eta^{2} \ll \chi \ll 1\),

\[d\eta^{2}/d\chi \sim 4B^{2}i^{2}\eta^{2}(0)\ln\chi,\]

which clearly indicates a less than linear growth rate. Thus for small \(\eta(0)\), the initial linear-growth phase is an insignificant part of the Smith-Hay curve and, in fact, the less-than-linear regime dominates. This feature is clearly evident from Fig. 1, and although alluded to, was not examined closely by Smith and Hay (1961). According to Eq. (36) there is a linear dependence of \(\eta(0)\) in this region, and although the numerical results in Fig. 1 show a slightly less than linear dependence, it is certainly stronger than square root. The Smith-Hay curves approach the Taylor linear regime as \(\chi\) tends to zero, more rapidly so for larger \(\eta(0)\).

For intermediate values of \(\chi\), both dispersion forms show, for sufficiently small \(\eta(0)\), an accelerated growth phase with slope greater than 1. While both approach the total dispersion curve at large \(\chi\), the Taylor form does so more rapidly.

It seems that the Smith-Hay form never overcomes the lag caused by the less-than-linear growth region at small \(\chi\). Ultimately both forms reach the limiting \(\chi^{1/2}\) behaviour of the total dispersion curve. The greatest difference between the two forms is at intermediate times, and is accentuated with decreasing \(\eta(0)\).

It is of interest to compare the present results with the similarity predictions made by Batchelor (1950). Quantitative agreement is not to be expected since the exponential Eulerian space correlation function adopted here does not obey the Kolmogorov inertial range law (Pasquill 1974, p. 47). However at high wave-numbers, the energy spectrum corresponding to the exponential correlation function behaves as \(k^{-2}\), not very different to the inertial range \(k^{-5/3}\) law, so qualitative agreement at least might be expected. Recalling that Batchelor's (1950) analysis predicts (in space variables)

\[\eta^{2}(\chi) - \eta^{2}(0) \sim \chi^{2}\quad \text{for} \ \chi \ '\small{\text{small}}',\]

and

\[\eta(\chi) \sim \chi^{3/2}\quad \text{for} \ \chi \ '\text{intermediate}',\]

this is so, at least for the Taylor form. Batchelor's theory does not predict the region of less-than-linear growth of the Smith-Hay curves. Coincidentally, for \(\eta(0)\) equal to \(10^{-2}\), the slope in the accelerated growth region is about \(3/2\), but for smaller values of \(\eta(0)\) is even steeper. The proportionality constants in Batchelor's theory are not the same as, for
example, in Eq. (35), since the only relevant parameters in the inertial range theory are the rate of dissipation of energy and \( \sigma(0) \).

(b) Alternative relations for \( R_{21}(t, \tau) \)

As shown in Section 2, the alternative forms of the Taylor and Smith-Hay relations result in identical expressions for the dispersion relation, but with \( \bar{\sigma} \) taking different values. The most striking feature of results for the cases \( \bar{\sigma} \) equal to \( \sigma(\chi) \) and \( \bar{\sigma} \) equal to \( \sigma(\chi - \chi') \) is the closeness to the case \( \bar{\sigma}^2 \) equal to \( \frac{1}{2} (\sigma^2(\chi) + \sigma^2(\chi - \chi')) \). Indeed, the \( \sigma(\chi) \) and \( \sigma(\chi - \chi') \) curves depart from those shown in Fig. 1 by at most 30% for the Taylor set and 15% for the Smith-Hay set. The reason for this lack of sensitivity is straightforward. For small \( \chi \), the solutions depend essentially on \( \eta(0) \) and are thus independent of the form of \( \bar{\sigma} \), while for large \( \chi \) they approach the total dispersion curve, again independent of \( \bar{\sigma} \). For intermediate values of \( \chi \), we see from Eq. (29), that because of the exponential weighting factor \( \exp(-\chi') \), by far the greatest contribution to the integral and hence the growth of the cluster comes from values of \( \chi' \) near zero. That is, although \( \sigma(\chi - \chi') \) varies from \( \sigma(0) \) to \( \sigma(\chi) \), the greatest contribution to the integral comes from values near \( \sigma(\chi) \) (in time variables, from \( \sigma(t) \) near \( \sigma(\tau) \)), and consequently there is little practical difference between the formulations Eqs. (7)–(9). While analysis of the Smith-Hay expression Eq. (25) is less obvious, it can easily be shown numerically that relative contributions to the integral are weighted even more strongly to values of \( \chi' \) near zero.

(c) Comparison of cluster and two-particle dispersion

As expected the general behaviour of the two-particle curves follows closely that of the cluster curves. Thus, as \( \chi \) tends to zero, for small \( \eta(0) \),

\[
d(\eta^2(\chi))/d\chi = (2\sqrt{2}/6)\beta^2 t^2 \eta(0) \chi, \tag{39}
\]

which differs from the corresponding cluster result Eq. (35) only in the numerical constant. Similarly for large \( \chi \), the two-particle curves approach the single particle curve and consequently are close to the cluster curves. For intermediate values, numerical evaluation shows the two-particle curves for the upper limit of \( \bar{\sigma} \) are almost indistinguishable from the cluster results, while those for the lower limit lie appreciably below the cluster curve, particularly in the Smith-Hay case.

Now the discussion above concerning the effects of varying the form of \( \bar{\sigma} \) applies equally well to the two-particle results, so that values of \( \chi \) near zero dominate the growth. Since for small \( \chi' \) (that is small \( (t - \tau) \)) the correlation between particle positions at times \( t \) and \( \tau \) (see Eq. (17) and the ensuing discussion) is close to unity, the upper limit of \( \bar{\sigma} \) which assumes total correlation, is an accurate representation of the correct \( \bar{\sigma} \) for two-particle dispersion. Indeed for the parametrizations Eqs. (7), (8), (4) or (5) it is correct. We thus conclude that there is little practical difference between the two-particle and cluster relative dispersion curves.

4. Comparison with observation

While there are many reported observations of relative diffusion in the atmosphere (see for example the summaries by Gifford (1977) and Hefter (1965)), few are appropriate for comparison with the present theory. Many, such as the tetroon trajectory studies (Angell et al. 1971) or the nuclear debris data (Randerson 1972) are mesoscale or large-scale studies and as such out of the range of applicability of the boundary-layer microscale processes being considered here. These larger scale observations are also well removed from the likely range of applicability of Batchelor's (1950) inertial range theory. Of the small scale studies, two which are sufficiently well documented to allow estimation of the intensity and integral
scale of turbulence and the initial cluster size are those of Smith and Hay (1961) and Högström (1964), which we analyze here in some detail.

(a) Högström’s smoke puff data

Högström (1964) measured both the relative and total spread of smoke puffs for two separate sites in stable conditions. Although he measured both the horizontal and vertical lateral spread, sufficient detail for the present analysis was given only for the latter. We note however, that the spread was not isotropic.

Högström (1964) tabulates his results for vertical spread, which represent a synthesis of many experiments, as a function of distance and of a stability parameter. He shows that by a suitable rescaling the total dispersion curves, with the exception of the most stable curve at Studsvik which was affected by local terrain, collapse onto a single curve. Thus here we use the total dispersion curves to evaluate a Lagrangian length scale and the intensity of turbulence for each case, and then use those parameters to scale the relative dispersion curves. As suggested by Pasquill (1974, p. 222), we determine the intensity of turbulence from the total spread at the shortest distance for which results are presented, by assuming the small-time limit,

$$\sigma_t = \kappa x$$  \hspace{1cm} (40)

to apply there. The Lagrangian length scale, $$\beta l_e$$, is then determined from the spread at large distances assuming the asymptotic law

$$\sigma_t^2 = 2\kappa^2 \beta l_e x$$  \hspace{1cm} (41)

It is readily seen that the data essentially attain this behaviour at the largest distances reported.

<table>
<thead>
<tr>
<th>TABLE 1. Scaling parameters for Högström’s (1964) data</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>AGESTA</strong></td>
</tr>
<tr>
<td>$\lambda$</td>
</tr>
<tr>
<td>$t$ (rad)</td>
</tr>
<tr>
<td>$\beta$</td>
</tr>
<tr>
<td>$\beta l_e$ (m)</td>
</tr>
<tr>
<td>$l_e$ (m)</td>
</tr>
</tbody>
</table>

The parameters derived from this fitting procedure are shown in Table 1. Measured values of intensity of turbulence at Studsvik of 0.08, 0.06 and 0.04 for $\lambda$ equal to 0.3, 1.5 and 2.25 respectively (see Högström's (1964) Fig. 12) lend confidence to the values derived here. In addition, we note that the length-scales, being of the order of 0.1 to 0.5 times the height at which the experiments were performed are also quite reasonable (Pasquill 1974, p. 60). The effectiveness of the parameters of Table 1 in scaling the total dispersion data is clear from Fig. 2 (the trace length-scale has been used to scale $\sigma_v$), which also shows for comparison the total dispersion curves Eq. (33) for $\eta_1$ (0) equal to 0.1 and 0.01. The actual $\sigma(0)$ is unknown, but the data suggest $\sigma(0)$ less than about 1 m, and in view of the magnitude of the length-scale these values of $\eta(0)$ represent a likely range. For relative dispersion, the elongated nature of the clouds increases the uncertainty of the initial size.

Figure 3 shows the relative dispersion data together with theoretical curves for $\eta(0)$ equal to 0.1 and 0.01. The use of non-dimensional variables in scaling the data for the neutral and slightly stable runs is remarkably effective and indirectly supports the assumption of a constant $\beta l$ under these conditions. However the very stable results remain consistently about 30% lower. The stable Studsvik run is of course doubtful because of the terrain effects, although the centre-of-mass dispersion rather than the relative dispersion seems to be
Figure 2. Non-dimensional plot of total dispersion data of Högström (1964) for a range of stabilities at two locations. Scaling parameters are taken from Table 1. Also shown for comparison are theoretical curves (Eq. 33) for initial sizes $\eta(0) = 10^{-1}$ and $10^{-2}$.

Figure 3. Non-dimensional plot of relative dispersion data of Högström (1964) for a range of stabilities at two locations. Scaling parameters are taken from Table 1. Also shown are Taylor and Smith-Hay theoretical curves for $\eta(0) = 10^{-1}$ and $10^{-4}$. Other parameters take the values $N = \infty$, $\sigma^2 = \frac{1}{2} \{\sigma^2(x) + \sigma^2(x - x')\}$ and $\beta_i = 0.33$. 

affected (see Fig. 8 in Högström 1964). In any case, the very stable Agesta total dispersion curve (Fig. 2) shows no such abnormality.

Let us consider the effect of possible errors in estimating the various parameters in Table 1. Any error in \( i \) (most probably an underestimation) is reflected in an inverse square manner in \( \beta e \) (through Eq. (41)), inversely in \( \beta \) (through Eq. (32)) and thus inversely in \( l_e \). For example, if \( i \) is 10\% low, \( \beta e \) is 20\% high and \( \beta \) and \( l_e \) are both 10\% high. The resultant effect is to translate the data points along a \( \chi^2 \) line. Similarly, a direct error in estimating \( \beta e \) is reflected as a proportional error in \( l_e \), and thus merely translates the points along a \( \chi^2 \) line. Clearly, although these errors are undoubtedly present, they do not account for the observed difference in Fig. 3.

Another possibility is that \( \beta i \) is not constant, but varies with stability, so that for given \( \beta e \) and \( i \), the assumption Eq. (32) leads to an error in \( \beta \), an error in \( l_e \) and hence a vertical displacement of the data points. However, to bring the stable points into coincidence with the others requires \( \beta i \) to be larger than for the near-neutral runs, that is for \( \beta i \) to increase with decreasing \( i \), which is opposite to the apparent trend shown by the data of Angell (1964) (see also Pasquill (1974), p. 92).

Finally, we note that the difference is not a result of variation in \( \eta(0) \) due to the change in \( l_e \) with stability, since for fixed \( \sigma(0), \eta(0) \) is larger for the more stable cases.

In view of the foregoing discussion, in comparing the observations with the theoretical curves we restrict attention to the less stable runs. The Taylor curves, with \( \eta(0) \) between \( 10^{-1} \) and \( 10^{-2} \) are in good qualitative agreement with observation, although the relative spread is overestimated at large \( \chi \). Indeed, the curve for \( \eta(0) \) about 0.05 represents the data to within about 20\% over virtually the whole range, but in view of the uncertainty in assigning the initial cluster size it would be unwise to attach too much significance to this fact, since in any case, different values of \( \eta(0) \) are implied by the different length-scales for the neutral and slightly stable cases. Obviously however, the data imply a value of \( \eta(0) \) somewhat less than \( 10^{-1} \).

In contrast, as also noted by Pasquill (1974, p. 222) from a rather simpler analysis, the Smith-Hay curves do not agree at all well with observation. This is particularly so at small and intermediate \( \chi \), and although they are closer than the Taylor curves at large \( \chi \), still do not show the same trend as the observations. There is no evidence from the observations for the slow-growth regime of the Hay-Smith curves at small \( \chi \).

In view of the uncertainty involved in the constant in Eq. (32), the comparison between theory and observation has also been made assuming \( \beta i \) equal to 0.44. As mentioned in Section 3, such a change does not alter the shape of the dispersion curves and in practice the rescaling of the data and theoretical curves cancels to a large extent. The relative comparison between the two theories and observation is thus little different from that shown in Fig. 3 for \( \beta i \) equal to 0.33.

(b) Smith and Hay's data

Smith and Hay (1961) conducted a series of short-range experiments measuring the crosswind spread of clouds of Lycopodium spores at distances up to 300 m. Supporting measurements included intensity of turbulence, mean wind speed and an estimate of the scale of turbulence obtained directly from spatial correlations between wind direction vanes. The experiments covered only a small range of stabilities and although the length-scale varied from 5 to 25 m there was no detectable correlation with any meteorological factor. Thus, in analyzing their data we take \( l_e \) equal to 15 m and since Smith and Hay (1961) estimate \( \sigma(0) \) about 1-2 m, we take \( \eta(0) \) = 0.1. As before we use \( \beta i \) = 0.33. For each run, Table 2 shows the parameters used to non-dimensionalize the data, which are then shown in Fig. 4 together with theoretical curves for \( \eta(0) \) equal to \( 10^{-1} \) and \( 10^{-2} \). The data show considerable scatter, with a suggestion that Runs 1, 4, 5 and Runs 2, 3, 6 fall on separate curves differing by a factor of about 1.5. Although there is no consistent difference in
### Table 2. Scaling parameters for Smith and Hay's (1961) data

<table>
<thead>
<tr>
<th>Run</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) (rad)</td>
<td>0.136</td>
<td>0.153</td>
<td>0.151</td>
<td>0.126</td>
<td>0.147</td>
<td>0.140</td>
<td>0.113</td>
<td>0.085</td>
<td>0.091</td>
<td>0.095</td>
</tr>
<tr>
<td>(\beta)</td>
<td>2.43</td>
<td>2.16</td>
<td>2.19</td>
<td>2.62</td>
<td>2.24</td>
<td>2.36</td>
<td>2.92</td>
<td>3.88</td>
<td>3.63</td>
<td>3.47</td>
</tr>
<tr>
<td>(\beta i_\infty) (m)</td>
<td>36.4</td>
<td>32.4</td>
<td>32.8</td>
<td>39.3</td>
<td>33.7</td>
<td>35.4</td>
<td>43.8</td>
<td>58.2</td>
<td>54.4</td>
<td>52.1</td>
</tr>
</tbody>
</table>

\(i_\infty = 15\) (m), \(\beta i = 0.33\)

#### Figure 4. Non-dimensional plot of relative dispersion data of Smith and Hay (1961). Scaling parameters are taken from Table 2. The theoretical curves shown are as for Fig. 3.

Conditions for the two sets of runs, Pasquill (1974, p. 220) suggests the difference may be a result of sampling errors on \(i\). Because of the scatter, the data are rather inconclusive, since the two theoretical forms for \(\eta(0)\) equal to \(10^{-1}\) tend to lie at the upper and lower extremes of the data. The difference between the cases of \(\eta(0)\) equal to \(10^{-1}\) and \(10^{-2}\) is not resolved for the Taylor curves, although it clearly is for the Smith-Hay curves.

It is encouraging to note however that the data are in broad agreement with those of Högström (1964). Indeed, the upper range is in excellent agreement with Högström’s neutral and slightly stable data, while fortuitously the lower range is coincident with Högström’s stable data.

Again, the comparison is not substantially altered by changing \(\beta i\) to 0.44.

#### 5. Conclusions

The novel aspect of the statistical theory of relative dispersion is the occurrence of the two-particle Lagrangian velocity correlation function. In the absence of direct knowledge of the form of this function, in this paper some different parametrizations in terms of simpler functions have been examined. Suggestions made by Brier (1950) lead to physically undesirable features and so were not pursued but approximations due to Taylor (Batchelor 1952) and Smith and Hay (1961) were examined in detail.

Close examination of the separate processes of ensemble averaging and averaging over pairs of particles in a cluster led to the recognition of further inherent approximations, and
of formal differences between the two-particle and many particle cases, the ultimate status of which is presented below.

The properties of the relative dispersion equations were analyzed mainly through numerical solutions. These show that the Taylor solution in particular, agrees qualitatively with Batchelor's (1950) inertial range theory, in that the increase over the initial size is linear for small $\chi$ and then shows a region of accelerated growth. While the Smith-Hay solution also shows the linear behaviour at suitably small $\chi$ and the accelerated growth at intermediate $\chi$, these regions are separated by a region of less-than-linear growth which is mainly responsible for the differences between the two approximations. The existence of such a retarded-growth region is neither predicted by Batchelor's theory nor supported by observation. Thus while the two approximations agree asymptotically at small and large $\chi$ (where the total dispersion $\chi^{1/2}$ law is approached) they differ significantly at intermediate ($10^{-1}$ to $10^1$) values of $\chi$, the difference increasing with decreasing initial cluster size, $\eta(0)$.

Both approximations were shown to be relatively insensitive to variation of the spatial separation argument in the correlation function, due to the strong (at least exponential) weighting of the time-lag argument. Since lags near zero dominate the growth then, as assumed by Smith and Hay (1961), the effect on the particle distributions of expansion of the cluster over the period of the lag can safely be ignored, thus justifying a posteriori the neglect of eddy motion over that period, at least for the purpose of calculating relative dispersion.

This same weighting effect means that the upper of the two limits of growth for the two-particle case is close to the correct result, which is thus in turn almost indistinguishable from the many particle curve except for small $\chi$, where the solutions differ by a factor of order unity.

As with absolute dispersion (Pasquill 1974, p. 137), the relative dispersion theories described here depend only weakly on $\beta$ or in particular on the value of the constant assumed in Eq. (32).

The difference between the two approximations is generally considerably larger than that arising from the variation of other parameters. An important exception is the initial cluster size which has a large effect for small $\chi$. This is consistent with a recent series of papers by Chatwin and Sullivan (1979a, b, 1980) which using rather general conservation principles, also emphasizes the importance of the source size on relative dispersion.

Comparison with observation, although not completely definitive, suggests the Taylor approximation to be the more appropriate. This conclusion is more firmly based on Högström's (1964) data which, being averaged over many experiments represent ensemble mean behaviour. However, the deviation of the most stable data in this set has not been explained. Smith and Hay's (1961) observations on the other hand, represent individual experiments and these show considerable scatter which, together with the small range covered, makes a clear distinction between the two approximations difficult. The consistency between the two data sets is, however, encouraging.

Finally, we note that the Taylor approximation has a number of practical advantages which stem from the fact that the time and space variables are separated. The first is the simplification of the resulting relative dispersion expressions, as can be seen by comparing Eqs. (25) and (29). Secondly the Lagrangian and Eulerian parts of the two-particle correlation function can be specified separately, so that it is straightforward to incorporate into the Taylor form more appropriate formulations for the Lagrangian or Eulerian correlation functions. For example, the exponential form could be retained for the Lagrangian part and a form satisfying the $5/3$ inertial range law used for the Eulerian part, thus producing agreement with Batchelor's (1950) similarity predictions. Thirdly, in Monte Carlo simulations of the motions of pairs of particles, using the Taylor approximation correlation between the velocities of the two particles can be incorporated simply and directly in contrast to the ad hoc methods used at present (Lamb et al. 1979).
COMPARISON OF APPROXIMATIONS IN DISPERSION THEORY

APPENDIX

Important notation

\begin{align*}
\langle f(r) \rangle & \quad \text{ensemble average over all realizations of turbulent field; see Batchelor (1953)} \\
\langle f(r) \rangle & \quad \text{average over ensemble probability distribution for } r \\
\text{average over particles in cluster} & = \frac{1}{N} \sum_{i=1}^{N} \hat{n}_i \hat{\sigma} \\
\text{average over particle pairs in cluster} & = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j \neq i} \hat{n}_i \hat{\sigma} \\
\text{deviation from cluster mean; exceptions } \chi', s' \\
an & \quad \text{vector quantity of magnitude } a \\
i & \quad \text{intensity of turbulence: } i^2 = \langle u^2 \rangle / 3 \Omega^2 \\
\hat{i}, \hat{j}, \hat{k} & \quad \text{unit vectors in } x, y, z \text{ directions; } \hat{i} \text{ downwind} \\
l_E, l_L & \quad \text{Eulerian and Lagrangian integral length scales} \\
l_{ij} & \quad \text{vector separation of particles } i, j \text{ at time } t \\
N & \quad \text{number of particles in cluster} \\
r_i(t) & \quad \text{position vector of particle } i \text{ at time } t \\
R_{\text{L}}(t, \tau) & \quad \text{Lagrangian velocity correlation function: } \langle u_i(t) \cdot u_i(\tau) \rangle / \langle u^2 \rangle \\
R_{\Omega}(t, \tau) & \quad \text{trace Eulerian space velocity correlation function viz. } \langle u_i(t) \cdot u_i(\tau) \rangle / \langle u^2 \rangle \\
R_{\text{LL}}(t, \tau) & \quad \text{two-particle Lagrangian velocity correlation function viz. } \langle u_i(t) \cdot u_i(\tau) \rangle / \langle u^2 \rangle \\
l_E, l_L & \quad \text{Eulerian and Lagrangian integral time scales} \\
u_i(t) & \quad \text{eddy velocity of particle } i \text{ at time } t \\
U & \quad \text{mean wind speed; } U = U \hat{i} \\
\text{var}(x) & = \langle x^2 \rangle - \langle x \rangle^2, \text{variance over ensemble of quantity } x \\
x, y, z & \quad \text{Cartesian components of position vector } r \\
\beta & \quad \text{ratio of Lagrangian to Eulerian integral scales} \\
\eta & \quad \text{stability parameter used by Högström (1964)} \\
\lambda_{ij} & = l_{ij} / l_E \\
\sigma^2 & \quad \text{ensemble expectation of variance of distribution of particles about the centre of mass} \\
\sigma_i^2 & \quad \text{ensemble expectation of total variance of distribution of particles} \\
\bar{\sigma} & \quad \text{see Eqs. (20) and (21) or (25) and (27)} \\
\chi & = x_i \beta l_E = U t / \beta l_E \\
\epsilon & \quad \text{rate of dissipation of energy} \\
\nu & \quad \text{kinematic viscosity}
\end{align*}

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