A model of the quasi-biennial oscillation on an equatorial beta-plane

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SUMMARY

An equatorial beta-plane model of equatorial wave-mean flow interaction in the lower stratosphere is described. Kelvin and mixed Rossby-gravity waves generated by a specified forcing on the lower boundary radiate upward through the model domain where they are subject to thermal and mechanical dissipation. The wave-mean flow feedback problem is solved on the assumption that changes in the zonal mean state occur slowly enough for the wave field to be steady, locally in time.

A quasi-biennial oscillation develops in the model with a period of about 1000 days. A novel feature of this model is the incorporation of latitudinal structure and of thermal and meridional wind components of the oscillation. The influence of the meridional circulation on the structure of the zonal wind oscillation is discussed in some detail.

1. INTRODUCTION

The essential features of the remarkable quasi-biennial oscillation (QBO) of zonal wind in the equatorial stratosphere were successfully modelled by Holton and Lindzen (1972) on the basis of momentum transfer by upward-propagating equatorial waves, the westerly phase being driven by an eastward-propagating Kelvin wave and the easterly phase by a mixed Rossby-gravity wave. The Holton-Lindzen theory has received strong support from a momentum budget analysis of the equatorial lower stratosphere (Lindzen and Tsay 1975) and a laboratory simulation (Plumb and McEwan 1978).

The Holton-Lindzen model (and a subsequent study by Plumb 1977) was one-dimensional and thus did not address the problem of the latitudinal structure of the phenomenon. For the most part, the zonal wind QBO is confined to within about 20° latitude of the equator, although there appears to be a significant signal at high latitudes (Tucker 1979; Holton and Tan 1980) and there are even persistent reports of a tropospheric component (e.g. Trenberth 1980). Tucker identified an amplitude minimum and phase change at about 30°S. Reed (1964) reported a thermal component of the QBO apparently in thermal wind balance with the zonal wind. He also found a phase shift with latitude, with the temperature oscillation in the subtropics being out of phase with that at the equator. Reed and authors of a number of subsequent papers discussed the role of the concomitant mean meridional circulation in the overall dynamics of the QBO and in the generation of the observed QBO in ozone amount. A two-dimensional study is clearly necessary to model these aspects of the oscillation.

Numerical models which explicitly resolve the wave structure in space and time are not well suited to this problem. The fine resolution needed to represent equatorial waves of about 5 km vertical wavelength is prohibitive, because of the small timestep required to ensure stability and the need to integrate over several QBO cycles, i.e. 10 years or so. In the present approach, the waves and mean flow are analysed by separate, though coupled, calculations, thus permitting a much longer timestep of 1 d.

A numerical scheme for calculating the latitude-height structure of equatorial waves on an equatorial beta-plane in the presence of a steady vertical and latitudinal shear flow was described by Plumb and Bell (1982; hereafter EW); results were presented of wave structures and the wave-induced mean flow acceleration for a few cases of equatorial jets. In this paper the mean flow is allowed to evolve in response to this wave-driving. The evolution of the wave field proceeds on the assumption that the mean flow is steady, locally in time; thus the problem (as far as the wave solutions are concerned) is one of a sequence of steady states at each timestep. This approach is in accord with earlier one-dimensional
models of Holton and Lindzen (1972) and Plumb (1977) although more recent studies have questioned its validity. Holton (1979b) found that the effects of wave transience (neglected in the quasi-steady approach) could be responsible for the observation that the east wind maximum of the QBO is found on the equator, in conflict with theoretical prediction based on forcing by steady thermally-damped mixed Rossby-gravity waves. Previously, following Andrews and McIntyre (1976), it had been thought that this observation indicated the influence of mechanical wave dissipation. Further, Dunkerton (1981a, b) has shown that wave transience may be of fundamental importance in the basic dynamics of the equatorial wave-mean flow interaction process.

In the present model, incorporation of mechanical as well as thermal dissipation is not only desirable (in order to ensure that the easterly regime be realistic) but necessary—otherwise the resolution requirements become prohibitive. The relation between dissipation mechanisms and resolution requirements is discussed in EW). This being the case, the model is incapable of addressing the problem of the basic cause of the observed structure of the easterly regime. Those aspects of wave transience discussed by Dunkerton are also implicitly ignored; an a posteriori check shows that the 'steady waves' assumption is a self-consistent one for these calculations, although it is recognized that this is probably an indication of the shortcomings of the model rather than a demonstration that such effects are not important in the equatorial stratosphere.

2. The Dynamical Basis of the Model

(a) The zonal mean motion

The basic equations for wave, mean-flow interaction on an equatorial beta-plane were presented in EW, to which the reader is referred for details. In the presence of small amplitude waves (formally $O(\varepsilon)$ in amplitude, say, where $\varepsilon \ll 1$) the equation for the Lagrangian-mean acceleration is, correct to $O(\varepsilon^2)$,

$$\frac{\partial \bar{u}^L}{\partial t} + v^L \left( \frac{\partial \bar{u}}{\partial y} - \beta y \right) + \bar{w}^L \frac{\partial \bar{u}}{\partial z} = -\nabla \cdot \Psi + P_1^L$$

where $(\cdot)^L$ is the Lagrangian-mean operator as defined by Andrews and McIntyre (1978a), $\Psi$ is the wave flux of zonal pseudo-momentum (an $O(\varepsilon^2)$ quantity) and $P_1$ is the viscous momentum sink. For an introduction to generalized Lagrangian-mean theory, see the excellent reviews by McIntyre (1980) and Dunkerton (1980). The Lagrangian-mean is related to the Eulerian-mean ($\bar{\cdot}$) by the relation

$$\bar{X}^L = \bar{X} + \bar{X}^S$$

where the 'Stokes' correction', $\bar{X}^S$, is an $O(\varepsilon^2)$ quantity associated with the wave motion. Therefore Eq. 1 may be written

$$\frac{\partial \bar{u}}{\partial t} + v^L \left( \frac{\partial \bar{u}}{\partial y} - \beta y \right) + \bar{w}^L \frac{\partial \bar{u}}{\partial z} = -\Delta + \bar{P}_1$$

where

$$\Delta = \nabla \cdot \Psi + \frac{\partial \bar{u}^S}{\partial t} - P_1$$

is the direct wave-induced acceleration of the Eulerian mean zonal wind $\bar{u}$.

The mean flow dissipation, $\bar{P}_1$, is written as a second order diffusion

$$\bar{P}_1 = \frac{\partial}{\partial y} \left( \nu_H \frac{\partial \bar{u}}{\partial y} \right) + \frac{\partial}{\partial z} \left( \nu_T \frac{\partial \bar{u}}{\partial z} \right)$$

The values adopted for the experiment described in detail in this paper (Run 1) were $\nu_H = 5 \times 10^3 \text{ m}^2 \text{ s}^{-1}$ and
\[ v_y = \begin{cases} v_1, & z > 20 \text{ km} \\ v_1 (5 - a/b), & a = z - 18 \text{ km}, \ b = 0.5 \text{ km}, \ z < 20 \text{ km} \end{cases} \] (5)

with \( v_1 = 0.04 \text{ m s}^{-1} \). Thus the vertical component of viscosity is enhanced near the lower boundary of the model at \( z = 18 \text{ km} \) in order to minimize the generation of strong shears there (for reasons that will be discussed in Section 2(b)).

Now, the mean motion is assumed to be symmetric about the equator \( y = 0 \) (provided this is so initially, then the wave-induced acceleration is symmetric and thus the symmetry is preserved). Elsewhere, the domain is bounded by rigid no-slip walls on \( z = z_B = 18 \text{ km} \) and \( y = y_0 = 5000 \text{ km} \) and a stress-free upper boundary on \( z = z_T = 36 \text{ km} \). Therefore the boundary conditions on \( \bar{u} \) are

\[
\begin{align*}
\frac{\partial \bar{u}}{\partial y} (0, z, t) &= 0, \\
\frac{\partial \bar{u}}{\partial x} (y, z_T, t) &= 0, \\
\bar{u}(y_0, z, t) &= 0, \\
\bar{u}(y, z_B, t) &= 0.
\end{align*}
\]

(b) The wave structure and wave forcing of the mean flow

The forcing of the mean flow is provided in the model by a Kelvin wave and a mixed Rossby-gravity wave; these modes are the major wave motions observed in the tropical stratosphere and are those accredited by current theory with the forcing of the QBO (Holton and Lindzen 1972). Generation of these waves within the model is accomplished by specifying at \( z = z_B \) a geopotential fluctuation

\[
\phi(z_B, y, t) = \text{Re } \phi_K \exp(-y^2/2L_K) \exp\{i(k_K(x - c_K t)) + \text{Re } \phi_{RG} (y/L_{RG}) \exp(-y^2/2L_{RG}^2) \exp[ik_{RG}(x - c_{RG} t)]
\]

(6)

where \( (\phi_K, \phi_{RG} ), (k_K, k_{RG} ), (c_K, c_{RG} ) \) and \( (L_K, L_{RG} ) \) are respectively the geopotential amplitudes, wavenumbers, phase speeds and meridional length scales for the Kelvin and mixed Rossby-gravity wave components. These quantities, taken to be representative of the observed waves (within quite a large range), are listed in Table 2 of EW. The relative magnitudes \( \phi_K : \phi_{RG} \) are chosen such that the net input of wave pseudo-momentum into the model is zero. The structure functions are based on theoretical results for equatorial waves in zero zonal flow; it is for this reason that it was found necessary to restrict the generation of strong shear near the lower boundary by increasing the vertical component of viscosity (otherwise numerical problems resulted).

The waves are dissipated in the model by both thermal and mechanical processes parametrized by Newtonian cooling and Rayleigh friction with rate coefficients \( \mu \) and \( \lambda \) respectively. As discussed in EW, Rayleigh friction is incorporated as a crude parametrization of possible mechanical dissipation processes and/or of the effects of wave transience which may be very important in determining the lateral structure of the wave forcing (Holton 1979b) and which, as discussed below, are not incorporated in this model. It was shown in EW that the resolution requirements of the model are very sensitive to the minimum values of \( \mu \) and \( \lambda \). The minimum practicable vertical grid increment was 500 m which led to the choice of a constant value of \( \mu = 1.653 \times 10^{-6} \text{ s}^{-1} \). This is too large to be realistic in the lower regions of the model causing exaggerated wave attenuation there; the effect of this on the results will be discussed in Section 5. Preliminary calculations (including those described in EW) suggested that an appropriate value of \( \lambda \), one that is large enough to allow a vertical grid increment as large as 500 m and to generate wave-induced accelerations that maximize on the equator in agreement with observation yet small enough not to dominate the overall attenuation of the waves, was about one half of value of \( \mu \). In addition, however, in order to minimize unrealistic wave reflection from the upper boundary, an enhanced mechanical dissipation was imposed at high levels, and therefore the profile
\[ \lambda = 8.267 \times 10^{-7} + 1.0 \times 10^{-5} \exp\left( \frac{z - z_T}{d} \right) \text{ s}^{-1}, \quad d = 2 \text{ km} \]

was adopted.

Now, the time-scale on which mean flow changes take place is \( O(\varepsilon^{-2}) \) (Plumb 1977); this is assumed to be long compared with the time for the waves to propagate through the region of interest. Therefore the wave problem is solved assuming that the mean state is steady, locally in time. Further, the wave pseudo-momentum satisfies a conservation relation (Andrews and McIntyre 1978b)

\[ \frac{\partial \tilde{p}}{\partial t} + \nabla \cdot \mathbf{\Psi} = -D \quad \ldots \quad (7) \]

where the density \( \rho \) and dissipation \( D \) of wave pseudo-momentum are \( O(\varepsilon^2) \) quantities, defined for this problem in EW. Now, since \( \rho \) and \( \tilde{u}^s \) are \( O(\varepsilon^2) \), it follows that \( \partial \rho / \partial t \) and \( \partial \tilde{u}^s / \partial t \) are \( O(\varepsilon^4) \) and therefore formally negligible if \( \varepsilon \ll 1 \). From Eqs. 4 and 7 then

\[ \Delta = -D - \tilde{p}_l^s \quad \ldots \quad (8) \]

Under this 'steady waves' approximation, then, at each timestep the wave field is solved separately for each wave for a steady mean flow \( \bar{u}(y, z, t) \) and hence \( \Delta(y, z, t) \) is determined. This calculation is precisely that described in EW, to which the reader is referred for details.

Dunkerton (1981a, b) has questioned the 'steady waves' assumption on the grounds that some neglected quantities (in particular \( \partial \rho / \partial t \)) may not be negligible. Even though \( \rho \) may be small at low levels, it may increase with height because of decreasing density and can also become large in regions of strong shear where the Doppler-shifted phase speed of the wave becomes small. The reliability of the steady waves assumption in the present model will be assessed \textit{a posteriori} in Section 7.

\[(c) \quad \text{The meridional circulation}\]

The meridional component of the Lagrangian-mean equation of motion is

\[ \left( \frac{\partial}{\partial t} + \tilde{v} \frac{\partial}{\partial y} + \bar{w} \frac{\partial}{\partial z} \right) \tilde{v}^L + \beta y \tilde{u}^L - \beta \eta \tilde{u}^L = - \left( \frac{\partial \widetilde{\phi}^L}{\partial y} \right) \quad (9) \]

(Andrews and McIntyre 1978a) where \( \tilde{v}^L \) is the Lagrangian disturbance zonal velocity and \( \eta \) the meridional disturbance parcel displacement. Now, as will be seen, the height and length scales of the mean zonal flow are about \( h = 5 \text{ km} \) and \( L = 1000 \text{ km} \) respectively; the time scale is about \( 10^8 \text{ s} \) while the Lagrangian mean meridional velocity is of order \( 10^{-2} \text{ m s}^{-1} \). Therefore it follows that

\[ \left| \left( \frac{\partial}{\partial t} + \tilde{v} \frac{\partial}{\partial y} + \bar{w} \frac{\partial}{\partial z} \right) \tilde{v}^L \right| \sim 10^{-10} \text{ m s}^{-2}, \quad \text{whereas} \]

\[ |\beta y \tilde{u}^L| \sim \beta L |\tilde{u}| \sim 2 \times 10^{-4} \text{ m s}^{-2}, \quad \text{and hence Eq. 9 may be approximated} \]

\[ \beta y \tilde{u}^L + \left( \frac{\partial \widetilde{\phi}^L}{\partial y} \right) = \beta \eta \tilde{u}^L \quad \ldots \quad (10) \]

The operator \((-)^L\) does not commute with partial differentiation (Andrews and McIntyre 1978a); however

\[ \left( \frac{\partial \widetilde{\phi}^L}{\partial y} \right) = \left( \frac{\partial \widetilde{\phi}}{\partial y} \right)^L + \left( \frac{\partial \widetilde{\phi}}{\partial y} \right)^S \]

\[ = \frac{\partial \widetilde{\phi}^L}{\partial y} - \frac{\partial \widetilde{\phi}^S}{\partial y} = \left( \frac{\partial \widetilde{\phi}}{\partial y} \right)^S \]

The time derivatives of the Stokes' corrections are zero on the steady wave assumption and therefore
\[
\frac{\partial}{\partial t} \left( \beta y \ddot{u}^L + \frac{\partial \ddot{P}^L}{\partial y} \right) = 0.
\] (11)

Similarly, from the hydrostatic relation \( \theta = \partial \phi / \partial z \),
\[
\frac{\partial}{\partial t} \left( \beta y \ddot{u}^L - \frac{\partial \ddot{P}^L}{\partial z} \right) = 0.
\]

Therefore
\[
\frac{\partial}{\partial t} \left( \beta y \ddot{u}^L + \frac{\partial \ddot{P}^L}{\partial y} \right) = 0.
\] (12)

Then, from Eq. (12), with Eq. (3) and the thermodynamic equation
\[
\left( \frac{\partial}{\partial t} + \ddot{v}^L \frac{\partial}{\partial y} \right) \theta^L + \ddot{w}^L N^2 = - \mu \theta^L.
\] (13)

(where \( N \) is the buoyancy frequency),
\[
\beta y \frac{\partial}{\partial z} \left( \frac{\partial \ddot{u}}{\partial y} - \beta y \ddot{v}^L + \frac{\partial \ddot{u}}{\partial z} \ddot{w}^L \right) + \frac{\partial}{\partial y} \left( \frac{\partial \ddot{P}^L}{\partial y} \ddot{v}^L + N^2 \ddot{w}^L \right) = \beta y \frac{\partial}{\partial z} (\ddot{P}_1 - \Delta) - \mu \frac{\partial \ddot{P}^L}{\partial y}.
\] (14)

Now, for steady waves, the Lagrangian-mean flow is non-divergent (Andrews and McIntyre 1978a) and therefore we may define a Lagrangian mass streamfunction \( \chi \) where
\[
\rho \ddot{u}^L = - \frac{\partial \chi}{\partial z}; \quad \rho \ddot{w}^L = \frac{\partial \chi}{\partial y}
\] (15)

when Eq. (14) becomes
\[
\beta y \left( \frac{\partial \ddot{u}}{\partial y} - \beta y \right) \frac{\partial}{\partial z} \left( \frac{\partial \chi}{\partial z} - \frac{\chi}{H} \right) - 2 \beta y \frac{\partial \ddot{u}}{\partial z} \frac{\partial \chi}{\partial y} \left( \frac{\partial \chi}{\partial z} - \frac{\chi}{2H} \right) - N^2 \frac{\partial \chi}{\partial y} = \beta y \rho \frac{\partial}{\partial z} (\ddot{P}_1 - \Delta) + \mu \rho \frac{\partial \ddot{P}^L}{\partial y}.
\] (16)

(the coefficients of \( \chi \) have been approximated to leading order, since \( \chi \) is itself \( O(\varepsilon^2) \)).

While there is, in principle, no reason not to solve Eq. (16) as it stands, there is a major conceptual simplification to be made on the assumption that \( N^2 h^2 \gg \beta^2 L^4 \). For the parameters of this study (q.v., Table 1), \( N^2 h^2 / \beta^2 L^4 \approx 40 \), and therefore the assumption

\begin{table}[h]
\centering
\caption{Details of Basic Numerical and External Parameters of the Model}
\begin{tabular}{ll}
Model domain & \( 0 \leq y \leq 5000 \text{ km} \)  \\
& \( 18 \text{ km} \leq z \leq 36 \text{ km} \)  \\
Horizontal grid increment & \( \Delta y = 200 \text{ km} \)  \\
Vertical grid increment & \( \Delta z = 500 \text{ m} \)  \\
Timestep & \( \Delta t = 1 \text{ day} \)  \\
Buoyancy frequency squared & \( N^2 = 4.667 \times 10^{-4} \text{ s}^{-2} \)  \\
Gradient of Coriolis parameter & \( \beta = 2.28 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1} \)  \\
Density scale height & \( H = 6 \text{ km} \)  \\
\end{tabular}
\end{table}

is a good one. Now, the role of the meridional circulation is to maintain thermal wind balance in the face of perturbing influences (see Holton 1979a, Ch.10.4). From Eq. (16) these influences are

(i) the wave-induced mechanical term \( \beta y \rho \frac{\partial A}{\partial z} \),

(ii) the mean mechanical term \( - \beta y \rho \frac{\partial \ddot{P}_1}{\partial z} \),
(iii) the wave-induced thermal term
\[ \mu \rho \frac{\partial \theta}{\partial y}, \]
and
(iv) the mean thermal term
\[ \mu \rho \frac{\partial \theta}{\partial y}. \]

The meridional circulation due to (i) and (iii) is negligible when \( N^2 h^2 \gg \beta^2 L^4 \) (Andrews and McIntyre 1976). Now, \( \partial \bar{\theta}/\partial z = \beta (\partial \bar{u}/\partial z) \partial z^2 \sim uU/h^2 \) where \( U \) is a velocity scale. But \( \bar{u}/\partial y \sim \beta \bar{u}/\partial z \sim \beta L U/h \). Therefore the ratio of (ii) to (iv) is \( N^2 \mu \rho \sim 2 \times 10^{-3} \). Therefore mean thermal dissipation is the dominant factor driving the Lagrangian mean meridional circulation, and Eq. (16) may be approximated by

\[ \beta \left( \gamma - \frac{\partial \bar{u}}{\partial y} \right) \frac{\partial}{\partial z} \left( \gamma - \frac{\partial \gamma}{\partial z} \right) + 2 \beta \gamma \frac{\partial \bar{u}}{\partial z} \frac{\partial}{\partial y} \left( \frac{\partial \gamma}{\partial z} - \frac{x}{2H} \right) + N^2 \gamma \frac{\partial^2 \gamma}{\partial y^2} = -\mu \rho \frac{\partial \theta}{\partial y}. \]  

(17)

This equation is solved subject to the boundary condition of no normal motion at any of the boundaries, i.e. \( \gamma = 0 \) on \( y = 0, y_0, z = z_B, z_T \).

Note that wave driving, mean mechanical dissipation and mean thermal dissipation now appear separately in Eq. (3) as \(-\Delta, \bar{P}, \) and Lagrangian mean advection, respectively.

3. The Role of the Meridional Circulation

It is worthwhile digressing here to note some features of the Lagrangian mean meridional circulation to be expected on the zonal flow evolution. Since no wave forcing appears in Eq. (17) the problem is precisely the same as that for the mean circulation (Lagrangian or Eulerian) in the absence of waves. This was given considerable attention in the years following the discovery of the QBO. The structure of the mean meridional circulation was deduced by Reed (1964); analyses by Lindzen (1966), Wallace (1967a, b) and Dickinson (1968) suggested that the effect of this circulation could explain the downward propagation of the alternating wind regimes from some (unspecified) high level source. Numerical calculations by Holton (1968) and Wallace and Holton (1968), however, could not reproduce the observed, unattenuated descent of the regimes. Following the success of the Holton-Lindzen (1972) theory in explaining the QBO as a wave-driven phenomenon, it appeared that these studies of the role of the meridional circulation had been superseded. However, many of the arguments put forward in those earlier papers will be seen to be relevant to this study; indeed the influence of the meridional circulation on the zonal wind evolution is the major dynamical difference between this model and one-dimensional models. This being the case, a discussion and extension of these arguments is in order here.

Under the assumption \( N^2 h^2 \gg \beta^2 L^4 \), Eq. (17) may be further approximated to

\[ \bar{\bar{w}}^L = \frac{1}{\rho} \frac{\partial \gamma}{\partial y} = -\mu \frac{\theta}{N^2} \]  

(assuming \( \gamma \to 0 \) and \( \bar{\bar{w}} \to 0 \) at large \( y \)), except close to horizontal boundaries. Near the equator where \( \beta y \) and \( \partial \bar{u}/\partial y \) vanish, Eq. (3) becomes

\[ \left( \frac{\partial}{\partial \ell} + \bar{\bar{w}}^L \frac{\partial}{\partial z} \right) \bar{u} = -\Delta + \bar{P}, \]  

(19)

Reed (1964) estimated that \( |\bar{w}| \sim 10^{-4} \text{ m s}^{-1} \) which is comparable with the rate of descent (about \( 3 \times 10^{-4} \text{ m s}^{-1} \)) of the QBO wind regimes. Therefore vertical advection by the meridional circulation could be a significant factor, especially since Eq. (18) shows that the advection is asymmetric; warm regions (westerly shear) descend while cool regions (easterly shear) ascend (Reed 1964, Wallace 1967a).
At high latitudes, where the Coriolis term is dominant, and thus \( y \gg (U/\beta)^2 \sim 10^3 \text{ km} \), the acceleration due to the mean circulation is

\[
\frac{\partial \bar{u}}{\partial t} \approx f \bar{u}^L 
\]  

(20)

where \( f = \beta y \). Using Eqs. (18) and (15) and assuming thermal wind balance \( \partial \bar{v}/\partial y \approx -\bar{v}/\partial z \), this gives

\[
\frac{\partial^3}{\partial y^2 \partial t} \left( \bar{u} \right) = \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{\mu f \partial \bar{u}}{N^2} \right) 
\]  

(21)

(Dickinson 1968). This is akin to a vertical diffusion equation; if \( f \approx f_o \) is constant and \( \partial^2 \bar{u}/\partial y^2 \sim \bar{u}/L^2 \), then Eq. (21) has the form

\[
\frac{\partial \bar{u}}{\partial t} \approx \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho v_{\text{eff}} \frac{\partial \bar{u}}{\partial z} \right) 
\]  

(22)

when the effective ‘viscosity’ is

\[
v_{\text{eff}} = \mu f_o^2 L^2 / N^2 
\]  

(23)

These properties are illustrated schematically in Fig. 1 for positive and negative thermal anomalies on the equator. The non-linearity of the vertical advection on the equator is clearly evident, as is the role of the Coriolis acceleration in diffusing the vertical shear at higher latitudes.

The result Eq. (23) answers what might be perceived as a possible difficulty with the maintenance of the QBO in the presence of thermal dissipation. It was demonstrated theoretically for one-dimensional models by Plumb (1977) (and confirmed experimentally by Plumb and McEwan (1978)) that no QBO-like oscillation is generated by the wave-driving mechanism unless the time-scale of mean momentum dissipation exceeds a time of the order of one oscillation period. In reality, of course, thermal dissipation acts on a time scale of \( \mu^{-1} \), i.e. 10 to 20 days or so. However, the mean meridional circulation maintains the available potential energy at the expense of the much larger reservoir of kinetic energy (Wallace 1967b); thus the relevant time-scale for decay is not \( \mu^{-1} \) but \( h^2 / v_{\text{eff}} \), i.e.

\[
\tau_{\text{diss}} \sim N^2 h^2 / \mu^{-1} f_o^2 L^2 
\]  

(24)

The factor \( N^2 h^2 / f_o^2 L^2 \) is the ratio of kinetic energy to available potential energy and, on present scaling assumptions, has a value of about 100. If \( \mu^{-1} = 10 \text{ d} \), \( \tau_{\text{diss}} \sim 10^3 \text{ d} \). Dissipation is weak enough, therefore, to allow the QBO to develop; nevertheless the time-scale on which it acts is not much larger than a QBO period and it is likely that this mechanism is a significant factor in the overall structure of the oscillation.

4. DETAILS OF THE MODEL

Quantitative features of the numerical model and values of external parameters are listed in Table 1. The basic procedure was as follows. With the given basic state at each time-step, the wave-induced accelerations \( \Delta \) were evaluated, separately for the Kelvin and mixed Rossby-gravity wave, by the method described in EW. The Lagrangian mean circulation was then obtained by solving Eq. (17) for \( \chi \) over the model grid by the method of Lindzen and Kuo (1969). \( \bar{u} \) was then stepped forward in time via (2.2) using a third order predictor-corrector method of R.O.R.Y. Thompson (private communication) used previously by Bell and Thompson (1980). It was found expedient to control numerical noise by applying a two-point time filter every 25 d. The calculation took about 4.5 s CPU time per model day on the CSIRO CDC Cyber 76.

The model was integrated from an initial condition of \( \bar{u} = 0, \bar{v} = 0 \) at \( t = 0 \); the run described was terminated after 2000 d, by which time the mean wind oscillation appeared to be in equilibrium.
5. RESULTS AND DISCUSSION

Examples of the structure of Kelvin and mixed Rossby-gravity waves in the model and with realistic mean flows were presented in EW. Here we shall concentrate on describing the characteristics of the zonal mean state.

(a) **Time-height structure of the mean zonal wind**

The evolution of the mean zonal wind at the equator over the 2000 d of the main experiment (Run 1) is shown in Fig. 2. A well-defined QBO develops over the first 1000 days or so, which appears to equilibrate by 2000 d, with no significant growth in amplitude over the final cycle. Figure 2 bears a strong structural resemblance to similar figures for the
atmosphere (Wallace 1973; Coy 1979) and those obtained from one-dimensional models (Holton and Lindzen 1972; Plumb 1977). As compared with observations, the major deficiencies are weak amplitude ($-8.5 \text{ m s}^{-1} \leq \bar{u} \leq 10.5 \text{ m s}^{-1}$; cf. a range of about $\pm 20 \text{ m s}^{-1}$ in reality), insufficient decay of the wind regimes at the lowest levels (below 20 km) and insufficient amplitude at high levels (maximum winds at 23 to 25 km, whereas maximum zonal wind amplitude is found at or above about 30 km in the stratosphere). The period, about 900 d, is within the range of variability (about 700 to 1100 d) of the observed QBO. The descent rates of successive shear zones are on average somewhat slower than observed; however the asymmetry of the descent of easterly and westerly shears, with easterlies descending more slowly (especially at lower levels), is in qualitative agreement with observation; the opposite asymmetry was found to obtain in one-dimensional models (Holton and Lindzen 1972; Plumb 1977). Associated with this disparity in descent rates is the result that easterlies are of shorter duration than westerlies below about 23 km and of longer duration above.

(b) The meridional structure

The meridional structure of the mean zonal wind, potential temperature departure and Lagrangian mean streamfunction are shown in Figs. 3 and 4 at $t = 1500$ and $1800$ d, about the times of absolute easterly and westerly wind maxima respectively. The easterlies are restricted to within about 1200 km of the equator; the westerlies are more extensive, having half-width of about 900 km, with the flow being westerly throughout the cycle in $y > 1200$ km. In fact the wind oscillation almost vanishes at $y \approx 1000$ km and reappears weakly with a change of phase at higher latitudes. Qualitatively, this feature is indicated by observations (Tucker 1979; Holton and Tan 1980) although the reversal of phase seems to be too close to the equator in the model results.

The thermal structure (Figs. 3b, 4b) is consistent with the zonal wind structure through thermal wind balance. As for the zonal wind, the amplitude of the temperature oscillation (about $\pm 0.45$ K) is a factor of two or more smaller than that observed (Reed 1964, Angell and Korshover 1978). There is a phase reversal of the thermal component at $y \approx 1000$ km (i.e. about 9° lat.); with a secondary maximum at $y \approx 1500$ km. Reed (1964) observed a similar phase shift at 16° latitude with a secondary thermal maximum at about 27°. While the absolute values of primary and secondary temperature maxima are too weak in the model, their relative magnitudes are much the same as Reed found.
Figure 3. Height-latitude structure of the mean state at $t = 1500$ d (time of east wind maximum) (a) Zonal wind (m s$^{-1}$) (b) Potential temperature deviation from initial state (K) (c) Mass streamfunction $\chi$ of the mean meridional circulation (m$^2$ s$^{-1}$). In all three cases, the amplitude is smaller than one-half of the contour interval in the undisplayed region $y > 2000$ km.

The Lagrangian mean meridional circulation (Figs. 3c, 4c) is thermally indirect and similar to that depicted in Fig. 1, although at 1800 days $\chi$ has a poleward-downward tilt which appears to be influenced by interaction with the lower boundary. The vertical velocity has maximum amplitude on the equator of about $\pm 10^{-5}$ m s$^{-1}$, again about one-half of the value deduced by Reed (1964). This circulation appears to be responsible for driving the secondary thermal maximum in the subtropics; thus at a given level the diabatic heating (cooling) at the equator is balanced by diabatic cooling (heating) at higher latitudes, with the available potential energy being maintained by the mean meridional circulation.

Figure 4. As Fig. 3, but at $t = 1800$ d (time of west wind maximum).
Figure 5. Contributions to the momentum budget as a function of height at $t = 1500 \, \text{d}$ and at (a) $y = 100 \, \text{km}$, (b) $y = 900 \, \text{km}$, (c) $y = 1700 \, \text{km}$. Heavy solid: wave forcing term $A$. Light solid: mean meridional circulation term. Dashed: vertical diffusion. Dotted: horizontal diffusion. The diffusion terms are not plotted in regions where their magnitudes do not exceed $5 \times 10^{-4} \, \text{m} \, \text{s}^{-2}$.

(e) Components of the acceleration

The components of the mean zonal acceleration at $t = 1500$ and $1800 \, \text{d}$ are shown in Figs. 5 and 6 respectively at latitudes $y = 100, 900$ and $1700 \, \text{km}$. The individual Kelvin and mixed Rossby-gravity wave-induced accelerations are much larger than their sum, particularly at low levels, and so only the sum is presented here. The vertical structure of $A$
is as found in the model of Holton and Lindzen (1972) and discussed in more detail by Plumb (1977). In easterly (westerly) equatorial wind regimes the forcing is dominated by the mixed Rossby-gravity (Kelvin) wave giving easterly (westerly) acceleration a little below the jet maximum. In latitude, \( \Delta \) falls off away from the equator with a half-width of about 1200 km, reflecting the latitudinal scale of the waves themselves, as shown in detail in EW. Vertical diffusion is important near the lower boundary where it maintains a quasi-steady state during each phase of the cycle (Plumb 1977); it is small in magnitude elsewhere. Horizontal diffusion is everywhere relatively weak with largest values in the vicinity of the jet maxima.

At the level of maximum wind, the net acceleration is close to zero at these times, since the jets then reach their maxima. The wave-induced component, however, is such as to amplify the jets. Plumb (1977) showed that for small viscosity each jet reaches a maximum value almost equal to the phase speed of the forcing wave, in this case \( \pm 25 \text{ m s}^{-1} \). With increasing viscosity, the jet maxima are reduced until some threshold value of viscosity is passed when no ‘QBO’ occurs at all. In the present model the effects of viscosity, though small, are clearly of paramount importance at the equator in counteracting the wave driving and thus limiting the jet magnitudes. Off the equator the meridional circulation is the dominant restraining mechanism, as it is, indeed, in a latitudinally integrated sense (Table 2).

**Table 2. Latitudinally-Averaged (over 0 \( \leq y \leq 2000 \text{ km} \)) Contributions to the Momentum Budget at the Level of Jet Maximum (10\(^{-4}\) m s\(^{-2}\))**

<table>
<thead>
<tr>
<th></th>
<th>( t = 1500 \text{ d} )</th>
<th>( t = 1800 \text{ d} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wave forcing ( \Delta )</td>
<td>-14.20</td>
<td>9.41</td>
</tr>
<tr>
<td>Vertical diffusion</td>
<td>3.00</td>
<td>-1.35</td>
</tr>
<tr>
<td>Horizontal diffusion</td>
<td>-0.52</td>
<td>-0.29</td>
</tr>
<tr>
<td>Meridional circulation</td>
<td>12.04</td>
<td>-9.15</td>
</tr>
</tbody>
</table>

It should not be deduced, however, that the meridional circulation is solely responsible for limiting the jet maxima; it will be shown in Section 6 that the magnitude of the maximum windspeed is sensitive to the values of viscosity used in the model. It seems, therefore, that the contribution of these diffusion terms near the equator is of overall importance in limiting the wind maxima.

Angular momentum advection by the mean meridional circulation manifests itself in the manner foreshadowed in Section 3. Near the equator, vertical advection is significant in strong shear zones where both \( \partial \vec{u} / \partial z \) and \( \vec{w}^L \) are large and thus modulates the descent rates of these shear zones. The level of zero wind descends at a rate

\[
- \frac{dz}{dt} = \left( \frac{\partial \vec{u}}{\partial t} \right)_{\vec{w}=0} \left( \frac{\partial \vec{u}}{\partial z} \right)_{\vec{w}};
\]

the wave contribution is \( \Delta (\partial \vec{u} / \partial z)^{-1} \) while that from the mean meridional circulation is just \( -\vec{w}^L \). These contributions and their sums are shown in Table 3 at the two times. From

**Table 3. Contributions to Descent Rates of the Zero Wind Line (10\(^{-4}\) m s\(^{-1}\))**

<table>
<thead>
<tr>
<th></th>
<th>( t = 1500 \text{ d} )</th>
<th>( t = 1800 \text{ d} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wave driving ( \Delta (\partial \vec{u} / \partial z) )</td>
<td>1.3</td>
<td>2.0</td>
</tr>
<tr>
<td>Meridional circulation ( -\vec{w}^L )</td>
<td>-0.6</td>
<td>0.4</td>
</tr>
<tr>
<td>Net</td>
<td>0.7</td>
<td>2.4</td>
</tr>
</tbody>
</table>

the wave driving alone, there is a weak tendency for the westerly shear (at 1800 d) to descend more rapidly than the easterly (at 1500 d). The meridional circulation enhances this asymmetry, especially by retarding the descent of the easterly shear.

At \( y = 900 \text{ km} \) (Figs. 5b, 6b) the dissipative role of the meridional circulation is apparent. This term cancels about half of the wave driving at these latitudes and therefore
seems responsible for the result that the length scale of the mean jets is rather less than that of the wave driving. At 1700 km, the meridional circulation term is so large that it is this term which largely determines the sign of ∂u/∂t. Therefore it seems that the zonal winds at these latitudes are driven remotely by the equatorial QBO (via the meridional circulation) rather than by local wave driving.

Another aspect of the oscillation in which the mean meridional circulation plays a role is the removal of the low-level jet, a necessary precursor to the reappearance of the jet at high levels (Holton and Lindzen 1972). Plumb (1977) showed that in one-dimensional models this ‘switching’ proceeds via momentum diffusion across the internal shear layer separating the low-level jet from the oppositely-directed, descending jet above. In Holton and Lindzen’s calculations, satisfactory agreement with observation was obtained with a mean flow viscosity of 0.3 m²s⁻¹, which is much larger than the value of νᵣ used in the present model. It is interesting, therefore, that the ‘effective viscosity’ associated with the meridional circulation, given by Eq. (24) with f₀ = bL as an average value of the Coriolis parameter, is νₑff ~ 0.4 m²s⁻¹. This mechanism cannot account directly for the dissipation of the low-level jet at all latitudes; since its influence is small (and not always of the required sign) near the equator. At higher latitudes, however, it clearly is important; the integrated components of ∂u/∂t at the level of the lowest equatorial jet are shown for t = 1500 and 1800 d in Table 4. The tendency of the wave driving, ∂, to accelerate the jet is more than

<table>
<thead>
<tr>
<th>TABLE 4. LATIDUINALLY AVERAGED (OVER 0 &lt; Y &lt; 200 km) CONTRIBUTIONS TO THE MOMENTUM BUDGET AT THE LEVEL OF THE LOWEST EQUATORIAL JET (10⁻⁴ m²s⁻²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>t = 1500 d</td>
</tr>
<tr>
<td>Wave forcing δ</td>
</tr>
<tr>
<td>Vertical diffusion</td>
</tr>
<tr>
<td>Meridional circulation</td>
</tr>
</tbody>
</table>

balanced by the effects of advection and vertical diffusion. The advection term is the dominant one overall, especially in destroying the westerly jet at 1500 d. Whether or not turbulent mixing as parametrized by the vertical diffusion term is essential here is not entirely clear, although it would appear to be necessary at the equator.

6. OTHER EXPERIMENTS

To test the sensitivity of these calculations to the parametrization of viscous diffusion, other cases were studied with different values of νᵣ and νᵥ. These are summarized in Table 5 (Run 1 is that described in Section 5).

<table>
<thead>
<tr>
<th>TABLE 5. VALUES OF MEAN VISCOSITY (m²s⁻¹) USED IN THE THREE EXPERIMENTS DISCUSSED IN SECTION 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run No.</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
</tbody>
</table>

With νᵥ and νᵣ increased by a factor of less than 2 in the interior of the domain (run 2) the maximum westerly jet speed attained was 6.2 m s⁻¹, demonstrating the sensitivity of the QBO amplitude in the model to the parametrization of viscosity. Conversely, decreasing νᵥ and νᵣ allows stronger, and hence more realistic, jets to develop; however, the model proved incapable of resolving stronger jets without becoming unstable. This is shown in Fig. 7 in which the maximum (westerly) velocity at the model equator is shown as a function of time for runs 1 and 3. (Run 3 was initiated at 1300 d with the basic state generated by run 1 at that time). Significant differences appear after 1600 d with the westerly jet in run 3
7. THE STEADY WAVE APPROXIMATION

The assumption that the waves respond to the mean state as if the latter were locally steady in time was associated with the neglect of such quantities as $\partial \tilde{u}/\partial t$ and $\partial \tilde{p}/\partial t$ in comparison with $\tilde{u}$ in Eq. (7) and the neglect of other time derivatives of $O(\varepsilon^2)$ wave quantities in the derivation of Eq. (15). To calculate these terms correctly, of course, would require solution of the complete time-dependent problem. However, we may use the results of the quasi-steady analysis to evaluate temporal changes in $\tilde{u}^S$, $p$, $\theta^S$, etc., and thereby to assess the self-consistency of the approximation.

Dunkerton (1981a, b) showed that the effects of wave transience could be important in the generation of the QBO as a result of the build-up of wave pseudo-momentum in regions of strong shear where the vertical group velocity decreases sharply with height.
From Eqs. (1) and (7)

\[
\frac{\partial}{\partial t}(\bar{u} - p) + \bar{v}'(\frac{\partial \bar{u}}{\partial y} - \beta y) + \bar{w}' \frac{\partial \bar{u}}{\partial z} = D + P_t
\]  

(25)

Thus Dunkerton argued that the term \(\partial p/\partial t\) could not be neglected if \(|p|\) becomes comparable with \(|\bar{u}'|\). As an indicator of the possible importance of these effects in the model we compare changes in \(p\) with those in \(\bar{u}\). Fig. 8 shows \(p\) for the Kelvin and mixed Rossby-

![Figure 8](image-url)  

Figure 8. Wave pseudo-momentum densities for run 1 as a function of height and latitude. (a) Kelvin wave, \(t = 1500\,\text{d}\)  (b) mixed Rossby-gravity wave, \(t = 1500\,\text{d}\)  (c) Kelvin wave, \(t = 1800\,\text{d}\)  (d) mixed Rossby-gravity wave, \(t = 1800\,\text{d}\). Amplitudes are insignificant in \(y > 2000\,\text{km}\).
gravity waves at 1500 and 1800 d. Typically, \(|p|\) decays with height (a consequence of the large thermal dissipation which more than cancels the tendency for \(p\) in increase as \(\rho^{-1}\)) although there is a secondary maximum of Kelvin wave pseudo-momentum in the westerly jet at 1800 d. The maximum change in \(p\) between opposite phases of the cycle is about 0.5 m s\(^{-1}\) for the Kelvin wave and 0.2 m s\(^{-1}\) for the mixed Rossby-gravity wave. These are small in comparison with the changes in \(\overline{u}\) and therefore Eq. (8) appears to be a suitable approximation to Eq. (4). Note, however, with stronger jets the concentration of \(p\) is enhanced. For run 3 at 1900 d, for example \(p_k\) reaches a value of 1.54 m s\(^{-1}\) near the westerly maximum of 16.7 m s\(^{-1}\); \(|p|_{\text{max}}\) increases faster than \(|\overline{u}|_{\text{max}}\) and therefore transience effects could become important in stronger jets. Further, and probably more importantly, the unrealistically large wave dissipation causes enhanced wave attenuation at low levels and thus reduces \(p\) at high levels – Dunkerton (1981a, b) discusses this point in some detail. Therefore, although the steady waves approximation is a self-consistent one for the present model, it seems that the neglected effects of wave transience would be important in a model that reproduced more faithfully the observed conditions of the equatorial lower stratosphere.

8. Conclusions

The model has successfully reproduced a QBO of realistic period with structural features that agree qualitatively with those observed in the equatorial stratosphere. Quantitative aspects, in particular the weak amplitude and details of the latitudinal structure, are not so satisfactory.

Possibly the most unrealistic of the model parameters is the wave dissipation which was set to unrealistically large values (particularly at lower levels) in order to avoid resolution problems. This is reflected in several ways. The exaggerated wave attenuation at low levels and consequent depletion of wave activity at middle and higher levels causes the mean wind structure to be concentrated at low levels (see Plumb 1977); such a discrepancy (as compared with observations) was noted in Section 5(a). The weakening of wave activity in the interior of the model domain might also explain the small amplitude of the mean zonal winds and, concomitantly, the sensitivity of the results to the rather small values of viscosity used in the model.

A novel feature of this model – the major dynamical difference from previous studies – is the incorporation of thermal dissipation of the QBO via the (Lagrangian) mean meridional circulation. The contributions to the momentum budget displayed in Figs. 5 and 6 show that the meridional circulation plays a major role in the local dynamics off the equator and/or in regions of strong vertical shear. Therefore it is clear that the wave contributions themselves cannot entirely satisfy local budget requirements and hence that there is no need to invoke the presence of additional wave modes to explain the budget deficit found by Lindzen and Tsay (1975) in these regions.

The meridional circulation provides communication between the equatorial thermal structure and the dynamics at higher latitudes (as well as providing a means of generating an ozone QBO – see Lindzen 1966); it actually reduces the amplitude of the zonal wind oscillation at \(y = 1000\) km and reverses its phase at higher latitudes. These features are observed in the atmosphere, but the phase reversal occurs at higher latitudes and the mid-latitude signal seems to be larger than the model predicts (Reed 1964, Tucke 1979, Holton and Tan 1980). The latitudinal scale of the zonal wind regimes appears to be determined by two major factors: the latitudinal length scale of the equatorial waves themselves and the dissipative role of the meridional circulation at latitude 10° or so. The latitudinal length scale of the waves is a function of many factors not well determined by existing data; adopting a value of \(\pm 30\) m s\(^{-1}\), for example, rather than \(\pm 25\) m s\(^{-1}\) for the wave phase speeds would increase their length scale by a factor of about 1.4. Further, as a result of the large value of Newtonian cooling, the forcing of the meridional circulation Eq. (16)
is exaggerated; this could also be a factor in causing the phase reversal to be located too close to the equator.

At high latitudes, the meridional circulation is very weak, decaying approximately as \( \exp(-\frac{1}{2}by^2/Nh) \) where \( h \) is an appropriate height scale. This is probably the weakest feature of the beta-plane geometry of the model; the Hough functions describing such zonally symmetric motions on a sphere have a secondary maximum at mid-latitudes (Plumb 1982) so the response would probably be more intense there in a model with spherical geometry. (This was pointed out to us by J. R. Holton.)

Despite the relatively large magnitude of wave dissipation adopted here a vertical resolution of 500 m was found to be inadequate in the presence of realistically strong jets. The resolution problem is probably the major reason why general circulation models fail to generate a QBO; the sort of resolution required to resolve the waves adequately would seem to be out of the question. (Another factor may be excessive mechanical dissipation; as noted earlier the time scale on which such dissipation acts on the mean flow must exceed about 2 years to allow a QBO to develop.) This problem can be overcome through the use of ‘two-scaling’ methods which bypass the need for explicit resolution of the wave structure; such an approach was used in the one-dimensional models of Holton and Lindzen (1972) and Plumb (1977). Boyd (1978a, b) showed that the method could be applied to a two-dimensional mean state; the accuracy of Boyd's method was confirmed in EW. The method can also be generalized to incorporate the effects of wave transience (Dunkerton 1981a, b). The application of these techniques in a mechanistic model of the QBO, and perhaps even as a parametrization scheme for general circulation models, remains a challenge for the future.

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