The addition of heat to a stratified airstream with application to the dynamics of orographic rain

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(Received 10 March 1981; revised 13 October 1981. Communicated by Dr K. A. Browning)

SUMMARY

The response of a stratified airstream to combined thermal and orographic forcing is investigated theoretically using the linearized hydrostatic equations of motion. The magnitude of the heating aloft is computed from observed rainfall rates. The elevated heating is shown to produce vertically propagating waves whose amplitude, relative to the mountain waves, is determined by the parameter \( \frac{gQb/c_p T U^4}{Nh} \) (symbols Appendix I). For typical wind speeds and rainfall rates the thermally generated waves will equal or exceed the orographically generated waves. These waves produce large vertical variations in the wave momentum flux at the levels where heat is being added and, or even reverse, the mountain drag. In hydrostatic flow, the phase relationship between the heating rate and the induced vertical displacements is found, unexpectedly, to be negative. In the vicinity of the heating, air parcels are displaced downwards. This result can be explained by wave propagation ideas but not by parcel arguments. This helps to explain why mountain wave amplitudes are sometimes reduced in moist atmospheres. The broad heat-induced descent may act to limit the amount of condensation-precipitation occurring in a stable middle-level orographic cloud. Other types of orographic precipitation involving embedded convection in stratus layers, low level feeder clouds, or deep cumulus convection may not be adversely affected by the heat-induced descent.

1. INTRODUCTION

The distribution of rainfall on the earth is strongly controlled by the orography (e.g. Bonacina 1945; Bergeron 1960; Smith 1979; Browning 1980; Price 1981). The modification to the airflow by mountains acts in ways which are not yet understood to produce precipitation where there would otherwise be none or, more commonly, to locally enhance or locally focus existing precipitation. In the process of condensation and precipitation an enormous amount of latent heat is released in the vicinity of the mountain which may cause additional modifications to the airflow.

A number of authors have considered the theoretical problem of combined latent heat orographic forcing (Sarker 1966, 1967; Raymond 1972; Fraser et al. 1973; Gocho 1978; Barcilon et al. 1979). These studies demonstrate various ways to describe mathematically the disturbance caused by orographic forcing and the release of latent heat but the results bear little resemblance to observed cases of orographic rain. There seem to be two reasons for this. First, the models (cf., Barcilon et al. 1980) do not consider the excess of heating over cooling which occurs if liquid water is lost by precipitation. We shall see that this excess heat can change the nature of the airflow. Second, the models assume that the condensation arises from smooth orderly ascent of the type resolved by their equations. This ‘smooth ascent hypothesis’ may be appropriate in a few stable solutions (Browning et al. 1975; Browning 1980) but there is now considerable evidence that most orographic rain is enhanced by convective motions. In mid-latitude winter-time situations this may be shallow convection embedded within frontal clouds which are being modified by orography (Browning 1974; Hobbs et al. 1975; Browning 1980; Marwitz 1980). In the tropics, in regions with a strong wind approaching a mountain range, it is more likely to be closely packed deep convection triggered by forced ascent in an unstable atmosphere. In either case, the condensation occurring in the convection can exceed the value one would predict using the mountain height as a measure of air ascent. This is probably the reason why the estimates of precipitation efficiency – the ratio of observed rainfall to the computed condensation – are often close to, or exceeding, 100%.

It is immediately evident that to construct a model of orographic rain without the
'smooth ascent hypothesis' would be enormously difficult. The interaction of large and small scales, the small-scale inhomogeneity of the initial moisture field, and the continual unsteadiness of the small-scale motion field would have to be considered. Some progress might be made with CISK-like cumulus parametrization or with a large numerical model, but for the present we report on a more modest semi-empirical approach. To avoid having to treat the details of the cloud system, and to avoid becoming too unrealistic, we assume that the distribution of rainfall is known from observations. We take this rainfall as a measure of the condensation aloft and proceed to calculate the effect of the latent heat of condensation on the stratified airstream. The computed vertical motion induced by this heating, and by the mountain, can then be added to construct a consistent view of the orographic rain system.

The solution to the problem of heat addition to a stratified airstream has application to other problems in dynamical meteorology – the heat island problem, atmospheric tides, the moving flame problem, and wave-CISK. It is surprising that the literature does not include a clear description of what it is that determines the phase relationship between the heating and the induced vertical motion. This relationship is especially important for orographic rain and for wave-CISK (Bolton, 1980) where the heat source is due to the condensation of water vapour in rising air. The first part of this paper (Sections 2–5) is therefore devoted to developing a conceptual model of how a stratified airstream responds to an elevated heat source. Section 2 describes the governing equations. In Section 3 the role of hydrostatic relationship and the usefulness of parcel arguments are discussed. Two useful solutions which incorporate the effect of a rigid lower boundary are given in Section 4. The difficulties involved with the addition of a net amount of heat to an airstream are discussed in Section 5. In the last Section 6 we will apply these ideas to the problem of orographic rain.

2. THE GOVERNING EQUATIONS WITH THERMAL FORCING

Consider the equations describing the inviscid flow of a perfect gas, in a rotating system

\[ \rho \frac{Du}{Dt} - \rho f v = - \frac{\partial p}{\partial x} \]  \hspace{1cm} (1)

\[ \rho \frac{Dv}{Dt} + \rho f u = - \frac{\partial p}{\partial y} \]  \hspace{1cm} (2)

\[ \rho \frac{Dw}{Dt} = - \frac{\partial p}{\partial z} - \rho g \]  \hspace{1cm} (3)

\[ \frac{1}{\rho} \frac{D\rho}{Dt} = - \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \]  \hspace{1cm} (4)

\[ p = \rho c_T \]  \hspace{1cm} (5)

where all variables are defined in Appendix I. The energy equation can be written as

\[ c_v \frac{DT}{Dt} + p \frac{D\alpha}{Dt} = \dot{H}, \]  \hspace{1cm} (6)

where \( \dot{H} \) represents the diabatic heating rate expressed in units of energy per unit mass per unit time (\( J \text{kg}^{-1} \text{s}^{-1} \)). Taking total derivative of Eq. (5) and using \( c_p = c_v + R \), Eq. (6) becomes

\[ c_v \frac{DT}{Dt} + \alpha \frac{Dp}{Dt} = \dot{H}, \]  \hspace{1cm} (7)

For the present purposes we reduce the system Eqs. (1–5), (7), by considering steady, two-dimensional, non-rotating, low Mach number flow where the perturbations from a back-
ground horizontal flow are small. Each dependent variable is represented as the sum of a background value and a perturbation.

\[ u(x, z) = \bar{U} + u'(x, z) \]  \hspace{1cm} (8a)
\[ w(x, z) = 0 + w'(x, z) \]  \hspace{1cm} (8b)
\[ \rho(x, z) = \bar{\rho}(z) + \rho'(x, z) \]  \hspace{1cm} (8c)
\[ p(x, z) = \bar{p}(z) + p'(x, z) \]  \hspace{1cm} (8d)
\[ T(x, z) = \bar{T}(z) + T'(x, z) \]  \hspace{1cm} (8e)

where the background flow variables \( \bar{U} \) and \( \bar{T}(z) \) could be specified and \( \bar{\rho}(z) \) and \( \bar{p}(z) \) determined from

\[ \frac{d\bar{p}}{dz} = -\bar{\rho}g \]  \hspace{1cm} (9)
\[ \bar{p} = \bar{\rho}RT \]  \hspace{1cm} (10)

Using the above assumptions and putting Eq. (8) into Eqs. (1–5), (7),

\[ \bar{\rho}\bar{U} \frac{\partial u'}{\partial x} = -\frac{\partial \rho'}{\partial x} \]  \hspace{1cm} (11)
\[ \bar{\rho}\bar{U} \frac{\partial w'}{\partial x} = -\frac{\partial \rho'}{\partial z} - \rho'g \]  \hspace{1cm} (12)
\[ \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \]  \hspace{1cm} (13)
\[ \frac{\bar{U}}{\bar{\rho}} \frac{\partial \rho'}{\partial x} = \beta w' - \dot{H}/c_p\bar{T} \]  \hspace{1cm} (14)

where \( \beta = \frac{1}{\bar{\rho}} \frac{d\bar{\rho}}{dz} \). These can be combined to give a single equation for the vertical velocity

\[ w''_{xx} + w''_{zz} + l^2(z)w' = g\dot{H}/c_p\bar{T}U^2 \]  \hspace{1cm} (15)

where \( l^2(z) \equiv (N^2/\bar{U}^2) \), and the partial derivatives are denoted by subscripts. The variation with height of \( \bar{\rho} \) in Eqs. (11), (12), and (14) has been neglected following the Boussinesq approximation. Equation (15) is identical to that used by Malkus and Stern (1953) in their study of the heat island problem.

The most straightforward way to treat Eq. (15) is to assume that in each layer of the atmosphere the heating rate \( \dot{H} \) is proportional to the vertical velocity \( w' \), i.e. \( \dot{H} \sim w' \). This would be appropriate for small perturbations in a layered cloudy atmosphere. In this case the heating term in Eq. (15) can be combined with the buoyancy term on the left hand side, (Sarker, 1966; and several other authors) giving a modified value for the Scorer parameter \( l^2 \) and making Eq. (15) homogeneous. This particular parametrization of \( \dot{H} \) in terms of \( w' \) is, of course, a very special one. It requires that the heat added to a parcel (i.e. the downstream integral of \( \dot{H}(x, z) \) (see Section 3) lags the vertical velocity by a quarter cycle. It follows that the buoyancy forces can do no net work and therefore that the heat can modify, but not
generate, internal gravity waves. This point is also discussed by Moncrieff and Green (1972). In the study of orographic rain the requirement \( \dot{H} \sim w' \) is certainly violated in the descending air, and probably elsewhere as well. Thus we keep Eq. (15) as it is and proceed to investigate its solution with various specified heating functions. A solution to Eq. (15) in a planar geometry can be most easily constructed for a separable heating function,

\[
\dot{H}(x, z) = Q g(x) f(z),
\]

(16)

where \( f(z) \) is normalized according to

\[
\int_{0}^{\infty} f(z) \, dz = 1
\]

(17)

so that

\[
\bar{\rho} \int_{0}^{\infty} \dot{H}(x, z) \, dz = \bar{\rho} Q g(x)
\]

(18)

represents the total power added in a vertical column of the atmosphere. For reasons which will be discussed later we restrict \( q(x) \) by

\[
\int_{-\infty}^{+\infty} q(x) \, dx = 0
\]

(19)

so that the net heating at each level is zero. In what follows we will consider the response to the heating functions

\[
q(x) = \cos(kx)
\]

(20)

\[
q(x) = b \frac{d}{dx} \left( \frac{b^2}{b^2 + x^2} \right) = \frac{-2b^3 x}{(x^2 + b^2)^2}
\]

(21)

\[
q(x) = \frac{b_1^2}{x^2 + b_1^2} - \frac{b_2}{x^2 + b_2^2}
\]

(22)

each satisfying Eq. (19). The simple function (20) will help us to understand the phase relationship between elevated heating and vertical motion. Equation (21) will be used to simulate the heating and cooling associated with stable saturated airflow over a mountain, without precipitation. Equation (22) will allow us to simulate the locally intense release of latent heat associated with orographic precipitation. The method of solution is to first find the response to a heating distribution which is concentrated at a height \( z_H \), according to

\[
\dot{H}(x, z) = Q g(x) \delta(z - z_H),
\]

(23)

and then to integrate with respect to \( z_H \) with the weighting function \( f(z) \) from Eq. (16) i.e., we use a Green’s function method.

Putting (23) into (15) and integrating from just below to just above \( z = z_H \) gives

\[
\Delta w' = Q g q(x)/c_p \bar{T} U^2,
\]

(24)

and by integrating again

\[
\Delta w' = 0.
\]

(25)
Away from the layer, Eq. (15) reduces to Scorer's equation

\[ w'_{xx} + w'_{xz} + l^2 w' = 0. \]

(26)

From this point on, the method of solution Eqs. (24)–(26) with appropriate boundary conditions, very closely parallels the methods of mountain wave theory reviewed by Queney et al. (1960) and Smith (1979). Some of the mathematical details are shown in Appendix II.

3. THE SOLUTION WITHOUT SOLID BOUNDARIES AND THE USEFULNESS OF PARCEL ARGUMENTS

The phase relationship between heating and vertical motion, and the validity of simple parcel arguments, are most clearly discussed by considering an unbounded fluid with a heating layer at \( z = 0 \). The response to the heating function (20) is:

When \( l^2 > k^2 \)

\[
\begin{align*}
\{ \\
\quad w'(x, z) = (gQ/2cpTU^2m) \sin(kx + mz), \quad & \text{for } z > 0 \\
\quad w'(x, z) = (gQ/2cpTU^2m) \sin(kx - mz), \quad & \text{for } z < 0,
\}
\]

(27)

where

\[ m = (l^2 - k^2)^{1/2} \]

(28)

This flow field satisfies a radiation condition at \( z = \pm \infty \).

When \( l^2 < k^2 \)

\[
\begin{align*}
\quad w'(x, z) = - (gQ/2cpTU^2m) e^{-mz} \cos kx, \quad & \text{for } z > 0 \\
\quad w'(x, z) = - (gQ/2cpTU^2m) e^{mz} \cos kx, \quad & \text{for } z < 0,
\}
\]

(29)

where

\[ m = (k^2 - l^2)^{1/2} \]

(30)

This flow satisfies a boundedness conditions at \( z = \pm \infty \). Equations (27) and (28) include the hydrostatic limit for which \( k^2 \ll l^2 \) and \( m = l \). The vertical displacement can be computed from (27) and (29) using

\[ \eta(x, z) = \frac{1}{U} \int_{-\infty}^{x} w'(x) \, dx. \]

(31)

The vertical displacement for the hydrostatic case is shown in Fig. 1.

In the heating layer, the amount of heat which has been received by a parcel at position \((x, z_H)\) is proportional to

\[ \frac{1}{U} \int_{-\infty}^{x} qx \, dx = (\sin kx)/Uk, \]

(32)
using Eq. (20). The vertical displacement in that layer $\eta(x, z_H)$ from Eqs. (23) and (27) is

$$\eta(x, z_H) = -gQ(\cos kx)/2c_p T U^3 mk$$  \hspace{1cm} (33)$$

for $l^2 > k^2$. In this case the maximum upward velocity occurs at that part of the cycle when the parcel has received the maximum amount of heat – just before cooling begins. From Eqs. (25) and (27),

$$\eta(x, z_H) = -gQ(\sin kx)/2c_p T U^3 mk$$  \hspace{1cm} (34)$$

for $l^2 < k^2$. In this case the vertical displacement is just opposite to the received heat. The hottest air is at the bottom of its swing and accelerating upwards.

It is tempting to try to explain these and later results using direct parcel arguments. The equation for the vertical displacement $\eta(t)$ of a parcel in a quiescent stratified environment and subject to heating and cooling at a rate $q(t)$ is

$$\ddot{\eta} + N^2 \eta = \frac{g}{c_p T} \int_0^t q(t) \, dt$$  \hspace{1cm} (35)$$

If

$$q(t) = \cos \omega t$$  \hspace{1cm} (36)$$

then

$$\eta(t) = g(\sin \omega t)/c_p T(N^2 - \omega^2)$$  \hspace{1cm} (37)$$

If we consider the forcing frequency $\omega$, to be equivalent to the intrinsic frequency $Uk$ in Eqs. (24) and (26), then we can directly compare (37) with (33) and (34). For high frequency forcing the parcel argument (37) predicts vertical displacement exactly opposite to the integrated received heat. The hottest parcels are at the bottom of their swing and accelerating
upwards – the coolest are at the top and accelerating downwards. This agrees with the continuum result Eq. (34). For low frequency forcing Eq. (37) with \( \omega^2 < N^2 \) describes vertical displacement in phase with the integrated received heat. The parcel is nearly successful in positioning itself so that its density always matches that of its environment. This phase relationship differs by \( \pi/2 \) from the continuum result (33).

The situation is not improved by using the idea of added mass (Batchelor 1977, p. 407) to account for the pressure forces acting on a parcel as it accelerates. This only changes the effective mass of the parcel, thus altering the natural frequency. To make the parcel argument work we must modify Eq. (35) by including a damping force proportional to velocity. If this term dominates, Eq. (35) becomes

\[
D\dot{\eta} = \frac{g}{c_p \bar{T}} \int_{0}^{t} q(t) \, dt,
\]

where \( D \) is a drag coefficient. Equation (37) becomes

\[
\eta(t) = -g(\cos \omega t)/Dc_p \bar{T} \omega^2
\]

in qualitative agreement with (33). The introduction of the strong damping force is not done to simulate viscosity or turbulence as these were not included in the continuum equations. Instead, the local damping force in the parcel argument represents the tendency for local motion in a stratified continuum to decay as energy propagates away by pressure-velocity correlations. We thus note that the phase relationship between vertical velocity and heating found in Eqs. (27) and (33) is analogous to that of a strongly damped system and that this is because the heated and cooled air is heavily engaged in doing mechanical work on the layers of air above and below it. This phase relationship carries over directly to the other heating functions Eqs. (21) and (22), in the absence of reflective boundaries. Downward displacement is in phase with a positive heating rate and upward velocity is in phase with the downstream integral of the heating rate, that is, the received heat. This relationship would not seem to allow a consistent pattern of vertical motion driven by condensation in rising air. We will return to this point later.

On the whole it does not appear that parcel arguments, even modified ones, are very useful in these types of problems. The roles of far-reaching pressure fields and distant boundary conditions are too important to be ignored. A well known example of this is the local criterion of Sheppard (1956) for determining when stratified air can surmount an obstacle. According to this criterion a low level air parcel, converting kinetic energy to potential energy as it rises, should reach its minimum speed at mountain top but this is incorrect (Smith 1980). The importance of boundary conditions is shown by the results of Malkus and Stern (1953). By using an incorrect radiation condition at infinity in the heat island problem, they compute vertical velocities in the wrong direction.

The transport of mechanical energy away from the layer of forcing is accompanied by a flux of momentum towards the layer, just as in mountain wave theory (Eliassen and Palm 1960). This momentum flux is

\[
F/\lambda = -\frac{\bar{p}}{2k} \left( gQ/2c_p \bar{T} U^2 \right)^2 \left( \begin{array}{c} +1 \\ -1 \end{array} \right), \quad \begin{array}{c} z > 0 \\ z < 0 \end{array}
\]

The vertical momentum flux (40) is convergent at the heating layer and must act to accelerate the flow there – even though this acceleration is not explicitly accounted for in linear theory. This acceleration may have relevance to the problems of wave-CISK, heat islands, atmospheric tides, and orographic rain but has been discussed most directly with regard to the ‘moving-flame’ problem in which heat sources are moved through a quiescent fluid. This situation is identical to the analysis we have done here, if we put ourselves in a reference
frame moving with the mean flow. It follows then, at least for linear theory, that the result of moving heat sources in a stratified fluid is to generate a mean flow in the opposite direction to the motion of the sources. At some distance from the layer, where the generated internal gravity waves dissipate, a mean flow will eventually arise moving in the same direction as the heat sources. The overall momentum of the fluid must remain zero as there is no net momentum imparted to the fluid by the moving sources of heat.

The direction of these circulations is just opposite to that found by Schubert and Whitehead (1969) in an unstratified, thermally conducting fluid. In their case, the disturbance outside the heating layer is associated with the conduction of heat and the tilt of the phase lines of the motion (and hence the momentum flux) is therefore opposite to that predicted by internal gravity wave theory. Either model could be used to explain a mean flow such as the 4-day rotation rate of ultra-violet markings on Venus, as one does not know whether the generation region or the dissipation region of the disturbance is being viewed.

Later on, when we compute the combined effect of thermal and orographic forcing, we will note that while the disturbance fields are additive, the non-linearly computed momentum flux is not. This will give rise to an interesting variety of momentum flux profiles.

4. TWO USEFUL SOLUTIONS

For applications to meteorological problems, the influence of the rigid earth surface, the localization of the horizontal heating distribution, and the distribution of the heating over a range of heights, must be considered. Furthermore, it is appropriate now to consider horizontal scales of more than a few kilometres so that the hydrostatic approximation is valid. Using the methods discussed above, the following useful solutions are obtained. For the heating function

\[ H(x, z) = \begin{cases} 
0 & \text{for } z > z_H + d, \\
\frac{Q}{2d}\left\{-2b^3x(x^2 + b^2)^{-2}\right\} & \text{for } z_H + d > z > z_H - d, \\
0 & \text{for } 0 < z < z_H - d,
\end{cases} \]  

(41)

the vertical displacement field is

\[ \eta(x, z) = -\frac{gQb^2}{c_p T U^3 l^2} \times \]
\[ \times \left[ \frac{b \{\sin l(z_H + d) - \sin l(z_H - d)\} + x \{\cos l(z_H + d) - \cos l(z_H - d)\}}{x^2 + b^2} \right] \]

for \( z < z_H - d \), (a)

\[ \eta(x, z) = \frac{gQb^2}{c_p T U^3 l^2} \left( b \cos l z - x \sin l z \right) \frac{\{\cos l z - \cos l(z_H - d)\}}{x^2 + b^2} - \]
\[ - \frac{gQb^2}{c_p T U^3 l^2} \left[ \frac{b \{\sin l(z_H + d) - \sin l z\} + x \{\cos l(z_H + d) - \cos l z\}}{x^2 + b^2} \right] \]

for \( z_H - d < z < z_H + d \), (b)

\[ \eta(x, z) = \frac{gQb^2}{c_p T U^3 l^2} \left[ \left( b \cos l z - x \sin l z \right) \{\cos l(z_H + d) - \cos l(z_H - d)\} \right] \]

for \( z > z_H + d \). (c)
If the depth of the heating region is small \((2dl \ll 1)\), (42) becomes

\[
\eta(x, z) = -\frac{gQb^2\sin(lz)(b\cos lz_H - x\sin lz_H)}{c_p\overline{T}U^3I(x^2 + b^2)} \quad \text{for} \quad z < z_H, \\
\eta(x, z) = -\frac{gQb^2\sin(lz_H)b\cos lz - x\sin lz)}{c_p\overline{T}U^3I(x^2 + b^2)} \quad \text{for} \quad z > z_H.
\]

(43)

The pressure distribution at the ground can be computed from (43) using either Bernoulli's equation (with Eqs. (13) and (31))

\[
p'(x, 0) = -\frac{\rho U'U}{x} = +\frac{\rho U^2}{x} \frac{\partial \eta}{\partial z} \bigg|_{z = 0},
\]

(44)
or the hydrostatic equation (with Eq. (14))

\[
p'(x, 0) = g \int_{0}^{\infty} \rho' \, dz = g\rho U \int_{0}^{\infty} \eta(x, z) \, dz - \frac{g\rho U}{c_p\overline{T}} \int_{-\infty}^{x} q(x) \, dx.
\]

(45)

Either way the result is

\[
p'(x, 0) = -\frac{g\rho Qb^2}{c_p\overline{T}U} \left(\frac{b\cos lz_H - x\sin lz_H}{x^2 + b^2}\right).
\]

(46)

Similar formulae can be derived for the heating function

\[
\dot{H}(x, z) = 0, \quad \text{for} \quad z > z_H + d, \\
= \frac{Q}{2d} \left(\frac{b_1^2}{x^2 + b_1^2} - \frac{b_1 b_2}{x^2 + b_2^2}\right), \quad \text{for} \quad z_H + d > z > z_H - d, \\
= 0, \quad \text{for} \quad 0 < z < z_H - d.
\]

(47)

The solution for the vertical displacement is

\[
\eta(x, z) = -\frac{gQb_1\sin lz}{c_p\overline{T}U^3I^2} \left[\left(\tan^{-1}\frac{x}{b_1} - \tan^{-1}\frac{x}{b_2}\right) \sin(lz_H + d) - \sin(lz_H - d)\right] - \frac{1}{2} \ln \left(\frac{x^2 + b_2^2}{x^2 + b_1^2}\right) \{\cos(lz_H + d) - \cos(lz_H - d)\} \\
\text{for} \quad z < z_H - d, \quad (a)
\]

(48) cont. on next page
\[ \times \{ \sin l(z_H + d) - \sin l \} \frac{1}{2} \ln \left( \frac{x^2 + b_2^2}{x^2 + b_1^2} \right) \{ \cos l(z_H + d) - \cos l \} \]

for \( z_H - d < z < z_H + d \), \( b \) \( (48) \) cont.

\[ \eta(x, z) = \frac{gQb_1}{c_p \bar{T} U^3 l^2} \left[ \{ \cos l \} \left( \tan^{-1} \frac{x}{b_1} - \tan^{-1} \frac{x}{b_2} \right) + \frac{1}{2} \{ \sin l \} \ln \left( \frac{x^2 + b_2^2}{x^2 + b_1^2} \right) \right] \times \{ \cos l(z_H + d) - \cos l(z_H - d) \} \]

for \( z > z_H + d \). \( c \)

When the depth of the heating region is small \((2dl \ll 1)\) Eq. \((48)\) becomes

\[ \eta(x, z) = -\frac{gQb_1 \sin l}{c_p \bar{T} U^3 l^2} \left\{ \{ \cos l \} \left( \tan^{-1} \frac{x}{b_1} - \tan^{-1} \frac{x}{b_2} \right) + \frac{1}{2} \{ \sin l \} \ln \left( \frac{x^2 + b_2^2}{x^2 + b_1^2} \right) \right\}, \]

for \( z < z_H \), \( a \) \( (49) \)

\[ \eta(x, z) = -\frac{gQb_1 \sin l}{c_p \bar{T} U^3 l^2} \left\{ \{ \cos l \} \left( \tan^{-1} \frac{x}{b_1} - \tan^{-1} \frac{x}{b_2} \right) + \frac{1}{2} \{ \sin l \} \ln \left( \frac{x^2 + b_2^2}{x^2 + b_1^2} \right) \right\}, \]

for \( z > z_H \). \( b \)

The momentum flux associated with \((49)\) is

\[ F = -\frac{\pi \bar{\rho}}{l} \left( \frac{gQb_1 \sin l}{c_p \bar{T} U^2} \right) \ln \left( \frac{(b_1 + b_2)^2}{4 b_1 b_2} \right), \]

for \( z > z_H \), \( a \) \( (50) \)

\[ F = 0, \]

for \( 0 < z < z_H \). \( b \)

The perturbation surface pressure is

\[ p'(x, 0) = -\frac{g \bar{\rho} Qb_1}{c_p \bar{T} U} \left\{ \{ \cos l \} \left( \tan^{-1} \frac{x}{b_1} - \tan^{-1} \frac{x}{b_2} \right) + \frac{1}{2} \{ \sin l \} \ln \left( \frac{x^2 + b_2^2}{x^2 + b_1^2} \right) \right\}, \]

\( (51) \)

The solutions Eqs. \((42\), \((43\), \((48\) and \((49)\) all satisfy the rigid boundary condition \( w' = 0 \) at \( z = 0 \), and thus the downgoing wave produced by the elevated thermal forcing is totally reflected. This makes the flow field somewhat more complex and local parcel arguments become even less reliable. The momentum flux is zero between the heating layer and the ground, due to the flux cancellation of the up and downgoing wave. The possibility of having the reflected wave add either constructively or destructively, makes the flow sensitive to the altitude at which the heat is added. Heating very near the ground, \( z_H l \ll 1 \), produces a vanishingly small disturbance, as expected. Cancellation of the direct upgoing wave and the reflected upgoing wave above \( z_H \), can also occur with other special values of \( z_H \) (see Eqs. \((43\), \((49)\) for \( z > z_H \))

\[ l z_H = 0, \pi, 2\pi, \ldots, n\pi. \]

This effect is less evident if the heating is spread over a range of altitudes as in Eqs. \((42\) and
THE ADDITION OF HEAT TO A STRATIFIED AIRSTREAM

Figure 2. The horizontal distribution of heating given by Eqs. (21) with \( b = 20 \) km and (22) with \( b_x = 20 \) and \( b_y = 100 \) km. The balanced heating and cooling function (curve 1) will be used to simulate the addition and removal of heat as saturated air passes over a mountain without precipitation. Curve 2 is used to simulate the heating associated with orographic rain, with isolated heating and widespread cooling.

Figure 3. The hydrostatic response of a stratified airstream, with a flat rigid lower boundary, to balanced heating and cooling concentrated at an altitude of 1.5 km. Regions with large heating and cooling are marked \( + + + + \) and \( - - - - \) respectively. This flow is given by Eq. (43) with \( Q = 1107 \text{ W m}^{-2} \text{ kg}^{-1} \), \( b = 20 \) km, \( U = 10 \text{ m s}^{-1} \), \( N = 0.01 \text{ s}^{-1} \). Vertically propagating waves are present above the heating level.

The surface pressure disturbance (in Pascals) is shown in the lower part of the figure.

(48). The response to heating at the quarter wave point \( lz_H \approx 
\pi/2 \) is shown in Figs. 3 and 4. For either heating function \( q(x) \), strong forward-tilting gravity waves are generated aloft and downward displacement is evident near the heating region. The sense of the vertical displacements in Fig. 4 is just the opposite of that shown by Malkus and Stern (1953, their Fig. 2) due to the incorrect upper boundary condition used in that paper.

The bottom section of Figs. 3 and 4 show the perturbation surface pressure associated with the disturbance aloft. At each point on the surface the pressure is an integral measure of the density anomalies aloft. Equation (14) shows that these density anomalies arise in two ways; directly from the heating and indirectly from the thermally generated vertical motion. In Fig. 4 for example, the direct effect of the heating would be to produce a surface pressure \( p'(x,0) \) which decreased monotonically with increasing \( x \). The total effect of the heating on \( p'(x,0) \) is quite different, as shown in Fig. 4.

The same situation exists in a stagnant atmosphere with a local heat source such as a
Figure 4. Like Fig. 3 but for the isolated heating-widespread cooling function (22). This flow is given by Eq. (49) with $Q = 900 \text{ W m kg}^{-1}, b_1 = 20 \text{ km}, b_2 = 100 \text{ km}, U = 10 \text{ m s}^{-1}, N = 0.01 \text{ s}^{-1}$. Note that the surface pressure is reduced directly below the region of heating. According to Bernoulli’s equation, the wind speed is increased and the streamlines are closer together.

growing cumulus cloud. The region of low pressure at the ground is much broader than the heat source aloft due to the widespread descent and warming surrounding the cloud.

The relationship between the thermal response in a system with and without an ambient flow is discussed by Thorpe et al. (1980). The large Froude number results in that paper are in qualitative agreement with results presented herein.

5. THE PROBLEM OF NET HEATING

Whenever condensed water falls from a cloud, the latent heat of condensation released in the cloud must exceed the evaporative cooling and the atmosphere will receive a net amount of heat. If one tries to obtain a solution to Eq. (15) with a heating function which allows a net heating at any level (violating Eq. (19)), for example

$$
\dot{H}(x, z) = Q \delta(x) \delta(z - z_H),
$$

(52)

the perturbation vertical velocity decays downstream only as $1/x$ and the vertical displacement grows as $\log x$. This implies that a net heating does not produce a localized disturbance, instead, the flow must undergo a permanent change. It is of interest then to consider what would actually happen far downstream of a region of net heating. One possibility is that through the slow process of radiative conduction, the heated air would eventually lose its excess heat and the vertical displacements would decay. This could be taken into account by including a small conductivity (possibly Newtonian cooling, see Kuo, 1956) but we have chosen the simpler way of including a prescribed weak, widely distributed cooling (i.e., the second term in (22) or (47)), so that Eq. (19) is satisfied. We are free to make the width of the cooling distribution ($b_2$) quite large so that the local response to the heating is not strongly affected, but we cannot take $b_2 \to \infty$ in (48) – (51). We cannot escape the fact that the flow is at least slightly sensitive to the shape of the cooling function.

We can only avoid the requirement of a compensatory cooling if we include both
three-dimensionality and the Coriolis force. These two together, allow the heated air to approach a permanent thermal-wind balance. To see how this works, we return to Eqs. (1)–(6) and rederive Eq. (15) retaining the Coriolis force in Eqs. (1) and (2) and allowing variations in the $y$-direction. Then

$$
\frac{U^2}{\partial x^2} (\nabla^2 w') + f \frac{\partial^2 w'}{\partial z^2} + N^2 (\nabla_H^2 w') = \frac{g}{c_p T} \nabla_H^2 \{H(x, y, z)\} \ . \tag{53}
$$

Integrating with respect to $x$ and using Eq. (27) gives

$$
f^2 \eta_{ooz} + N^2 \eta_{ Gry} = \frac{g}{c_p T} U \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} \hat{H}(x, y, z) dx , \ . \tag{54}
$$

where $\eta_{oo}(y, z)$ is the vertical displacement of parcels from their initial level after they have passed far downstream. Alternatively Eq. (54) can be derived from a set of equations which govern the final geostrophically adjusted flow, far downstream. From Eq. (1), the final speed increment $u_{oo}'$ is shown to be caused by the Coriolis force acting on the lateral displacement $\delta_{oo}(y, z)$ and acceleration associated with a permanent pressure change $p_{oo}'(y, z)$ according to

$$
U u_{oo}' - f U \delta_{oo} = - \frac{p_{oo}'}{\rho} \ . \tag{55}
$$

The final state must be in a thermal wind balance given by (from Eqs. (2) and (3))

$$
f u_{oo}' = N^2 \eta_{oo} \ , \ . \tag{56}
$$

and, from Eq. (4) the mass flux in a streamtube must be constant according to

$$
u_{oo}'/U + \delta_y + \eta_z = 0 \ . \ . \ . \ . \ . \tag{57}
$$

The solution to (54) for a source of heat localized in the $y-z$ plane and distributed arbitrarily in $x$

$$
H(x, y, z) = Q q(x) \delta(y) \delta(z) , \ . \tag{58}
$$

with

$$
\int_{-\infty}^{\infty} q(x) dx = 1 , \ \text{is}
$$

$$
\eta_{oo}(y, z) = \left( \frac{g Q}{2\pi c_p T U N f} \right) \frac{(N^2/f^2)z^2 - y^2}{\{y^2 + (N^2/f^2)z^2\}^2} \ . \tag{59}
$$

Using Eq. (56)

$$
u_{oo}'(y, z) = \left( \frac{g Q}{2\pi c_p T U f} \right) \frac{2(N/f)z}{\{y^2 + (N^2/f^2)z^2\}^2} \ . \tag{60}
$$

Using Eq. (55) or Eq. (57), the final lateral displacement is

$$
\delta_{oo}(y, z) = \left( \frac{g Q}{2\pi c_p T U f N} \right) \left[ \left( \frac{2N}{f} \right) \frac{(N/f)z}{\{y^2 + (N^2/f^2)z^2\}^2} + \left( \frac{N}{U} \right) \frac{(N/f)z}{\{y^2 + (N^2/f^2)z^2\}^2} \right] \ . \tag{61}
$$
The pattern of displacements in the $y-z$ plane, given by (59) and (61), is shown in Fig. 5. This pattern is qualitatively similar to the results of Eliassen (1951) and Kuo (1956) for the meridional circulations forced by zonally distributed heating, but there are fundamental differences. First, with (59) and (61) we have finite final displacements driven by a localized heat source rather than a steady meridional circulation. Because the source is compact it also generates a complicated pattern of internal gravity waves. The final state, however, is not influenced by these smaller scale features because of the linearization, but depends only on the net heat added to each fluid parcel.

The other difference is that unlike the circulations of Eliassen and Kuo, the displacement field in Fig. 5 is asymmetric and divergent ($\delta_y + \eta_z \neq 0$) in the $y-z$ plane. This is connected with the fact that as the flow goes from its undisturbed upstream conditions to a new more complicated thermal wind balance downstream, some air parcels permanently change their speed (i.e., $u'_z$) and thus all three terms in (55) and (57) play a role. This balance of terms in the $x$-momentum equation and the continuity equation are fundamentally different from that of Eliassen and Kuo.

A closer inspection of (61) shows that the two terms on the right hand side decay at different rates and thus a natural length scale, $U/f$, arises. The first term will dominate when $|y| < U/f$ or $(N/f)|z| < U/f$ and in this case the balances in the continuity (57) and $x$-momentum (55) equations are $\delta_{uv} + u'_o = 0$ and $U\delta_{uv} = fU\delta_{v}$. On the other hand, when $|y| > (U/f)$ or $(N/f)|z| > (U/f)$, the second term in (61) dominates and the balances are $u'_o/\bar{U} + \delta_{uv} = 0$ and $\bar{p}fU\delta_v = p_o$. This result is not unrelated to the behaviour at different scales found in other geostrophic adjustment problems (Blumen 1972).

The problem discussed in this section was not fully recognized in the recent paper by Barcilon et al. (1980). In response to isolated net heating they found solutions which in many respects are similar to the flow in Fig. 4 but, because there was no compensating cooling, the vertical displacement, pressure, and temperature fields were not bounded at infinity. This is not readily apparent from the figures in that paper because integrals such as (31) were evaluated numerically over a rather small range of $x$. With hindsight it seems that the vertical velocity field in that paper is probably correct but that the other variables and especially their behaviour in the downstream "wake" is incorrect.
6. **Combined thermal and orographic forcing and the problem of orographic rain**

To apply the foregoing results to the problem of orographic rain, we need to consider the case of combined thermal and orographic forcing. If we retain the linearized approach, and assume that we know both the mountain shape and the heating distribution, the two types of disturbance may simply be added. The solution of Queney (1947) for hydrostatic, 2-D flow over a bell shaped mountain

\[ h(x) = ha^2/(x^2 + a^2), \]

is most convenient for this purpose. Adding his solution to (43) and (49) gives

\[
\begin{align*}
\eta(x, z) &= \frac{ha(a \cos lx - x \sin lx)}{x^2 + a^2} - \frac{g Q b^2 (\sin lx) (b \cos lz_H - x \sin lz_H)}{c_p T U^3 l (x^2 + b^2)}, \text{ for } z < z_H. \\
\eta(x, z) &= \frac{ha(a \cos lx - z \sin lx)}{x^2 + a^2} - \frac{g Q b^2 (\sin lz_H) (b \cos lx - x \sin lz_H)}{c_p T U^3 l (x^2 + b^2)}, \text{ for } z > z_H.
\end{align*}
\]

(63)

and

\[
\begin{align*}
\eta(x, z) &= \frac{ha(a \cos lx - x \sin lx)}{x^2 + a^2} - \frac{g Q b_1 (\sin lx)}{c_p T U^3 l} \left[ \cos lz_H \right] \left( \tan^{-1} \left( \frac{x + c}{b_1} \right) - \\
- \tan^{-1} \left( \frac{x + c}{b_2} \right) \right) + \frac{1}{2} (\sin lz_H) \ln \left( \frac{\left( x + c \right)^2 + b_2^2}{\left( x + c \right)^2 + b_1^2} \right), \text{ for } z < z_H. \\
\eta(x, z) &= \frac{ha(a \cos lx - x \sin lx)}{x^2 + a^2} - \frac{g Q b_1 (\sin lz_H)}{c_p T U^3 l} \left[ \cos lx \right] \left( \tan^{-1} \left( \frac{x + c}{b_1} \right) - \\
- \tan^{-1} \left( \frac{x + c}{b_2} \right) \right) + \frac{1}{2} (\sin lz) \ln \left( \frac{\left( x + c \right)^2 + b_2^2}{\left( x + c \right)^2 + b_1^2} \right), \text{ for } z > z_H.
\end{align*}
\]

(64)

where ‘c’ is the upstream distance between the heating pattern and the mountain top. Some examples of these flows are shown in Figs. 6 to 10. Note that the relative magnitude of the response to thermal and orographic forcing can be written as the ratio of the two coefficients.

---

**Figure 6.** Hydrostatic adiabatic flow over a bell-shaped mountain, first derived by Queney (1947). The momentum flux in this flow is shown in Fig. 10.
in Eqs. (63) and (64). This is

\[ g Q b / c_p T U^3 h \]

where \( Q \) and \( b \) are crudely related to the intensity and the horizontal extent of the observed rainfall at the ground.

The mere addition of the thermal and orographic disturbances would not seem likely to give rise to interesting new phenomena but this is not quite true. In particular, we note that the momentum (and energy) fluxes, being non-linear quantities, are not just the sum of the pure thermal and orographic values. The momentum flux corresponding to (64) is

\[
F = - \frac{\pi}{4} \bar{\rho} h^2 N U \left( \frac{n \rho h a g Q b}{c_p T U} \right) \left[ \frac{((a + b_1) \cos l z_H - c \sin l z_H)}{(a + b_1)^2 + c^2} \right] - \frac{((a + b_2) \cos l z_H - c \sin l z_H)}{(a + b_2)^2 + c^2} \right), \quad \text{for } z < z_H,
\]

and

\[
F = - \frac{\pi}{4} \bar{\rho} h^2 N U \left( \frac{g Q b_1 \sin l z_H}{c_p T U^2} \right) \ln \left( \frac{(b_1 + b_2)^2}{4 b_1 b_2} \right) \right) + \left( h a g Q b_1 \sin l z_H \right) \left[ \frac{2c}{((a + b_1)^2 + c^2)} - \frac{2c}{((a + b_2)^2 + c^2)} \right], \quad \text{for } z > z_H.
\]

Note that besides the pure \( h^2 \) and \( Q^2 \) terms there are cross terms proportional to \( h Q \). The existence of this important contribution to the momentum flux below the heating level (Eq. (65)) is easily explained as arising from the thermally generated pressure disturbance at
the ground, acting on the topography. It could therefore be computed directly from (51) together with (62). If a large amount of heating occurs over the windward slope of a mountain, the pressure at the surface could be lowered sufficiently to cause a reversal of the expected downstream drag. An example of this is shown in Figs. 8 and 10. Note in (65), that the cross terms depend on the relative horizontal position of the mountain and the heating (described by c). As mentioned earlier, the part of the flux which is driven purely by heating (the $Q^2$ term in (65)) must vanish below the heating level because of wave reflection.
We now proceed to discuss in more detail the pattern of motion induced by combined thermal and orographic forcing. To simulate the smooth flow of initially saturated air over a ridge, without precipitation, we assume that the heat release in any column is proportional to the slope of the underlying terrain. The history and the conditions for validity of this assumption are reviewed by Smith (1979). With this assumption the heating function Eq. (21) is compatible with the ridge shape (62) so that the solution (63) applies with 'a' set equal to 'b'. An example of this type of flow is shown in Fig. 7. The comparison of the adiabatic flow (Fig. 6) with this diabatic flow shows that the effect of the heating and cooling is to reduce the amplitude of the wave disturbance aloft (and incidentally, the mountain drag). This result has been commented on by a number of previous investigators (Sarker 1966, 1967; Raymond 1972; Fraser et al. 1973; Gocho 1978; Barcilon et al. 1979), and their explanation – that the moist ascent and descent along wet adiabats reduces the effective stability of the atmosphere and its ability to propagate waves – seems essentially correct. The solutions of Barcilon et al., for example, are in fact much more elegant and correct than (63) as they have found fields of motion and latent heating which are mutually consistent within the 'smooth ascent hypothesis'. The present closed-form result is in qualitative agreement with those results but gives a slightly different interpretation. From the arguments in Section 3 we know that the heating over the windward slope will produce downward displacement while the cooling over the lee slope produces local upward displacement. This pattern is almost in direct opposition to the orographically forced motion – especially if the upstream phase shift of the orographic disturbance at the heating level is considered. The details depend on the precise nature of the heating function but in general (as remarked earlier) heating tends to restrict itself by producing vertical motion in the wrong sense.

The primary purpose of this section is to describe combined thermal- orographic flow with a heating function corresponding to cases of orographic rain. We will consider the extreme case where most of the condensate falls to the ground and therefore the local cooling by evaporation of cloud droplets can be neglected. In using the heating function Eq. (22) we will, however, be including a broadly distributed cooling, as discussed earlier, to keep the solution bounded at infinity.

The classical view of orographic rain is that the condensation is directly connected with forced orographic ascent, thus we begin by putting the condensation rate maximum directly over the steepest windward slope. The amplitude of the thermal forcing is estimated from the observed rainfall rate at the surface. A rainfall rate $R$ is associated with a heat release rate

$$ R \rho_{\text{water}} \lambda \]

in W m$^{-2}$. From Eq. (18)

$$ \bar{\rho} Q g(x) = R(x) \bar{\rho}_{\text{water}} \lambda. \] (66)

Since $g(x)$ is chosen to be about unity (see Fig. 2)

$$ Q \approx R(\rho_{\text{water}}/\rho) \lambda. \]

For $R = 1 \text{ mm h}^{-1} = 2.78 \times 10^{-7} \text{ m s}^{-1}$ and with $\lambda = 2.5 \times 10^{6} \text{ J kg}^{-1}$

$$ Q \approx 695 \text{ W m kg}^{-1}. \]

An example of this type of flow with $Q = 900$, is shown in Fig. 8. The differences between this flow, the non-precipitating flow (Fig. 7), and the dry flow (Fig. 6) are quite striking.

The choice of $Q = 900$ represents quite a small precipitation rate. Precipitation rates of 5 to 10 mm h$^{-1}$ would produce a disturbance so large that the computed streamlines would cross and assumptions of linear theory would be violated. This is an indication that with $U = 10 \text{ m s}^{-1}$, the observed orographic rain represents a thermal forcing which is larger than
the direct orographic forcing. The vertical displacements caused by heating (60) decrease strongly with increasing wind speed, thus the relative importance of thermal vis-a-vis orographic forcing becomes less at higher velocities. With observed precipitation rates of 5 mm h$^{-1}$ or so, and a wind speed of 20–30 m s$^{-1}$, the two effects are comparable as they are in Fig. 8. The momentum flux profile for the flow in Fig. 8 is shown in Fig. 10.

In some cases of orographic rain the region of maximum condensation rate occurs well upstream of the mountain. This condition can be simulated, by increasing the parameter 'c' in (64) thus moving the centre of the heating function Eq. (22) further upstream. For the sake of comparison the heating is still confined to a thin layer at $z = 1.5$ km. The result is shown in Fig. 9.

It is important to keep in mind that the solutions in this paper contain a large number of free parameters describing the geometry of the heating region and the orography. Most of these values are quite difficult to estimate from existing case studies. More guidance from observations is needed. It may also be necessary to extend the current work to include such complications as environmental wind shear and the turbulent boundary layer.

7. DISCUSSION

We believe that the method of specifying the heating distribution from observed rainfall rates and then computing the combined thermal-orographic disturbance is an essential first step in the investigation of orographic rain. The next step is much more difficult. The small-scale mechanisms which produce the precipitation and the way in which these are controlled by the mesoscale environment must be understood. If these processes can be parametrized, then a closed model of orographic rain might be possible using the solutions presented herein. Another possibility is that these present solutions can be used to verify and to diagnose more complicated numerical models of orographic rain.
8. Summary of Conclusions

a. The heat release associated with observed rates of orographic precipitation is found theoretically to have a significant influence on the air flow. A crude measure of the relative importance of thermal versus orographic forcing is given by the non-dimensional number $gQb/c_pT^3U^3h$ where $Q$, the vertically integrated heating rate, can be crudely estimated from the local rainfall rate. Of course, the geometry of the heating region and the topography are also important.

b. In a hydrostatic atmosphere the typical response to localized heating is a downward displacement in the vicinity of the heating. This can be modified only to a limited extent by the presence of a rigid lower boundary or by an increased depth of the heating region. This process may tend to limit the amount of condensation-precipitation which could occur in a standing orographic cloud aloft unless the condensation is occurring in small-scale convection of a sort which is not strongly suppressed by the broad scale (about 10-100 km) descent.

The existence of a low level 'feeder cloud' (Bergeron 1960; Browning et al. 1974; Bader and Roach 1977) would be little influenced because the input of latent heat at such a low altitude produces little dynamic response.

Heating at certain special levels (for example $z_H = 3 \text{ km}$ if $l = 10^{-3} \text{ km}^{-1}$) can produce upwards air motion at some distance below the region of heating (say at $z = 1 \text{ km}$). In a nearly unstable atmosphere this could trigger deep cumulus convection which could produce the required heating aloft. The possibility of such a positive feedback between broadscale thermally induced motion and smaller scale deep convection (analogous to wave-CISK) is currently under investigation in connection with orographic rain in the tropics.

c. The momentum flux and mountain drag are strongly altered by moist processes. The momentum flux will vary with altitude at the heating levels. The thermally induced surface pressure perturbation acts on the sloping sides of a mountain to alter and possibly reverse the mountain drag. The magnitude and sign of these changes depends on location of the heating relative to the mountain.

d. The effect of moisture on the airflow over mountains cannot be properly accounted for by using the moist adiabatic lapse rate in the Scorer parameter if there is precipitation. The non-precipitating case, with $H \sim w$ in each layer, is a restrictive special case in which the thermally induced buoyancy forces are exactly orthogonal to the vertical velocities. Thus gravity waves can be modified but not generated by the latent heating. In a more general case, as for example with precipitation, gravity waves will be generated by the heating and the momentum flux will vary with height.

e. The input of a net amount of heat at any level makes it impossible to find bounded solutions at $|x| \to \infty$ for the linearized, 2-D, steady state problem. This difficulty can be overcome within the linearized steady state assumptions, by including the Coriolis force and a third dimension. An alternative is to add a widely distributed compensating cooling. The implications of this fact on the numerical solutions of the orographic rain problem are not yet clear.

Acknowledgments

We would like to thank A. Barcilon and J. Jusen of Florida State University for stimulating conversations on the subject of moist air-flow over mountains. This research was supported by the National Science Foundation, Atmospheric Sciences Division, Grant ATM-7722175.

Appendix I

List of symbols in order of appearance

$x$ downstream coordinate
The addition of heat to a stratified airstream

\( y \) lateral coordinate
\( z \) vertical coordinate
\( t \) time
\( u \) downstream velocity
\( v \) lateral velocity
\( w \) vertical velocity
\( p \) pressure
\( \rho \) density
\( g \) gravitational acceleration
\( f \) Coriolis parameter
\( R \) gas constant
\( c_v \) specific heat capacity at constant volume
\( c_p \) specific heat capacity at constant pressure
\( \alpha \) specific volume
\( H \) heating rate (W kg\(^{-1}\))
\( U \) incoming velocity
\( \bar{\rho} \) incoming density
\( \bar{p} \) incoming pressure
\( \bar{T} \) incoming temperature
\( \theta \) potential temperature
\( l \) Scorer parameter
\( N \) Brunt-Vaisala frequency
\( Q \) amplitude of heating function
\( g(x) \) horizontal distribution of heating
\( f(z) \) vertical distribution of heating
\( k \) horizontal wavenumber
\( b \) horizontal scale of heating function Eq. (21)
\( b_1 \) width of the heating part of Eq. (22)
\( b_2 \) width of the cooling part of Eq. (22)
\( \delta \) Dirac delta function
\( z_H \) altitude of heating level
\( m \) vertical wavenumber
\( \eta \) vertical displacement
\( \omega \) parcel oscillation frequency
\( D \) linear drag coefficient
\( F \) vertical flux of horizontal momentum
\( d \) half depth of heating layer
\( c \) upstream displacement of heating function
\( \nabla^2 \) Laplacian operator
\( \nabla_H^2 \) horizontal Laplacian operator
\( \delta \) lateral displacement
\( h(x) \) mountain profile
\( h \) mountain height
\( a \) half width of mountain
\( R(x) \) rainfall rate
\( \rho_w \) density of liquid water
\( \lambda \) latent heat of condensation

**Appendix II Derivation of Eq. (42)**

First, let us consider the heating function in the form of Eq. (21) with the heat added at a certain height \( z_H \).
\[ \dot{H} = - Q \left\{ \frac{2b^3 x}{(x^2 + b^2)^2} \right\} \delta(z - z_H). \]

The governing equation can be written as

\[ w''_{xx} + w''_{zz} + l^2 w' = - \frac{gQ}{c_p T_0^2} \left\{ \frac{2b^3 x}{(x^2 + b^2)^2} \right\} \delta(z - z_H). \]  \hspace{1cm} (A1)

Let \( \tilde{w}(k, z) \) be the one-sided Fourier transform of \( w'(x, z) \) in \( x \), i.e.,

\[ \tilde{w}'(k, z) = \frac{1}{\pi} \int_{-\infty}^{\infty} w'(x, z) e^{-ikx} \, dx \]

\[ w'(x, z) = \text{Re} \int_{-\infty}^{\infty} \tilde{w}'(k, z) e^{ikx} \, dk. \]

Then the Fourier transform of (A1) becomes

\[ \tilde{w}''_{zz} + (l^2 - k^2) \tilde{w}' = \frac{igQ_m b^2 k}{c_p T_0^2} e^{-ikz} \delta(z - z_H). \]  \hspace{1cm} (A2)

For \( z \neq z_H \), (A2) becomes

\[ \tilde{w}''_{zz} + (l^2 - k^2) \tilde{w}' = 0. \]  \hspace{1cm} (A3)

For a vertical propagating \( (k^2 < l^2) \), hydrostatic \( (k^2 \ll l^2) \) wave, the solution of (A3) can be written as

\[ \tilde{w}'(k, z) = A e^{iz} + B e^{-iz} \quad \text{for} \quad z < z_H \] \hspace{1cm} (A4a)

\[ \tilde{w}'(k, z) = C e^{iz} + D e^{-iz} \quad \text{for} \quad z > z_H \] \hspace{1cm} (A4b)

At the ground, the flow is assumed to follow the terrain, thus

\[ \frac{w}{u} = \frac{w'}{U + u'} = \frac{dh(x)}{dx}. \]  \hspace{1cm} (A5)

where \( h(x) \) represents the terrain height. For a small amplitude topography and disturbance \( u' \), (A5) can be simplified to

\[ w' = U \frac{dh(x)}{dx} \text{ at } z = 0. \]  \hspace{1cm} (A6)

Since \( h(x) \) is assumed to be a bell-shaped function

\[ h(x) = h \frac{a^2}{(x^2 + a^2)}. \]  \hspace{1cm} (A7)

Substitute (A7) into (A6) and take the Fourier transform, we have

\[ \tilde{w}'(k, 0) = ikU h e^{-ka}. \]  \hspace{1cm} (A8)

Apply the lower boundary condition (A8) to (A4a), it follows that
\[ A + B = ik\bar{U}ae^{-ka}. \] (A9)

In order to allow the energy to propagate upward, in which the terrain and heating region are regarded as the source of disturbance energy, the coefficient D in (A4b) should be dropped. With (A9), (A4) may be rewritten as

\[ \hat{w}'(k, z) = 2iA\sin lz + ik\bar{U}ae^{-ka}e^{-ilz} \quad \text{for} \quad z < z_H, \] (A10a)

\[ \hat{w}'(k, z) = Ce^{ilz} \quad \text{for} \quad z > z_H. \] (A10b)

Coefficients A and C can be determined by applying the two matching conditions (Eqs. (24) and (25)) along the heating level \( z_H \). Taking the Fourier transform of (24) and (25),

\[ \Delta \hat{w}' = 0 \] (A11)

\[ \Delta \hat{w}'_z = \frac{igQb^2k}{c_pT\bar{U}^2}e^{-kb}. \] (A12)

Apply (A11) to (A10), one obtains

\[ 2iA\sin lz_H + ik\bar{U}ae^{-ka}e^{-ilz} = C\exp(iz_H). \] (A13)

Differentiate (A10) with respect to \( z \); it follows that

\[ \hat{w}' = 2ilA\cos lz + ik\bar{U}ae^{-ka}e^{-ilz} \quad \text{for} \quad z < z_H \] (A14a)

\[ \hat{w}'_z = iCe^{ilz} \quad \text{for} \quad z > z_H. \] (A14b)

Now we can apply (A12) to (A14) and obtain

\[ ilC\exp(iz_H) - 2ilA\cos lz_H - ik\bar{U}ae^{-ka}\exp(-ilz_H) = \frac{igQb^2k}{c_pT\bar{U}^2}e^{-kb}. \] (A15)

From (A13) and (A15), A and C can be obtained. Then substitute them into (A10), we have

\[ \hat{w}'(k, z) = i\bar{U}e^{-ka}e^{ilz} - \frac{igQb^2ke^{-kb}\exp(iz_H)\sin lz}{c_pT\bar{U}^2l} \quad \text{for} \quad z < z_H, \] (A16a)

\[ \hat{w}'(k, z) = i\bar{U}e^{-ka}e^{ilz} - \frac{igQb^2ke^{-kb}\sin lz_H e^{ilz_H}}{c_pT\bar{U}^2l} \quad \text{for} \quad z > z_H. \] (A16b)

From (A16) and Fourier relation, \( w(x, z) \) can be obtained,

\[ w'(x, z) = \text{Re}\left\{ \int_0^\infty \left[ i\bar{U}e^{-ka}e^{ilz} - \frac{igQb^2ke^{-kb}\exp(iz_H)\sin lz}{c_pT\bar{U}^2l} \right] e^{ikx}dx \right\} \] (A17a)

\[ \quad \text{for} \quad z < z_H. \]

\[ w'(x, z) = \text{Re}\left\{ \int_0^\infty \left[ i\bar{U}e^{-ka}e^{ilz} - \frac{igQb^2ke^{-kb}\sin lz_H e^{ilz}}{c_pT\bar{U}^2l} \right] e^{ikx}dx \right\} \] (A17b)

\[ \quad \text{for} \quad z > z_H. \]
The relation of the vertical displacement \( \eta \) and \( w \) can be written as

\[
w' = U \frac{\partial \eta}{\partial x}
\]

so

\[
\eta(x, z) = \frac{1}{U} \int_{x}^{x'} w'(x, z) \, dx.
\]  \hspace{1cm} (A18)

From (A17) and (A18), \( \eta(x, z) \) can be obtained,

\[
\eta(x, z) = \frac{h a (a \cos l z - x \sin l z)}{(a^2 + x^2)} - \frac{g Q b^2 \sin l z (b \cos l z H - x \sin l z H)}{c_p T U^3 (b^2 + x^2)} \quad \text{for } z < z_H, \quad (A19a)
\]

\[
\eta(x, z) = \frac{h a (a \cos l z - x \sin l z)}{(a^2 + x^2)} - \frac{g Q b^2 \sin l z H (b \cos l z - x \sin l z)}{c_p T U^3 (b^2 + x^2)} \quad \text{for } z > z_H. \quad (19b)
\]

Now let us consider the case with the heating uniformly added to a layer of \( z = z_H - d \) to \( z = z_H + d \), that is

\[
\dot{H} = Q \left\{ \frac{-2b^3 x}{(x^2 + b^2)^2} \right\} \quad \text{for } z_H - d < z < z_H + d, \quad . \quad . \quad (A20)
\]

\[
\dot{H} = 0 \quad \text{otherwise}.
\]

The solution can be obtained by superposition of the heating terms of (A19) (last terms). If we rewrite (A19a) and (A19b) as \( \eta(x, z) = \eta_1 + \eta_2 \) and \( \eta(x, z) = \eta_1 + \eta_3 \), respectively, then the superposition of heating will give us

\[
\eta(x, z) = \eta_1(x, z) + \int_{z_H - d}^{z_H + d} \eta_2(x, z, h) \, dh \quad \text{for } z < z_H - d, \quad . \quad . \quad (21a)
\]

\[
\eta(x, z) = \eta_1(x, z) + \int_{z_H - d}^{z_H + d} \eta_3(x, z, h) \, dh + \int_{z}^{z_H + d} \eta_2(x, z, h) \, dh \quad \text{for } z_H - d < z < z_H + d, \quad (21b)
\]

\[
\eta(x, z) = \eta_1(x, z) + \int_{z_H - d}^{z_H + d} \eta_3(x, z, h) \, dh \quad \text{for } z > z_H + d, \quad . \quad . \quad (21c)
\]

where \( h \) represents \( z_H \) in (A19) and acts as an integral variable in (A21). The calculation is long but straightforward and can be obtained as

\[
\eta(x, z) = \eta_1(x, z) - A (\sin l z) \times \left[ \frac{b \{ \sin l (z_H + d) - \sin l (z_H - d) \} + x \{ \cos l (z_H + d) - \cos l (z_H - d) \}}{(b^2 + x^2)} \right] \quad \text{for } z < z_H - d.
\]  \hspace{1cm} (42a)
\begin{equation}
\eta(x, z) = \eta_1(x, z) + A \frac{(b \cos lz \sin x - lz)(\cos lz - \cos l(z_H - d))}{(b^2 + x^2)} - A(\sin lz) \left[ \frac{b[l(z_H + d) - \sin lz] + x[\cos l(z_H + d) - \cos lz]}{(b^2 + x^2)} \right] \tag{42b}
\end{equation}

for \( z_H - d < z < z_H + d \),

\begin{equation}
\eta(x, z) = \eta_1(x, z) + A \left[ \frac{(b \cos lz - x \sin lz)(\cos l(z_H + d) - \cos l(z_H - d))}{(b^2 + x^2)} \right] \tag{42c}
\end{equation}

for \( z > z_H + d \),

where

\[ \eta_1(x, z) = \frac{ha(\alpha \cos lz - x \sin lz)}{(\alpha^2 + x^2)} , \]

and

\[ A = \frac{gQb^2}{c_pT U^3 I^2} . \]

\textbf{References}


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