Extremal principles for global climate models

By S. D. MOBBS

Department of Applied Mathematical Studies, University of Leeds, Leeds, LS2 9JT, England
and
Geophysical Fluid Dynamics Laboratory, Meteorological Office, Bracknell, Berkshire, RG12 2SZ, England

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SUMMARY

Paltridge has recently hypothesized that the mean global climate operates near a state of maximum entropy production. In this paper an attempt is made to justify this hypothesis by parametrizing the eddy heat fluxes in terms of eddy diffusivities and using the local potential method of Glansdorff and Prigogine. With eddy diffusivities independent of the temperature and its gradient a principle of minimum entropy production is in fact found. However, by using a more realistic parametrization, a quantity closely related to entropy production is found to be maximized. The relevance of this to global climate models is discussed.

1. INTRODUCTION

Two contrasting approaches have been used in attempts to predict the mean global climate and the response of the climate to changes in imposed conditions. One approach assumes that the climate can be predicted by modelling in detail all significant atmospheric and oceanic processes. To this end large and expensive numerical models of the global circulation have been developed.

The other approach looks for relatively simple laws governing the overall behaviour of the atmosphere and oceans. The simplest climate models of this type involve a meridional energy balance between incoming solar radiation, atmospheric meridional heat transport and outgoing long wave radiation. Such models have been used by Budyko (1969), Sellers (1969), North (1975) and others. A severe limitation of these models and their later developments is that feedback through effects such as cloud cover or precipitation cannot easily be included except by specifying at the start the amount of cloud or the precipitation rate and not allowing this to vary. This problem is related to one of closure encountered in most climate models. Most models based on sound physical principles involve the difficulty of having one more unknown than there are equations unless some assumption such as specified cloud cover is used. This problem is explained more fully by Paltridge and Platt (1976).

One method by which climate models may be closed and which allows quantities such as cloud cover to vary has been indicated by Paltridge (1975). He postulated that the global climate is constrained by a principle of minimum entropy exchange and applied this idea to a simple zonally averaged energy balance model in which cloud cover was an additional variable. Paltridge was able to obtain predictions of mean surface temperature, cloud cover and total atmospheric and oceanic heat fluxes in remarkable agreement with observed values. Paltridge (1978) applied a principle of maximum entropy production to a more detailed model and obtained predictions of energy fluxes which even showed structures resembling the Gulf Stream and other oceanic currents. Grassl (1981) showed that a principle of minimum entropy exchange is equivalent to one of maximum entropy production by meridional heat fluxes.

In spite of the apparent success of these extremum principles their value has been limited by the fact that no a priori justification for their use could be found. The total entropy production by atmospheric and oceanic heat fluxes is observed to be near its maximum value for present day solar heat input (see Grassl 1981). However, this does not guarantee that the entropy production would be a maximum if external or internal imposed

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conditions were changed and the principal aim of climate models is to predict the response to such changes. In addition, it would be useful to know whether any extremum principle could be applied to the atmospheres of other planets. Golitsyn and Mokhov (1978) have shown that the climate model of Budyko has a maximum of entropy production when the meridional heat flux is varied arbitrarily and that that due to North has a maximum entropy production when the diffusivity describing the meridional heat flux is varied. However there is no guarantee that these entropy production maxima correspond to a state of the atmosphere resembling that of the present day Earth. North et al. (1979) have obtained a variational principle for the simple climate model used in North (1975). This is a minimum principle but the quantity which is minimized is not entropy production.

Paltridge (1979) has attempted to justify the principle of maximum entropy production for the Earth's atmosphere. However, it will be explained in section 5 that Paltridge's argument, even if correct, does not justify the type of application to which the principle has been put.

In this paper it is shown a priori, using a zonally averaged model, that the Earth's atmospheric circulation is likely to operate near a state corresponding to a maximum of a quantity related to the entropy production by meridional and vertical heat fluxes. In order to do this it is assumed that the eddy heat fluxes can be parametrized by eddy diffusivities. The result lends support to the extremal principles used by Paltridge and Grassl.

2. DERIVATION OF A VARIATIONAL PRINCIPLE FOR THE MEAN ATMOSPHERIC TEMPERATURE

In order to make clear the method which is to be used, we will first derive a simple variational principle for the zonally and time averaged temperature of the atmosphere. In section 3 the variational principle will be reformulated in terms of the entropy production by meridional and vertical heat fluxes. In cartesian co-ordinates the Boussinesq energy equation for the atmosphere can be written in the form (see Spiegel and Veronis 1960)

\[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = -\Gamma w + Q, \]  

where $T$ is the temperature, $\Gamma$ is the adiabatic lapse rate and $Q$ is a heating term including the effects of solar radiation, long wave radiation, latent heat release and any other thermal effects. The co-ordinates $x, y, z$ are directed eastwards, northwards and upwards respectively and $u, v, w$ are the corresponding velocity components.

Taking into account the continuity equation

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \]  

Eq. (1) can be written in terms of the heat flux:

\[ \frac{\partial T}{\partial t} + \frac{\partial}{\partial x} (uT) + \frac{\partial}{\partial y} (vT) + \frac{\partial}{\partial z} (wT) = -\Gamma w + Q. \]  

We will distinguish between quantities averaged both over longitude and over a time long compared with the time scale of the fluctuations associated with the internal dynamics (denoted by an overbar) and eddy quantities (denoted by a prime). So, for example, $\overline{T} = \overline{T} + T'$. The averaged form of Eq. (3) is then
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\[
\frac{\partial}{\partial y} (\tilde{v_T}) + \frac{\partial}{\partial z} (\tilde{w_T}) + \frac{\partial}{\partial y} (\tilde{v_T'}) + \frac{\partial}{\partial x} (\tilde{w_T'}) = -\Gamma \tilde{w} + \tilde{Q} .
\]  

(4)

In this simple example we will assume that the eddy heat fluxes can be parametrized by means of positive horizontal and vertical eddy diffusivities \( K_H \) and \( K_V \) which may vary with position. Thus

\[
\tilde{v_T}' = -K_H \frac{\partial \tilde{T}}{\partial y}
\]

and

\[
\tilde{w_T}' = -K_V \frac{\partial \tilde{T}}{\partial z} .
\]

(5)

The final form of the energy equation is then

\[
\frac{\partial}{\partial y} (\tilde{v_T}) + \frac{\partial}{\partial z} (\tilde{w_T}) + \Gamma \tilde{w} - \tilde{Q} = \frac{\partial}{\partial y} \left( K_H \frac{\partial \tilde{T}}{\partial y} \right) + \frac{\partial}{\partial z} \left( K_V \frac{\partial \tilde{T}}{\partial z} \right) .
\]

(6)

We will employ the local potential technique developed by Glansdorff and Prigogine (1964). Suppose a system is described by functions of position and time \( v_m(x, t) \) for \( m = 1 \) to \( n \). Then the local potential method involves looking for a variational principle of the form

\[
\delta \iint L(v_1, v_1^*, v_2, v_2^*, \ldots, v_n, v_n^*) \, dx \, dt = 0 .
\]

(7)

When the variations are made, the variables \( v_m^* \) are regarded as having their stationary values and are therefore not varied. The Euler-Lagrange equations are then

\[
(\partial L/\partial v_m)_{v_m^*} = 0 ; m = 1, \ldots, n .
\]

(8)

Having made the variations we apply the subsidiary conditions

\[
v_m = v_m^* ; m = 1, \ldots, n .
\]

(9)

The variational principle (7) can be generalized to include problems involving the derivatives of the functions \( v_m(x, t) \) in the usual way. It must be emphasized that the local potential method is purely a mathematical technique. Starred variables are not constants in the physical sense. The method has no essential connection with either Glansdorff and Prigogine's principle of minimum entropy production or any theory of random thermal fluctuations.

For the atmosphere governed by Eq. (6), a possible variational principle of the local potential type (i.e. one whose Euler-Lagrange equation is Eq. (6)) is

\[
\delta \iint \left\{ \frac{\partial}{\partial y} (\tilde{v_T}^*) + \frac{\partial}{\partial z} (\tilde{w_T}^*) + \Gamma \tilde{w}^* - \tilde{Q}^* \right\} \, d\tilde{T} + \frac{K_H}{2} \left( \frac{\partial \tilde{T}}{\partial y} \right)^2 + \frac{K_V}{2} \left( \frac{\partial \tilde{T}}{\partial z} \right)^2 \, dy \, dz = 0 .
\]

(10)

The Euler-Lagrange equation formed by varying \( \tilde{T} \) (but not \( \tilde{T}^* \)) in (10) is easily shown to be Eq. (6) if the subsidiary conditions \( \tilde{T} = \tilde{T}^*, \tilde{v} = \tilde{v}^*, \tilde{w} = \tilde{w}^* \) and \( \tilde{Q} = \tilde{Q}^* \) are used.

In order to determine whether the extremal is a maximum, minimum or neither of these, we substitute for the terms in square brackets in (10) from Eq. (6), giving
\[
\delta \int \left[ \int \left( \frac{\partial}{\partial y} \left( K_H \frac{\partial T^*}{\partial y} \right) + \frac{\partial}{\partial z} \left( K_V \frac{\partial T^*}{\partial z} \right) \right) + \frac{K_H}{2} \left( \frac{\partial T}{\partial y} \right)^2 + \frac{K_V}{2} \left( \frac{\partial T}{\partial z} \right)^2 \right] \, dy \, dz = 0. \tag{11}
\]

Such a substitution from the Euler-Lagrange equation back into the variational integral is not always valid but in this case the Euler-Lagrange equation of the variational principle (11) is

\[
\frac{\partial}{\partial y} \left( K_H \frac{\partial T^*}{\partial y} \right) + \frac{\partial}{\partial z} \left( K_V \frac{\partial T^*}{\partial z} \right) - \frac{\partial}{\partial y} \left( K_H \frac{\partial T}{\partial y} \right) - \frac{\partial}{\partial z} \left( K_V \frac{\partial T}{\partial z} \right) = 0
\]

which is clearly true. However, in making the substitution some information concerning energy conservation has been lost. Integrating by parts the term in square brackets in (11) and noting that if there is to be no eddy flux of energy across the boundaries of the atmosphere \(K_H\) and \(K_V\) must vanish there, the variational principle becomes

\[
\delta I_1 = 0 \quad \text{ (12)}
\]

where

\[
I_1 = -\int \int \left\{ K_H \frac{\partial T}{\partial y} \frac{\partial T^*}{\partial y} + K_V \frac{\partial T}{\partial z} \frac{\partial T^*}{\partial z} - \frac{K_H}{2} \left( \frac{\partial T}{\partial y} \right)^2 - \frac{K_V}{2} \left( \frac{\partial T}{\partial z} \right)^2 \right\} \, dy \, dz \quad \text{ (13)}
\]

We now put \(T = T^* + \Delta T\) (\(\Delta T\) not necessarily small) in Eq. (13) giving

\[
I_1 = \int \int \left[ -\frac{K_H}{2} \left( \frac{\partial \Delta T}{\partial y} \right)^2 - \frac{K_V}{2} \left( \frac{\partial \Delta T}{\partial z} \right)^2 + \frac{K_H}{2} \left( \frac{\partial T}{\partial y} \right)^2 + \frac{K_V}{2} \left( \frac{\partial T}{\partial z} \right)^2 \right] \, dy \, dz. \quad \text{ (14)}
\]

For any distribution of temperature variations \(\Delta T\) we have therefore

\[
I_1 \geq -\int \int \left\{ \frac{K_H}{2} \left( \frac{\partial T^*}{\partial y} \right)^2 + \frac{K_V}{2} \left( \frac{\partial T^*}{\partial z} \right)^2 \right\} \, dy \, dz, \quad \text{ (15)}
\]

so the variational principle is a minimum principle.

It may be noticed at this point that the variational principle (12) could have been written down without knowledge of the energy equation (6). In this sense (12) is a trivial statement satisfied by any function \(T\) such that \(T = T^*\). In later sections, however, we will derive variational principles which are similar to (12) but which are expressed in terms of the entropy production and which retain physical significance due to restrictions on the allowed variations \(\Delta T\). The purpose of starting the analysis leading to (12) with the energy equation (6) is to give a straightforward and systematic method which can be repeated for the more complex cases studied in sections 3 and 4.

The variational principle (10) is similar to one found by North et al. (1979) for a different energy equation.

3. The Variational Principle expressed in terms of Entropy Production

The rate of entropy production per unit volume, \(\Sigma\), by meridional and vertical heat fluxes, \(J\), and by internal thermal processes is given by

\[
\frac{\Sigma}{c_p} = \sigma = -\frac{J \cdot \Delta T}{T^2} + \frac{Q}{T} \quad \text{ (16)}
\]
where $c_p$ is the specific heat capacity at constant pressure. In the remainder of the paper we will refer to $\sigma$ simply as the entropy production. With the heat fluxes parametrized as in Eq. (5) the entropy production by eddy fluxes, $\sigma_E$, takes the form

$$\sigma_E = \frac{K_H}{T^2} \left( \frac{\partial \bar{T}}{\partial y} \right)^2 + \frac{K_V}{T^2} \left( \frac{\partial \bar{T}}{\partial z} \right)^2 . \tag{17}$$

We now wish to modify the variational principle derived in section 2 so as to involve $\sigma$.

A possible variational principle for Eq. (6) is

$$\delta \int \left[ \frac{T}{T^*} \left( \frac{\partial \bar{T}^*}{\partial y} \right)^2 + \frac{\partial \bar{T}^*}{\partial z} \right] + \frac{\partial}{\partial z} \left( \bar{w}^* \bar{T}^* \right) + \Gamma \bar{w}^* - \bar{Q}^* = \frac{-K_H}{T^*} \left( \frac{\partial \bar{T}^*}{\partial y} \right)^2 - \frac{K_V}{T^*} \left( \frac{\partial \bar{T}^*}{\partial z} \right)^2 + \frac{K_H}{2T^2} \left( \frac{\partial \bar{T}}{\partial y} \right)^2 + \frac{K_V}{2T^2} \left( \frac{\partial \bar{T}}{\partial z} \right)^2 \right] dy \, dz = 0 . \tag{18}$$

Following the same procedure as in section 2, we simplify this using Eq. (6) giving

$$\delta I_2 = 0 \tag{19}$$

where

$$I_2 = \int \left[ \frac{T}{T^*} \left( \frac{\partial \bar{T}^*}{\partial y} \right)^2 + \frac{\partial \bar{T}^*}{\partial z} \right] + \frac{\partial}{\partial z} \left( \bar{w}^* \bar{T}^* \right) + \Gamma \bar{w}^* - \bar{Q}^* = \frac{-K_H}{T^*} \left( \frac{\partial \bar{T}^*}{\partial y} \right)^2 - \frac{K_V}{T^*} \left( \frac{\partial \bar{T}^*}{\partial z} \right)^2 + \frac{K_H}{2T^2} \left( \frac{\partial \bar{T}}{\partial y} \right)^2 + \frac{K_V}{2T^2} \left( \frac{\partial \bar{T}}{\partial z} \right)^2 \right] dy \, dz . \tag{20}$$

Integrating by parts and taking $K_H$ and $K_V$ to be zero on the boundaries we have

$$I_2 = \int \left[ \frac{K_H}{T^*} \left( \frac{\partial \bar{T}^*}{\partial y} \right)^2 + \frac{K_V}{T^*} \left( \frac{\partial \bar{T}^*}{\partial z} \right)^2 - \frac{K_H}{T^*} \frac{\partial \bar{T}}{\partial y} \frac{\partial \bar{T}^*}{\partial y} - \frac{K_V}{T^*} \frac{\partial \bar{T}}{\partial z} \frac{\partial \bar{T}^*}{\partial z} \right] dy \, dz . \tag{21}$$

Two questions must now be answered. Firstly, how is $I_2$ related to the entropy production $\sigma$ and secondly, is Eq. (19) a maximum or minimum principle? To answer the first question we consider the quantity

$$I_2 - \frac{1}{2} \int \sigma_E \, dy \, dz$$

$$= \int \left[ \frac{\Delta \bar{T}}{T^*} \left( \frac{\partial \bar{T}^*}{\partial y} \right)^2 + \frac{\partial \bar{T}^*}{\partial z} \right] - \frac{K_H}{T^*} \frac{\partial \bar{T}^*}{\partial y} \frac{\partial (\Delta \bar{T})}{\partial y} - \frac{K_V}{T^*} \frac{\partial \bar{T}^*}{\partial z} \frac{\partial (\Delta \bar{T})}{\partial z} \right] dy \, dz . \tag{22}$$

The variations of $\bar{T}$ will now be restricted to include only distributions of $\bar{T}$ which satisfy the energy equation (6). This is not a necessary condition for the existence of the variational principle Eq. (19) but it is necessary if $I_2$ is to be interpreted physically in terms of the entropy production. Furthermore, this constraint is used by Paltridge in his applications.

We can write Eq. (6) in two forms, namely both

$$\frac{\partial}{\partial y} (\bar{v}^* \bar{T}^*) + \frac{\partial}{\partial z} (\bar{w}^* \bar{T}^*) + \Gamma \bar{w}^* - \bar{Q}^* = \frac{\partial}{\partial y} \left( K_H \frac{\partial \bar{T}^*}{\partial y} \right) + \frac{\partial}{\partial z} \left( K_V \frac{\partial \bar{T}^*}{\partial z} \right).$$
and the same equation expressed in terms of the varied quantities $\tilde{T}, \tilde{v}, \tilde{w}$ and $\tilde{Q}$. Subtracting these equations we get

$$\frac{\partial}{\partial y} \{A(\tilde{v}\tilde{T})\} + \frac{\partial}{\partial z} \{A(\tilde{w}\tilde{T})\} + \Gamma \Delta \tilde{w} - \Delta \tilde{Q} = \frac{\partial}{\partial y} \left\{ K_H \frac{\partial(\Delta \tilde{T})}{\partial y} \right\} + \frac{\partial}{\partial z} \left\{ K_V \frac{\partial(\Delta \tilde{T})}{\partial z} \right\},$$

(23)

where $\Delta$ has the same meaning as in section 2. By straightforward use of the divergence theorem and noting that $K_H$ and $K_V$ vanish on the boundaries we have the following identity:

$$\int \int \left[ \frac{\Delta \tilde{T}}{\tilde{T}^{*2}} \left\{ K_H \left( \frac{\partial \tilde{T}^{*}}{\partial y} \right)^2 + K_V \left( \frac{\partial \tilde{T}^{*}}{\partial z} \right)^2 \right\} - K_H \tilde{T}^{*2} \frac{\partial(\Delta \tilde{T})}{\partial y} \frac{\partial T^{*}}{\partial y} - K_V \tilde{T}^{*2} \frac{\partial(\Delta \tilde{T})}{\partial z} \frac{\partial T^{*}}{\partial z} \right] dy \, dz = \int \int \left[ -\frac{1}{2\tilde{T}^{*2}} \left\{ \frac{\partial}{\partial y} \left( K_H \frac{\partial(\Delta \tilde{T})}{\partial y} \right) + \frac{\partial}{\partial z} \left( K_V \frac{\partial(\Delta \tilde{T})}{\partial z} \right) \right\} + \frac{\Delta \tilde{T}}{2\tilde{T}^{*2}} \left\{ \frac{\partial}{\partial y} \left( K_H \frac{\partial \tilde{T}^{*}}{\partial y} \right) + \frac{\partial}{\partial z} \left( K_V \frac{\partial \tilde{T}^{*}}{\partial z} \right) \right\} \right] dy \, dz. \tag{24}$$

Proof of this relation is given in Appendix A. Using Eqs. (23) and (24) and restricting $\Delta \tilde{T}/\tilde{T}^{*2}$ to be small, Eq. (22) can be written in the approximate form

$$I_2 - \frac{1}{2} \int \int \sigma_E \, dy \, dz = -\frac{1}{2} \Delta \left[ \int \int \left\{ \frac{1}{\tilde{T}} \frac{\partial (\tilde{v}\tilde{T})}{\partial y} + \frac{1}{\tilde{T}} \frac{\partial (\tilde{w}\tilde{T})}{\partial z} + \frac{\Gamma \tilde{w}}{\tilde{T}} - \frac{\tilde{Q}}{\tilde{T}} \right\} \, dy \, dz \right] = \frac{1}{2} \Delta \left[ \int \int \left\{ -\frac{\partial (\tilde{v}\tilde{T})}{\tilde{T}^{*2}} \frac{\partial \tilde{T}}{\partial y} - \frac{\partial (\tilde{w}\tilde{T})}{\tilde{T}^{*2}} \frac{\partial \tilde{T}}{\partial z} - \frac{\Gamma \tilde{w}}{\tilde{T}} + \frac{\tilde{Q}}{\tilde{T}} \right\} dy \, dz \right]. \tag{25}$$

The terms on the right-hand side of Eq. (25) are the variation in the entropy production by mean horizontal and vertical heat fluxes ($\sigma_M$, say) and by other thermal processes ($\sigma_T$, say). The term involving the adiabatic lapse rate $\Gamma$ makes no contribution when integrated latitudinally (see Stone 1974). We have therefore

$$\delta I_2 = \frac{1}{2} \delta \int \int (\sigma_E + \sigma_M + \sigma_T) \, dy \, dz = \frac{1}{2} \delta \int \int \sigma \, dy \, dz \quad \ldots \quad \ldots \tag{26}$$

and the quantity which is extremalized is the total entropy production.

The question of whether the variational principle $\delta I_2 = 0$ is a maximum or minimum principle can be answered by applying the Legendre test (see Forsyth 1927). (This test applies to local potential type variational principles as well as conventional ones as can easily be proved by following exactly the same method as employed by Forsyth.) For the present problem the Legendre test states that:

(i) if the relation

$$\frac{\partial^2 \sigma}{\partial T_y^2} \cdot \frac{\partial^2 \sigma}{\partial T_z^2} \geq \left( \frac{\partial^2 \sigma}{\partial T_x \partial T_z} \right)^2$$

is satisfied (where $T_y = \partial \tilde{T}/\partial y$ and $T_z = \partial \tilde{T}/\partial z$), then

(ii) the variational principle is a maximum principle if $\partial^2 \sigma/\partial T_y^2 < 0$ (and hence by (i) $\partial^2 \sigma/\partial T_z^2 < 0$) and a minimum principle if $\partial^2 \sigma/\partial T_y^2 > 0$. 

If the condition (i) is not satisfied the extremal is neither a maximum nor a minimum.

To apply the Legendre test we therefore need to know how \( \sigma \) depends on \( \bar{T} \), \( \partial \bar{T} / \partial y \) and \( \partial \bar{T} / \partial z \). We already have \( \sigma_\epsilon \) in terms of the temperature and its gradient but the contributions from \( \sigma_M \) and \( \sigma_T \) are unknown. It seems likely that these will be less strongly dependent on the temperature gradient than is \( \sigma_\epsilon \) and we will neglect the effect due to these terms when applying Legendre’s test. It must be emphasized that no formal justification for the neglect can be given. This problem is discussed in more detail in section 5. It is also important to state that \( \sigma_M \) and \( \sigma_T \) must not be neglected in any practical application which involves actually finding the values of \( \bar{T} \), etc. at the extremal.

Applying Legendre’s test we have

\[
\frac{1}{2} \frac{\partial^2 \sigma_\epsilon}{\partial \bar{T}_y^2} = \frac{K_H}{\bar{T}^4} > 0, \\
\frac{1}{2} \frac{\partial^2 \sigma_\epsilon}{\partial \bar{T}_z^2} = \frac{K_V}{\bar{T}^4} > 0
\]

and

\[
\frac{1}{2} \frac{\partial^2 \sigma_\epsilon}{\partial \bar{T}_y \partial \bar{T}_z} = 0.
\]

Therefore we have a principle of minimum entropy production. This is at variance with the principles used by Paltridge and Grassl.

We must now ask how we can explain this disagreement with the work of Paltridge and Grassl. In the next section it is shown that if more realistic parametrizations for the heat fluxes are used, the minimum principle found in this section can become a maximum principle.

4. More realistic parametrization of the eddy heat fluxes

Poleward of about 30° latitude the atmospheric meridional heat flux is dominated by the contribution from the baroclinic eddies (Newton 1970). Such heat fluxes are not well represented by constant diffusivities as used in section 3. Quite realistic eddy heat flux parametrizations have been obtained by Stone (1974) based on a cartesian Eady type model of baroclinic instability. Stone parametrized the heat fluxes in the following way:

\[
w' T' = -K \left\{ \frac{\partial \bar{T}}{\partial y} + \gamma \left( \frac{\partial \bar{T}}{\partial z} + \Gamma \right) \right\} \\
w' T' = -K \gamma \left( \frac{\partial \bar{T}}{\partial y} + \gamma \left( \frac{\partial \bar{T}}{\partial z} + \Gamma \right) \right).
\]

This type of parametrization was first suggested by Reed and German (1965). Stone derived expressions for \( K \) and \( \gamma \) which for the present problem can be written as

\[
K = 0.144 \; g \; z_T \frac{2}{\bar{T}} \left\{ \frac{g}{\bar{T}} \left( \frac{\partial \bar{T}}{\partial z} + \Gamma \right) \right\} \left( \frac{\partial \bar{T}}{\partial y} \right) \left\{ \frac{2 \Omega^2 T}{\bar{T}} \left( 1 - \frac{5}{2} \frac{z}{z_T} \left( 1 - \frac{z}{z_T} \right) \right) \right\}
\]

and

\[
\gamma = -\left( \frac{5}{2} \frac{\partial \bar{T} / \partial y}{\bar{T} / \partial z + \Gamma} + \frac{1.42 z_T \delta \bar{T} / \delta y}{R \bar{T} / \partial y} \right) \frac{z}{z_T} \left( 1 - \frac{z}{z_T} \right)^2,
\]

where \( z_T \) is the height of the tropopause, \( \Omega \) is the angular velocity of the Earth and \( R \) is the radius of the Earth. These parametrizations give meridional and vertical heat fluxes of the
observed order of magnitude and with qualitatively the observed spatial variations. The maxima in the meridional heat fluxes occur at about 15° nearer the equator than is observed (see Stone 1974).

In principle we can now rework the analysis of section 3 using these more realistic parametrizations. It should be noted that the variations will now include variations in the diffusivities $K$, $K_\gamma$ and $K_\gamma^2$. The extension of the method of section 3 is mostly straightforward although rather tedious. However, because some non-trivial generalizations of the analysis of section 3 are involved, the details are given in Appendix B.

We will quote here only the final form of the variational principle which is

$$\delta I_3 = 0 \quad \ldots \quad (29)$$

where

$$I_3 = \iint \left\{ \frac{1}{2}(\sigma_E + \sigma_M + \sigma_T) \frac{dT}{\gamma} \frac{\partial T}{\partial y} + \frac{K_\gamma \Gamma}{2T^2} \frac{\partial T}{\partial y} + \frac{K_\gamma^2 \Gamma}{2T^2} \frac{\partial T}{\partial z} + \right.$$ 

$$\left. + \frac{T}{T^*} \left( K_\star \gamma^* \frac{\partial T}{\partial y} + K_\star \gamma^{*2} \frac{\partial T}{\partial z} \right) - \frac{1}{T^*} \left( K_\star \gamma^* \frac{\partial T}{\partial y} + K_\star \gamma^{*2} \frac{\partial T}{\partial z} \right) \right\} dy \, dz. \quad (30)$$

$2I_3$ is closely related but not exactly equal to the total entropy production. We will postpone further discussion of this until the next section.

We can again use Legendre's test to determine whether (29) is a maximum or minimum principle. As in the previous section, Legendre's test was not applied to the second and third terms in (30) and as before, no formal justification is given, but the comments made concerning this point in the previous section still apply. Because of the dependence of $K$ and $\gamma$ on the temperature and its gradient, we need to know the stationary (in the calculus of variations sense) state of the atmosphere before we can apply the test. The observed zonal and time averaged cross sections of the temperature of the atmosphere quoted by Newell et al. (1970) were used. These cross sections apply to the periods December to February and June to August in the northern hemisphere. The results of applying Legendre's test for the troposphere are shown in Figs. 1 and 2. No attempt was made to apply the test in the stratosphere as the parametrizations do not apply there. It was found that conditions are such that the extremal of $I_3$ is a maximum over a large region of the troposphere, and this region is that where the heat transport by baroclinic eddies is dominant. Outside this region there was found to be no clear maximum or minimum. Strictly, Legendre's test should indicate conditions suitable for a maximum everywhere before $I_3$ can be considered to have a global maximum. However, in view of the approximate nature of the parametrizations of the heat fluxes and in particular their limited region of validity (i.e. the region of dominant baroclinic wave activity), the failure to find conditions for a maximum everywhere cannot be considered significant. This is discussed further in section 5.

The variational principle

$$\delta \int \int \sigma_E \, dy \, dz = 0 \quad \ldots \quad (31)$$

with the parametrizations (27) was also tested for the existence of a maximum or minimum using Legendre's test. No clear maximum or minimum was found for either summer or winter conditions. However, no a priori reason why the variational principle (31) should apply has been found.
Figure 1. Cross section of the troposphere in the northern hemisphere showing mean isotherms for the period December to February (adapted from Newell et al. 1970). The crosses represent the region where the variational principle (29) is a maximum principle. The temperature is in °C and the heavy lines are the tropopause.

5. DISCUSSION

In the previous sections we have considered the energy balance of the atmosphere and deduced variational principles for the zonal and time averaged temperature. If the eddy heat fluxes are parametrized by eddy diffusivities independent of the temperature and its gradient then the variational principle is one of minimum entropy production. For eddy heat fluxes parametrized by more realistic diffusivities which depend on the temperature and its gradient and also on height, we have found that a quantity related to the entropy production is maximized over a large part of the troposphere. In both cases variations are subject to conservation of energy and we have assumed that only the eddy part of the heat flux is strongly dependent on the temperature gradient.

The effect of taking into account only the eddy contributions when checking for a maximum or minimum is seen in Figs. 1 and 2. It can be seen that in the northern hemisphere the region where conditions allow a maximum of $I_3$ is reduced in area in summer. This corresponds to the decreased heat transport by baroclinic eddies in summer. It is not surprising that no clear maximum or minimum is found outside the region of influence of the baroclinic eddies as outside this region the heat fluxes are not properly taken into account when applying Legendre's test. If the effect on the existence of a maximum or minimum of the other contributions to $I_3$ could be taken into account the regions of maximum $I_3$ would be altered but the extent of the alteration is uncertain. In order to assess the importance of the other contributions to the entropy production, these could be parametrized in some way. However it is unlikely that this could be done without first knowing the stationary (in the calculus of variations sense) state of the atmosphere; the reason being that the atmosphere is a highly nonlinear system in which the mean state depends on the short time scale fluctuations. Hence the non-eddy contributions to the entropy pro-
duction depend not only on the mean temperature distribution but also on the structure of the eddies and are therefore extremely difficult to parametrize. For the same reason, no parametrization of the eddy heat flux in terms of the mean temperature and its gradient can be entirely satisfactory as the quantities upon which the parametrizations are based are to some extent the result of the baroclinic eddies as well as being their cause. That a parametrization with constant eddy diffusivities is inadequate can be demonstrated by the following example. Suppose one of the varied (in the calculus of variations sense) states of the atmosphere is one with slightly greater meridional heat flux than the stationary state. (Remember here that these varied states must also satisfy the energy equation.) For constant solar heat input this requires less radiation loss near the equator and greater radiation loss near the poles, i.e. a cooler equator and warmer poles. However, this would imply a reduced average meridional temperature gradient which in turn implies a reduced meridional heat flux, contradicting the original assumption. It must be stated that Stone's parametrizations are an improvement on this, although they do not completely resolve the problem. The difficulty may be due to the fact that the parametrizations are based on models which do not have the heating term Q included.

In view of these limitations of the parametrizations of the heat fluxes it is not surprising that we do not find a principle of maximum entropy production which holds everywhere in the troposphere, even if in reality such a principle is operating. However, the results presented in section 4 suggest that a variational principle which is at least related to one of maximum entropy production is likely to hold. Of course, in any application of such a principle we would calculate the heat fluxes independently of the temperature using the variational principle (as Paltridge and Grasell have done) and parametrizations would be unnecessary. Parametrizations are only needed when attempting to justify the principle. Another point which needs to be made here is that in the variational principles (26) and (29), although the variations are made in such a way that the energy equation (6) is satisfied, we can take any variation $\Delta T$ if suitable variations $\Delta Q$, $\Delta \bar{u}$ and $\Delta \bar{w}$ are chosen. However, in a practical problem, we would need the same number of equations as variables. We
could, for example, include the three components of the momentum equations and the continuity equation in addition to the energy equation (6) giving five equations and six variables \( u, v, w, T, Q \) and the pressure). The problems could then be closed using one of the variational principles and in this problem the variations \( \Delta T, \Delta u \), etc. would not be independent.

It must be emphasized here that the variational principles derived in this paper have been proved (subject to certain stated assumptions) without any \textit{a priori} knowledge of the stationary state of the atmosphere, unlike the principles used by Paltridge and Grassl which are postulates.

(a) Differences between present work and that of Paltridge

It is worth noting here two other differences between the present work and that of Paltridge. Firstly, Paltridge calculated only surface temperature and height averaged heat fluxes whereas here the vertical structure was needed in order to parametrize the heat fluxes. For a one dimensional model such as in Paltridge (1975) the existence of at least one maximum or minimum is virtually guaranteed if the heat fluxes depend at all on the temperature gradient, but in a two dimensional model an extremal which is neither a maximum nor a minimum is possible. Secondly, Paltridge included the oceans in his model. The methods used in the present paper could equally be applied to the oceans (although different parametrizations would be needed), but if we wished to assess the effects of the non-eddy contributions on the entropy production it would be necessary to parametrize the atmosphere-ocean heat transfers. Thus we could look for an independent variational principle applying to the oceans which had only eddy heat fluxes or alternatively we could look for a single global variational principle applying to the atmosphere-ocean system in which all contributions to the entropy production were included.

It is interesting to note that when the heat fluxes are parametrized by constant diffusivities we have verified a principle of minimum entropy production, whereas Paltridge's and Grassl's results strongly suggest that a principle of maximum entropy production is operating. This may have implications for some of the simple climate models such as North's (1975) model and later developments of this which employ constant diffusivities.

(b) Other planetary atmospheres

It may be useful to enquire whether a principle of maximum entropy production could apply to the atmospheres of other planets. A related problem is that of whether for the Earth’s atmosphere the maximum entropy production principle, even if it applies for conditions close to those of the present day, could apply if imposed conditions were changed considerably. Although we cannot answer these questions conclusively, it is clear in the present work that the existence of a principle of maximum or minimum entropy production depends to some extent on the relationship between the heat fluxes and the temperature gradient, i.e. on the structure of the dynamics of the atmospheric circulation. There is therefore no reason to suppose that the atmospheres of other planets will obey the same extremal principle as the Earth’s atmosphere and it is possible that even for the Earth’s atmosphere, a different principle would hold if imposed conditions were changed sufficiently. Some atmospheres may have principles of \textit{minimum} entropy production operating, as was found to be possible with constant diffusivities.

(c) Invariance to Earth rotation

Rogers (1976) and Paltridge (1976) and others have been concerned that Paltridge's results are apparently invariant to the rate of Earth rotation. The first point to note here is that the principle of maximum entropy production itself must be invariant to the Earth’s
rotation since it involves only a thermodynamic quantity. However, the principle could be applied to many models of the atmosphere, some of which may involve the rotation rate. It is merely that Paltridge chose to use an energy balance model independent of the Earth's rotation. One may enquire how it is that Paltridge was able to reproduce structures resembling oceanic currents which are known to depend on the Earth's rotation. The reason is that Paltridge specified the global distribution of albedo corresponding to present day observed values and held this fixed in his model. Certain constraints on the energy fluxes were therefore implicit in his simulation of present day climatic conditions and it is not surprising that these gave rise to structures resembling the currents which are to a large extent responsible for producing the specified albedo distribution.

(d) Other Variational Principles

It is worth noting the connection (if any) between Paltridge's principle of maximum entropy production and other thermodynamic variational principles. Firstly, Rogers (1976) and Paltridge (1976) have discussed Glansdorff and Prigogine's (1964) principle of minimum entropy production in this context. However, this principle applies only to systems where convective heat transport is negligible and so it cannot apply to the atmosphere. Furthermore, it is an entirely different type of variational principle from the present one. It states that as a system evolves towards a steady state the entropy production will decrease with time until a minimum is reached at the steady state, whereas the present variational principle asserts that of all the energetically possible mean states, the one which is realized is the one with the maximum (or minimum in the case of constant diffusivities) entropy production. The variational principle of Palm (1972) is an extension of Glansdorff and Prigogine's principle in which a 'generalized dissipation' evolves towards a minimum in a final state of slightly supercritical Rayleigh-Bénard convection. Therefore both Glansdorff and Prigogine's principle and Palm's principle involve classes of allowed variations quite different from those permitted in the present paper.

The principle of maximum heat transport by steady state turbulent convection suggested by Malkus (1954) is more closely related to the present principle but even here there is a difference. This is that Malkus compares the realized state only with other steady states rather than all other energetically possible states. Shutts (1981) studied a quasi-geostrophic two level model of the atmosphere and applied a principle of maximum entropy production, defining entropy in a dynamical context by using methods of statistical mechanics. As in the present paper, Shutts found it was necessary to use a constraint of energy conservation and he also employed a constraint of no angular momentum transfer between the atmosphere and the Earth. His results were in reasonable agreement with those obtained previously by more conventional means for a similar model.

Paltridge (1979 and 1981) has attempted to justify his principle of maximum entropy production by considering a simple model of the atmosphere with feedback mechanisms. He concluded that if the atmosphere has a finite number of dynamically possible states then due to fluctuations in solar output (assuming they exist!), there would be a tendency over a long period of time to move towards the states with the highest entropy production. However, it must be emphasized that here Paltridge was comparing the realized state of the atmosphere only with other dynamically possible states which are assumed to have occurred at an earlier time. Thus the allowed variations are far more restricted than in Paltridge's applications and in the present work, where all energetically possible variations are allowed.

Sawada (1981) has discussed the differences between Glansdorff and Prigogine's principle and a proposed general principle of maximum entropy production describing non-equilibrium phenomena. Sawada suggests that Glansdorff and Prigogine's principle applies on a time scale comparable with that of the characteristic time scale of a single metastable state while the principle of maximum entropy production applies on a much longer time scale during which transitions between metastable states can occur. However, Sawada's maximum entropy production principle is of the same type as that discussed by
Paltridge (1979) in which the entropy production is assumed to increase with time until a maximum is reached. The allowed variations are therefore more restrictive than in either Partridge's earlier applications or the present work.

(e) Possible extensions

Finally, we mention how the present work could be extended in at least two ways. Firstly, better parametrizations for the eddy heat fluxes could be sought and parametrization of the non-eddy heat fluxes could be attempted. Secondly, there is a possibility that variational principles for simpler models such as that suggested by Shutts could be verified using the methods described in this paper.

6. Conclusions

If the eddy heat fluxes are parametrized by means of constant diffusivities then the atmosphere is found to obey a principle of minimum entropy production. If more realistic parametrizations are used then a quantity closely related to the entropy production is maximized. In both cases variations are subject to conservation of energy and only the eddy contribution to the entropy production is used to test for a maximum or minimum.

These results suggest, although they do not prove, that the atmosphere is likely to operate near a state of maximum entropy production, thus lending support to the variational principles used by Paltridge and Grassl. The essential point to note is that the results obtained in this paper are proved without a priori knowledge of the state of the atmosphere whereas the principles used by Paltridge and Grassl are postulates whose applicability is judged by their ability to reproduce known climatic conditions.

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Appendix A

Proof of Eq. (24)

Consider

\[
\frac{\partial}{\partial y} \left[ K_H \frac{\partial (\Delta T)}{\partial y} \left( \frac{\partial T}{\partial y} \right) \right] + \frac{\partial}{\partial z} \left[ K_V \frac{\partial (\Delta T)}{\partial z} \left( \frac{\partial T}{\partial z} \right) \right] =
\]

\[
= \frac{1}{T^*} \frac{\partial}{\partial y} \left[ K_H \frac{\partial (\Delta T)}{\partial y} \right] + \frac{1}{T^*} \frac{\partial}{\partial z} \left[ K_V \frac{\partial (\Delta T)}{\partial z} \right] - \frac{\Delta T}{T^{*2}} \frac{\partial}{\partial y} \left( K_H \frac{\partial T^*}{\partial y} \right) - \frac{\Delta T}{T^{*2}} \frac{\partial}{\partial z} \left( K_V \frac{\partial T^*}{\partial z} \right) - \frac{2K_H}{T^{*2}} \frac{\partial T^*}{\partial y} \frac{\partial (\Delta T)}{\partial y} \left( \frac{\partial T}{\partial y} \right) - \frac{2K_V}{T^{*2}} \frac{\partial T^*}{\partial z} \frac{\partial (\Delta T)}{\partial z} \left( \frac{\partial T}{\partial z} \right) + \frac{2K_H \Delta T}{T^{*3}} \left( \frac{\partial T^*}{\partial y} \right)^2 + \frac{2K_V \Delta T}{T^{*3}} \left( \frac{\partial T^*}{\partial z} \right)^2
\]

where the right-hand side has been obtained simply by expanding the left-hand side. By integrating this over \( y \) and \( z \), neglecting the divergence terms since \( K \) and \( \gamma \) vanish on the boundary and rearranging, we obtain Eq. (24).
Appendix B

Proof of the Variational Principle (29)

The variational principle can first be written in the form

$$
\delta \int \left[ \frac{T}{T^*} \left( \frac{\partial}{\partial y} (\delta * T^*) + \frac{\partial}{\partial z} (\delta * T^*) \right) + \Gamma \delta \left( \frac{\partial T}{\partial z} + 2\Gamma \right) \right] + 
+ \frac{1}{2T^2} \left( K \frac{\partial T}{\partial y} + K_y \left( \frac{\partial T}{\partial z} + 2\Gamma \right) \right) \frac{\partial T}{\partial y} + \frac{1}{2T^2} \left( K \frac{\partial T}{\partial z} + K_y \left( \frac{\partial T}{\partial z} + 2\Gamma \right) \right) \frac{\partial T}{\partial z} - 
- \frac{T}{T^*} \left( K^* \frac{\partial T^*}{\partial y} \right)^2 + K^* \frac{\partial T^*}{\partial z} \frac{\partial T^*}{\partial y} - 2K^* \frac{\partial T^*}{\partial y} \frac{\partial T^*}{\partial z} - 
- \frac{T}{2T^*} \left( \frac{\partial K^*}{\partial y} \left( \frac{\partial T^*}{\partial y} \right) + \frac{\partial (K^* \gamma^*)}{\partial T^*} \left( \frac{\partial T^*}{\partial z} \right)^2 + 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \frac{\partial T^*}{\partial y} \frac{\partial T^*}{\partial z} + 
+ 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \Gamma \frac{\partial T^*}{\partial y} + 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \Gamma \frac{\partial T^*}{\partial z} \right) + 
+ \frac{T}{2T^*} \left( \frac{\partial K^*}{\partial y} \left( \frac{\partial T^*}{\partial y} \right) + \frac{\partial (K^* \gamma^*)}{\partial T^*} \left( \frac{\partial T^*}{\partial z} \right)^2 + 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \frac{\partial T^*}{\partial y} \frac{\partial T^*}{\partial z} + 
+ 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \Gamma \frac{\partial T^*}{\partial y} + 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \Gamma \frac{\partial T^*}{\partial z} \right) + 
+ \frac{T}{2T^*} \left( \frac{\partial K^*}{\partial y} \left( \frac{\partial T^*}{\partial y} \right) + \frac{\partial (K^* \gamma^*)}{\partial T^*} \left( \frac{\partial T^*}{\partial z} \right)^2 + 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \frac{\partial T^*}{\partial y} \frac{\partial T^*}{\partial z} + 
+ 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \Gamma \frac{\partial T^*}{\partial y} + 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \Gamma \frac{\partial T^*}{\partial z} \right) \right] dy dz = 0.
$$

By varying $T$ it is easily shown that the Euler-Lagrange equation is Eq. (4) with the eddy heat fluxes parametrized as in Eq. (17). Substituting for the first term in square brackets from Eq. (4) we have after a little rearrangement and the neglect of divergence terms

$$
\delta \int \left[ \frac{1}{2T^*} \left( K \frac{\partial T}{\partial y} + K_y \left( \frac{\partial T}{\partial z} + 2\Gamma \right) \right) \frac{\partial T}{\partial y} + \frac{1}{2T^*} \left( K \frac{\partial T}{\partial z} + K_y \left( \frac{\partial T}{\partial z} + 2\Gamma \right) \right) \frac{\partial T}{\partial z} - 
- \frac{T}{T^*} \left( K^* \frac{\partial T^*}{\partial y} \right)^2 + K^* \frac{\partial T^*}{\partial z} \frac{\partial T^*}{\partial y} - 2K^* \frac{\partial T^*}{\partial y} \frac{\partial T^*}{\partial z} - 
- \frac{T}{2T^*} \left( \frac{\partial K^*}{\partial y} \left( \frac{\partial T^*}{\partial y} \right) + \frac{\partial (K^* \gamma^*)}{\partial T^*} \left( \frac{\partial T^*}{\partial z} \right)^2 + 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \frac{\partial T^*}{\partial y} \frac{\partial T^*}{\partial z} + 
+ 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \Gamma \frac{\partial T^*}{\partial y} + 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \Gamma \frac{\partial T^*}{\partial z} \right) + 
+ \frac{T}{2T^*} \left( \frac{\partial K^*}{\partial y} \left( \frac{\partial T^*}{\partial y} \right) + \frac{\partial (K^* \gamma^*)}{\partial T^*} \left( \frac{\partial T^*}{\partial z} \right)^2 + 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \frac{\partial T^*}{\partial y} \frac{\partial T^*}{\partial z} + 
+ 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \Gamma \frac{\partial T^*}{\partial y} + 2 \frac{\partial (K^* \gamma^*)}{\partial T^*} \Gamma \frac{\partial T^*}{\partial z} \right) \right] dy dz = 0.
$$
\[ + 2\left\{ \frac{\partial (K^* \gamma^e)}{\partial T^*} + \frac{\partial T}{\partial y} \frac{\partial (K^* \gamma^e)}{\partial T^*} + \left( \frac{\partial T}{\partial y} + \Gamma \right) \frac{\partial (K^* \gamma^e)}{\partial z} \right\} \left( \frac{\partial T}{\partial y} + \Gamma \right) \frac{\partial T^*}{\partial z} \right\} \right\} dydz = 0. \quad (B1) \]

We now need the generalization of Eq. (24) to the present case where the diffusivities are not constant. It is

\[
\begin{align*}
\int \int \left[ \left( \frac{T}{T^*} \frac{\partial T}{\partial y} - \frac{1}{T^*} \frac{\partial T^*}{\partial y} \right) \left\{ K^* \frac{\partial T^*}{\partial y} + K^* \gamma^e \left( \frac{\partial T^*}{\partial z} + \Gamma \right) \right\} + \\
+ \left( \frac{T}{T^*} \frac{\partial T}{\partial z} - \frac{1}{T^*} \frac{\partial T^*}{\partial z} \right) \left\{ K^* \gamma^e \frac{\partial T^*}{\partial y} + K^* \gamma^e \left( \frac{\partial T^*}{\partial z} + \Gamma \right) \right\} \right] dydz \\
= \int \int \frac{1}{2T^*} \left\{ (K - K^*) \frac{\partial T^*}{\partial y} + (K \gamma - K^* \gamma^e) \left( \frac{\partial T^*}{\partial z} + \Gamma \right) \right\} \frac{\partial T}{\partial y} + \\
+ \frac{1}{2T^*} \left\{ (K \gamma - K^* \gamma^e) \frac{\partial T^*}{\partial y} + (K \gamma^2 - K^* \gamma^e) \left( \frac{\partial T^*}{\partial z} + \Gamma \right) \right\} \frac{\partial T}{\partial z} - \\
- \frac{1}{2T^*} \left\{ \frac{\partial}{\partial y} \left( K \frac{\partial T}{\partial y} + K \gamma \left( \frac{\partial T}{\partial z} + \Gamma \right) \right) - \frac{\partial}{\partial y} \left( K^* \frac{\partial T^*}{\partial y} + K^* \gamma^e \left( \frac{\partial T^*}{\partial z} + \Gamma \right) \right) \right\} \\
- \frac{1}{2T^*} \left\{ \frac{\partial}{\partial z} \left( K \gamma \frac{\partial T}{\partial y} + K \gamma^2 \left( \frac{\partial T}{\partial z} + \Gamma \right) \right) - \frac{\partial}{\partial z} \left( K^* \gamma^e \frac{\partial T^*}{\partial y} + K^* \gamma^e \left( \frac{\partial T^*}{\partial z} + \Gamma \right) \right) \right\} + \\
+ \frac{(T - T^*)}{2T^*} \left\{ \frac{\partial}{\partial y} \left( K^* \frac{\partial T^*}{\partial y} + K^* \gamma^e \left( \frac{\partial T^*}{\partial z} + \Gamma \right) \right) \right\} + \\
+ \frac{\partial}{\partial z} \left( K^* \gamma^e \frac{\partial T^*}{\partial y} + K^* \gamma^e \left( \frac{\partial T^*}{\partial z} + \Gamma \right) \right) \right\} dydz. \quad (B2) 
\end{align*}
\]

This is proved in the same way as for Eq. (24) but this time starting by considering

\[
\begin{align*}
\frac{\partial}{\partial y} \left( K \frac{\partial T}{\partial y} \right) + K \gamma \frac{\partial}{\partial z} \left( \frac{T}{T^*} \right) + K \gamma \Gamma \frac{\partial}{\partial y} \left( \frac{T}{T^*} \right) + \\
+ \frac{\partial}{\partial z} \left( K \gamma \frac{\partial T}{\partial y} \right) + K \gamma^2 \frac{\partial}{\partial z} \left( \frac{T}{T^*} \right) + K \gamma^2 \Gamma \frac{\partial}{\partial z} \left( \frac{T}{T^*} \right) - \\
- \frac{\partial}{\partial y} \left( K^* \frac{\partial T^*}{\partial y} \right) + K^* \gamma^e \frac{\partial}{\partial z} \left( \frac{T^*}{T^*} \right) + K^* \gamma^e \frac{\partial}{\partial z} \left( \frac{T^*}{T^*} \right) - \\
- \frac{\partial}{\partial z} \left( K^* \gamma^e \frac{\partial T^*}{\partial y} \right) + K^* \gamma^e \left( \frac{\partial T^*}{\partial z} + \Gamma \right) \right\} \right\}.
\end{align*}
\]

We can substitute for the second, third, fourth and fifth terms in curly brackets in Eq. (B1) using Eq. (B2) (except for some terms involving \( \Gamma \)). If we make \( \Delta T / T^* \) small and neglect \( (\Delta T / T^*)^2 \) we then find that the first two terms in curly brackets on the right-hand side of Eq. (B2) cancel with the final term in curly brackets in Eq. (B1). This is because \( K \) satisfies
\[ \bar{\mathcal{T}} \frac{\partial K}{\partial \bar{T}} + \frac{\partial \bar{T}}{\partial y} \frac{\partial K}{\partial \bar{T}_y} + \left( \frac{\partial \bar{T}}{\partial z} + \Gamma \right) \frac{\partial K}{\partial \bar{T}_z} = 0 \]

with similar expressions with \( K_y \) and \( K_y^2 \) replacing \( K \). This is a special feature of this particular parametrization. Finally we note that the last three terms in square brackets in Eq. (B2) are the variations in the non-eddy contributions to the entropy production if \( \Delta \bar{T} \) is small. Using this fact, Eq. (B1) becomes the variational principle (29).

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