Airflow over hills of moderate slope

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(Received 8 July 1981; revised 19 January 1982)

SUMMARY

A three-layer model is presented. This describes the flow on an inversion capped boundary layer over hills of moderate size and slope. The linearized vorticity equation is solved using non-linear boundary conditions. The pressure field developed in the boundary layer is used to calculate a modified inner region velocity field (Jackson and Hunt 1975).

A low level inversion with height $l \lesssim 2h$ is found to decrease the speed-up near the ground. When $l \gtrsim 2h$ the flow of the inversion layer shows an increasing tendency to be subcritical; this leads to a much increased speed-up. Increasing the stability of air above the inversion increases the asymmetry of the flow. This causes an upwind lull and a subsequent rapid increase in the flow speed with very high wind speeds at the summit and downwind.

Comparison of the theory with simple observations made on Great Dun Fell, Cumbria shows encouraging agreement.

I. INTRODUCTION

There has been much recent investigation (both theoretical and experimental), of the flow of a neutrally stable boundary layer over hills. The linear theory of Jackson and Hunt (1975) and the numerical model of Taylor (1977) have been found to predict mean velocities well, except in the wake, even for hills of moderate slope (e.g. Mason and Sykes 1979; Bradley 1980; Britter et al. 1981). However, no previous attempt has been made to assess the effect of an elevated inversion on the neutral boundary layer as it flows over a hill. For hills of moderate size and slope such an inversion may be expected to have a considerable influence on the boundary layer flow. In this paper we develop a three-layer semi-analytic model which describes the inviscid flow of an inversion capped boundary layer over a hill. The pressure field maintained in the lowest neutral layer is employed to recalculate the inner mean velocity field of Jackson and Hunt (1975).

The motivation for the model was provided by recent field experiments performed at the UMIST field research station on the summit of Great Dun Fell, Cumbria. These are described in detail in Blyth et al. (1980) and Baker et al. (1982). The experiments were performed to investigate the effect of the mixing-in of unsaturated air on clouds, and typically took place in the atmospheric conditions outlined above, the only difference being that the inversion capped a cloudy boundary layer. The model can be used to provide a dynamical framework in which the microphysical processes which occur during the mixing process can be investigated theoretically.

Scorer (1949) introduced a multilevel description of the atmosphere for flow over a small hill; however, it was not until 1959 that Scorer and Klieforth addressed themselves to the problem of large amplitude perturbations, the main difficulty being that the linear boundary conditions usually employed are only satisfied at the undisturbed position of the layer interfaces. These boundary conditions result in the lowest streamlines cutting through the hill. They are inadequate for estimating the displaced height of an inversion interface especially when the hill is as high or higher than the depth of the boundary layer. Scorer and Klieforth overcame the difficulty by applying the most stringent conditions possible at the undisturbed interface heights: they required $\zeta$ (streamline displacement) and $\partial \zeta/\partial z$ to be continuous and also $\partial^2 \zeta/\partial z^2$. However Fraser et al. (1973) used a different approach: they used the standard boundary conditions, but applied them at the displaced positions of the interfaces. In effect they solved linear equations using non-linear boundary conditions at the ground and at the interfaces. It is this procedure we adopt (Section 2), and whilst it leads to some error since the calculated Fourier transforms used in the solution are slowly
dependent on \( x \), we show that this is unimportant for the aspect ratio of the hill \( h/L \leq 0.3 \). Our use of the radiative condition in the non-linear system also introduces some error since there may be non-linear interactions of upward progressing modes (Klemp and Lilly 1977); however comparisons of our solution in the hydrostatic limit with the exact hydrostatic one-layer solution of Lilly and Klemp (1979) and with a linear solution show much closer agreement with the former solution. These comparisons are presented in Section 3.

The advantage of the three layer model described herein is that it can be used with comparable accuracy for all hill sizes for which rotational effects are not important (the half-length \( L \leq 50 \, \text{km} \)) and for which \( h/L \leq 0.3 \). Of particular importance for our calculations is its suitability for hill sizes in the range \( 10^2 \, \text{m} < L < 10^4 \, \text{m} \) where the Jackson and Hunt analysis is also applicable, but where the hydrostatic approximation is not valid. The calculation of the inner region velocities, using the pressure field developed in the three layer model, is described in Section 5.

2. The three-layer model

The equation solved (two-dimensional vorticity equation) is

\[
\nabla^2 \zeta + \mu^2 \zeta = 0
\]

(1)

where \( \zeta \) is the vertical displacement of a streamline from its upstream height above ground, \( \mu^2 = g(d\theta/dz)_0/u_0^2 \), \( u_0 \) is the upstream windspeed and \( (d\theta/dz)_0 \) the upstream potential temperature gradient. Throughout the paper we refer to the parameter \( \mu \) as the stability. This equation is identical to that employed by Fraser et al. (1973), and is obtained from that used by Scorer and Klieforth (1959) by requiring zero upwind shear. The equation is linear and hence analytically soluble if the upstream stability \( \mu \) is constant with height. To comply both with this condition and to simulate the particular atmospheric situations in which we are interested we divide the atmosphere into three layers as follows:

Layer 1. This is the lowest layer and contains the mechanical boundary layer, \( \mu = 0 \).
Layer 2. An inversion capping the lowest layer. In this layer \( \mu = \mu_2 \).
Layer 3. This extends from the top of the inversion throughout the remainder of the atmosphere and is slightly stable. In this layer \( \mu = \mu_3 \).

![Diagram](image)  

Figure 1. The flow regimes for the analysis. Layer 2 is an inversion above layer 1 which is neutral.
A sketch of the flow is shown in Fig. 1. Layer 2 is enclosed between the two interfaces I and J; the inner region (Jackson and Hunt 1975) is in the lowest part of layer 1. Whilst requiring \( u_0, (\partial \ln \theta / \partial z)_0 \) to be constant in a layer are strong restrictions, upper air soundings show that these conditions are often well satisfied. Further for small shear Klemp and Lilly (1978) show that taking the average wind velocity in each layer results in a good approximation to the flow.

The solution of Eq. (1) is best obtained by using Fourier transforms. The equation becomes:

\[
\frac{\partial^2 \tilde{\zeta}}{\partial z^2} + (\mu^2 - k^2) \tilde{\zeta} = 0 \quad . \quad . \quad . \quad (2)
\]

where \( \tilde{\zeta} = \int_{-\infty}^{\infty} \zeta(x, z) e^{ikx} dx \) is the Fourier transform of the displacement. The Fourier transforms are of the form:

\[
\tilde{\zeta}_j = A_j \exp(-\lambda_j z) + B_j \exp(\lambda_j z) \quad . \quad . \quad . \quad (3)
\]

where the suffix \( j = 1, 2, 3 \) denotes the layer,

\[
\lambda_j = -i(\mu_j^2 - k^2) \quad \mu_j^2 > k^2
\]

\[
= (k^2 - \mu_j^2) \quad \mu_j^2 < k^2 \quad . \quad . \quad . \quad (4)
\]

(a) Boundary conditions

(i) At the surface, streamlines follow the hill surface, hence

\[
A_1 e^{-k f(x)} + B_1 e^{k f(x)} = \tilde{f} \quad (\lambda_1 = k) \quad . \quad . \quad . \quad (5)
\]

where \( \tilde{f} \) is the Fourier transform of the hill function \( f(x) \). Note that the boundary condition is non-linear: we stipulate that the displacement at the surface is equal to the height of the hill, rather than the linear condition that the displacement at \( z = 0 \) is equal to the height of the hill. As mentioned in the introduction this leads to a dependence of the Fourier coefficients on \( x \); however we show in Section 3 that the errors incurred are small for \( h/L \leq 0.3 \).

(ii) Upper boundary condition: the requirement that \( \zeta \) does not tend to \( \infty \) as \( z \to \infty \) \( (k^2 > \mu_3^2) \) and the radiative condition \( (k^2 < \mu_3^2) \) give \( B_3 = 0 \quad (k > 0) \) and \( A_3 = 0 \quad (k < 0) \). We use the radiative condition since in layer 3 it allows energy to propagate upwards without any reflection. Observational support is provided for this by Lilly and Kennedy (1973). The condition has the disadvantage that it does not allow for any non-linear interactions of upward progressing modes which result from the non-linear surface condition; however the comparisons in Section 3 show that the errors incurred are small.

(iii) At the interfaces: continuity requires that \( \tilde{\zeta} \) and \( \partial \tilde{\zeta} / \partial z \) are matched at the displaced position of each interface which we assume at this stage to be known. Having obtained expressions for the Fourier integrals, we iterate onto the correct values of \( I, J \) as explained in Section 3. We consider only \( k > 0 \) since

\[
\text{Re} \int_{-\infty}^{\infty} \tilde{\zeta} e^{ikx} dk = 2 \text{Re} \int_{0}^{\infty} \tilde{\zeta} e^{ikx} dk \quad . \quad . \quad . \quad (6)
\]
(b) The three-layer solution

The boundary conditions lead to the following expressions for the coefficients $A_j, B_j$,

$$A_2 = \frac{2\tilde{f}}{\gamma} \left( 1 + \lambda_2/\lambda_3 \right) \exp \{ k f(x) - \lambda_2 I(x) \} \quad . \quad . \quad . \quad (7)$$

$$B_2 = \frac{2\tilde{f}}{\gamma} \left( 1 - \lambda_2/\lambda_3 \right) \exp \{ k f(x) - \lambda_2 I(x) - 2\lambda_2 J(x) \} \quad . \quad . \quad . \quad (8)$$

$$A_3 = \exp(\lambda_2 J) \{ A_2 \exp(\lambda_2 J) + B_2 \exp(-\lambda_2 J) \} \quad . \quad . \quad . \quad (9)$$

$$B_1 = \frac{1}{\beta} \left[ A_2 \exp \{-\lambda_2 I(x)\} + B_2 \exp \{\lambda_2 I(x)\} - \tilde{f} \exp k \{ f(x) - I(x) \} \right] \quad . \quad . \quad . \quad (10)$$

$$A_1 = \tilde{f} \exp \{ k f(x) \} - B_1 \exp \{ 2k f(x) \} \quad . \quad . \quad . \quad . \quad (11)$$

where

$$\gamma = (\alpha + \lambda_2 \beta/k)(1 + \lambda_2/\lambda_3) \exp \{-2\lambda_2 I(x)\} - \frac{(\alpha - \lambda_2 \beta/k)(1 - \lambda_2/\lambda_3) \exp \{-2\lambda_2 I(x)\}}{} . \quad . \quad . \quad (12)$$

$$\alpha = \exp \{ k I(x) \} + \exp k \{ 2f(x) - I(x) \} \quad \{ \quad . \quad . \quad (13)$$

$$\beta = \exp \{ k I(x) \} - \exp k \{ 2f(x) - I(x) \}$$

Fourier integral for the streamline displacements is:

$$\zeta_j(x, z) = \Re \frac{1}{\pi} \int_0^{\infty} \{ A_j \exp(-\lambda_j z) + B_j \exp(\lambda_j z) \} \exp(ikx) \, dk \quad . \quad . \quad . \quad (14)$$

Scorer (1949) has shown that a suitable change of stability with height leads to atmospheric lee waves; these are signified by the presence of a pole in the integrand. We find there can be a solution to $\gamma = 0$ and hence a pole for $k$ in the interval $\mu_3 < k < \mu_2$. Letting $\gamma = 0$ gives

$$\tan(\lambda_2(J - I)) = -Q/P \quad (\mu_3 < k < \mu_2) \quad . \quad . \quad . \quad (15)$$

where

$$P = \alpha - \lambda^2 \beta/\lambda_3 k, \quad Q = \lambda_2 \beta/k + \alpha \lambda_2/\lambda_3 .$$

When condition (15) is satisfied the integral (14) cannot be calculated directly; however a solution can be obtained by invoking contour integration. This is most simply achieved by rotating the integrand onto the imaginary $k$ axis, the direction of the rotation being determined by the requirement that $\zeta$ does not tend to $\infty$ as $z \to \infty$. We must make the following rotations:

(i) $x < 0, k > 0 : \quad k \to -ik$

(ii) $x > 0, k > 0 : \quad k \to +ik$

The Fourier integrals are now of the form

$$\zeta(x, z) = \Re \frac{1}{\pi} \int_0^{\infty} \{ A_j \exp(i \lambda_j z) + B_j \exp(-\lambda_j z) \} \exp(-k|z|) \, dz + 2iR \quad . \quad . \quad (16)$$
where \( R \) is the residue of the pole. Assuming slight time dependence or the presence of small friction forces Crapper (1959, 1962) has shown that any singularities on the real axis (except at \( x = 0 \)) are displaced into the upper half of the \( k \) plane. This implies that the residue of the poles contribute for \( x > 0 \); i.e. the waves occur in the lee of the hill. These are the only waves we consider. In common with most other workers we ignore any contribution from the branch points which occur in the solutions for layers 2 and 3. These are discussed for the case of a one-layer model by Crapper (1959); for most likely atmospheric conditions they lead to waves with amplitude \( A < h/20 \).

For convenience in the discussion of results we shall call the solution evaluated on the real \( k \) axis (no lee waves) solution \( A \), whilst the solution evaluated on the imaginary \( k \) axis is referred to as solution \( B \). \( B \) can be evaluated for all values of \( \mu_2, \mu_3 (\mu_2 > \mu_3) \), \( I, J \), whereas \( A \) is only employable for cases in which condition Eq. (15) is not satisfied. For cases in which \( A \) and \( B \) are both employable the two solutions are different algebraic representations of the same flow.

3. comparisons with other models

As a preface to the results of the three-layer model we compare the results of an analogous one-layer model, using the non-linear surface and linear radiative conditions already described, with other one-layer models. This will illustrate when our formulation of the boundary conditions accurately determines the flow.

The one-layer model in the hydrostatic limit \( (\mu L \gg 2\pi) \) was first compared with a one-layer hydrostatic model with linear boundary conditions. This is a solution of

\[
\frac{\partial^2 \zeta}{\partial x^2} + \mu^2 \zeta = 0 \quad . \quad . \quad . \quad (17)
\]

The two flow patterns over a bell-shaped hill, \( f(x) = h/\{1 + (x/L)^2\} \), are shown in Fig. 2(a);
the large difference is due entirely to the difference in the surface boundary condition which is accurately represented in our solution but badly modelled by the linear condition. Substitution of the calculated values of $\nabla^2 \zeta, \zeta$ into the vorticity equation showed that the $x$ dependence of the Fourier coefficient caused negligible error for the large value of $L (> 20 \text{ km})$ and consequently small $h/L (< 1/10)$.

A comparison of our one-layer model in the hydrostatic limit with the exact hydro-
static solution of Lilly and Klemp (1979) is shown in Fig. 2(b). Lilly and Klemp's solution uses both non-linear radiative and surface conditions, so that the difference between the two curves is caused by the upper condition. The linear radiative condition used in our solution does not take account of any non-linear upward propagating modes which arise as a result of the non-linear surface condition. Nevertheless the difference in the two solutions is small and these results are typical of a number of computations which used different values of \(\mu h\). In the hydrostatic limit our solution always showed much closer agreement with the exact solution than with the wholly linear solution.

As \(\mu L\) decreases non-hydrostatic features become more important and this is shown in Fig. 2(c). The solid line shows the solution of our one-layer model with \(\mu L = 2; h/L = 0.25\). The now non-negligible \(\partial^2 \zeta / \partial x^2\) acts to decrease the slope of the streamlines. However substituting computed values of \(\nabla^2 \zeta, \zeta\) into the vorticity equation shows that the error introduced by the \(x\) dependent Fourier coefficient is not negligible. We must therefore distinguish between non-hydrostatic effects and the error induced by the boundary condition. We find that the ratio

\[
R_n = |(\nabla^2 \zeta + \mu^2 \zeta) / \mu^2 \zeta|_{\text{max}} \approx 0.1
\]

the suffix \(n\) being used to denote our non-hydrostatic solution. However the same ratio using the exact hydrostatic solution and the above values of \(\mu L, h/L\) gives

\[
R_h \approx 0.4
\]

where the suffix \(h\) is used to denote a hydrostatic solution. It therefore follows that most of the difference between the two flow patterns is non-hydrostatic with the error in our model causing a smaller change.

The error ratios \(R_n, R_h\) can be related to the error resulting in the streamline displacements by the use of a hydrostatic one-layer model with a non-linear surface boundary condition and a linear radiative condition (Lilly and Klemp, 1979). In the hydrostatic limit, this agrees exactly with our one-layer model. We find for \(\mu L \approx \pi, h/L \approx 0.1\) that \(R_h\) is significant (\(> 0.01\)) and rapidly changing with \(\mu L\) while \(R_n (\approx 0.001)\) is almost unchanged from its value in the hydrostatic limit (the error ratio is always non-zero due to errors in the numerical integration). This means that non-hydrostatic effects occur when the error in our solution is unimportant, and enables us to relate the ratio \(R_h\) and so \(R_n\) to changes in the streamline displacement. When \(R_h \approx 0.1\) the magnitude of the differences between our solution and the hydrostatic solution are similar to those shown in Fig. 2(b). Taking this to be an acceptable error requires that \(R_n \leq 0.1\). The computations show that this requires \(h/L \leq 0.3\); we use this approximate upper bound for the aspect ratio throughout the remainder of the paper. In effect this limits the error caused by the lower boundary condition to a value which is equal to or less than that incurred by our linear upper condition.

4. RESULTS OF THE THREE-LAYER MODEL

The evaluation of the integrals for the streamline displacements presupposes a knowledge of the displaced position of the interfaces \(I(x), J(x)\). These are given by solutions of

\[
I(x) = \zeta\{x, I(x)\} + I(-\infty)
\]

\[
J(x) = \zeta\{x, J(x)\} + J(-\infty)
\]

where \(I(-\infty), J(-\infty)\) are the upwind interface heights. Solutions of these equations are obtained using a double iteration procedure similar to that described by Fraser et al.
(1973): we iterate to $I(x)$, given a first estimate of $J(x)$, then iterate to $J(x)$ using the first calculated $I(x)$ value, and so on until Eqs. (20), (21) are satisfied. Using a 16-point Gaussian quadrature and the regula falsi iteration method convergence is usually rapid. The integrations were sometimes checked using a Romberg extrapolation with $2^k$ points; the errors were always very small ($< 0.5\%$). Once the interface positions have been established streamlines are calculated using a series of single iterations onto $z_i = \xi(z_i, x) + z_f(\infty)$; where $z_f(\infty)$ is the initial height of streamline $r$. Horizontal and vertical velocities are given by
\[
  u = u_0 (1 - \partial \xi / \partial z), \quad w = u_0 \partial \xi / \partial x. \quad (22)
\]

In the results which follow the hill function used in all cases is $f(x) = h/\{1 + (x/L)^2\}$. Firstly, as a check, solutions $A$ and $B$ were compared in situations in which both could be used. Accurate numerical integrations should yield identical results and good agreement was found except for small $x (-L/2 < x < L/2)$ where the numerical calculation of $B$ became inaccurate since the integrand decreases very slowly with increasing $k$ (at $x = 0$ solution $B$ is undefined). For these values streamlines are drawn in by hand when solution $B$ is employed. Solution $A$ does not suffer from this disadvantage; it is preferred when both $A$ and $B$ can be employed.

We present four specimen flow fields using $L = 2000 \text{m}$, $h = 665$; these illustrate the different types of flow encountered. Figure 3 shows the displacement of a low level inversion, the upwind interface height ($I(\infty) = 400 \text{m}$) is less than the height of the hill. The

![Figure 3. Calculated streamlines for three-layer flow over a hill. The inversion layer is shaded. $I(\infty) = 400 \text{m}$, $J(\infty) = 800 \text{m}$, $\mu_2 = 2.0 \times 10^{-3} \text{m}^{-1}$, $\mu_3 = 1.0 \times 10^{-3} \text{m}^{-1}$.](image)

stability of the inversion layer $\mu_2 = 2.0 \times 10^{-3} \text{m}^{-1}$ is typical of many low level inversions observed. The lee wave condition Eq. (15) is not satisfied so solution $A$ is used. The inversion bows up windward of the hill and then drops quickly below its equilibrium level on the lee side. The general features of the flow above the inversion are similar to those of, for example, Raymond (1972), with regions of 'jetting' separated by relatively stagnant zones. In Fig. 4 the displacement of a much thicker, more intense inversion is illustrated. The rapid descent down the leeside is even more noticeable. This is followed by a steep rise or jump in which the streamlines return to near their equilibrium positions; condition Eq. (15)
Figure 4. Calculated streamlines. \( I(-\infty) = 400 \text{ m}, J(-\infty) = 1500 \text{ m}, \mu_2 = 2.4 \times 10^{-2} \text{ m}^{-1}, \quad \mu_3 = 1.0 \times 10^{-2} \text{ m}^{-1}. \)

Figure 5. Calculated streamlines. \( I(-\infty) = 1200 \text{ m}, J(-\infty) = 1600 \text{ m}, \mu_2 = 3.4 \times 10^{-3} \text{ m}^{-1}, \quad \mu_3 = 2.8 \times 10^{-3} \text{ m}^{-1}. \)

is satisfied and lee waves appear downstream of the jump. In Fig. 5 the combination of very high stabilities and high upwind inversion height results in only very small perturbations to the interfaces as they flow over the hill; there are no lee waves. Note that the streamlines in the inversion are displaced for the most part below their equilibrium levels; this is termed subcritical flow. Flow patterns in which the displacements of the inversion is mostly above its equilibrium level are labelled supercritical (e.g. Fig. 3). Figure 6 shows the result of an iteration onto the interface heights in which there was convergence onto a double-valued solution. This is much more difficult to interpret than the previous examples; in Fig. 6 the actual interface heights are clearly unrealistic. We believe that the non-uniqueness of the
Figure 6. Example of a non-unique solution. Calculated interfaces.

$I(-\infty) = 400 \text{ m}, I(-\infty) = 800 \text{ m}, \mu_2 = 3 \times 10^{-3} \text{ m}^{-1}, \mu_3 = 1.2 \times 10^{-3} \text{ m}^{-1}$.

Figure 7. Flow regimes as a function of $\mu_2L$, $\mu_3L$.

(a) $h/L = 0.33$, $h/I(-\infty) = 1.66$, $h/I(-\infty) = 0.83$.
flow signifies a time dependence with hydraulic jumps propagating up and down-stream from close to the summit. Computations showed that these non-unique solutions occurred in the hydrostatic limit; comparisons with time-dependent hydrostatic models are therefore used to support our hypothesis. These used two layers separated by a density discontinuity to represent an inversion and whilst we therefore expect no exact similarity we find that the occurrence of time dependent solutions is governed by the relevant Froude numbers in a way which is analogous to the effect of the Froude numbers of the three-layer flow on the occurrence of non-unique solutions in that flow. The shallow water model (e.g. Lamb and Britter 1981) exhibits a time dependent solution with propagating jumps when the Froude number \( F = u_0/\sqrt{g'h} < 1 \). \( g' \) is the modified gravity and \( h \) the local height of the interface above the surface. The analogous Froude number in the three-layer flow is \( F_2 \), viz. \( 1/[\mu_2 (I(x) - f(x))] \), but \( F_3 = 1/[\mu_3 (J(x) - f(x))] \) is also important. We find for a given \( F_3 > F_3(\text{crit}) \) that decreasing \( F_2 \) sufficiently will always produce a non-unique solution; we see an analogy here with the shallow water flow if we assume that non-uniqueness signifies a time-dependence. In three-layer flow \( F_3 \) complicates the picture; however the numerical model of Klemp and Lilly (1975) helps to assess its effect. Unlike the shallow water model their formulation allows the upper-layer to be stable; the relevant Froude numbers are \( F \) and \( F_3 \). Considering only \( F = 1 \) Klemp and Lilly show that allowing \( F_3 \) to be finite \( (\mu_3 \neq 0) \) inhibits time dependence (jump formation) and remembering that increasing \( F_3 \) in the three-layer solution inhibits non-uniqueness the analogy is further supported.
Five dimensionless numbers $h/L$, $h/I$, $h/J$, $\mu_2 L$, $\mu_3 L$ characterize the flow. Because of the consequent complexity it is therefore difficult to relate any changes in the flow regime to a particular dimensionless number. We attempt this by classifying the flow into different types as a function of the inverse Froude numbers $\mu_2 L$, $\mu_3 L$ for three different sets of $h/I$, $h/J$. We keep $h/L = 0.33$. The three Figs. 7(a, b, c) show the values of $\mu_2 L$, $\mu_3 L$ when each of the flow types shown in Figs. 3–6 occur. No computations have been performed for $\mu_2 < \mu_3$ (this region is hatched) since our interest has been to assess the effect of the inversion on the flow. The dashed lines divide (i) the regions of well-behaved flow in which the iterations converged satisfactorily from (ii) the region in which there was no unique solution. Flows with stabilities in the region above the solid lines satisfy the lee wave condition Eq. (15). The lee waves are of large amplitude for $k^*L \sim 0(1)$, where $k^*$ is the solution of Eq. (15). For other values of $k^*L$ the waves are insignificant. In Fig. 7(a) the depth of the inversion is small and the initial height of the lower interface is below the level of the hill. Many combinations of stabilities lead to well-behaved super-critical flow, while the region of lee wave activity without non-unique convergence is small. By contrast Fig. 7(b) is dominated by the lee wave region. In this case the inversion is much deeper. Iterations showed that when $k^*L \sim 0(1)$ the lee waves were of large amplitude ($\approx h$) and stationary lee jumps were a frequent occurrence. In Fig. 7(c) the inversion is narrow with $I(-\infty) > h$. The most noteworthy feature is the transition from supercritical to subcritical flow types which occurs as $\mu_3 L$ is increased.
It is of interest to assess the relative importance of the height, thickness and stability of the inversion layer. Looking at Figs. 7(a), (b) and (c) together and using the lee wave condition Eq. (15) we find that the stability and thickness are of comparable importance in determining whether the solution is single-valued (increasing stability and thickness have opposite effects) and also whether lee waves occur. The height of the inversion is most important in determining whether the well-behaved flow is super- or subcritical. It has little effect on the occurrence of lee waves or non-unique flow.

5. THE INNER REGION

As outlined in the introduction, the theory of Jackson and Hunt (1975) (subsequently referred to as JH) for the turbulent flow of neutral air over a hill has been found to work well even when the conditions for the analysis \((h/L < 0.05, 10^2 \text{ m} < L < 10^4 \text{ m})\) are not well satisfied. Hunt (1980) suggests that the theory is useful for \(h/L \leq 0.4\). Prompted by the success of the theory for flow over these hills of moderate slope, we present an analysis which calculates the velocities in the turbulent inner region when the neutral boundary layer is capped by an inversion. The results of Section 4 suggest that such an inversion may seriously distort the structure of the boundary layer flow. The essential steps of the analysis are as follows: the displaced positions of the inversion interfaces are first calculated using the three-layer model described in Section 2. The pressure field maintained in the neutral lowest layer (layer 1) is then calculated using the previously calculated interface heights as boundary conditions, and a linearized expression relating the streamline displacement to the pressure (as in Section 2 we solve linear equations using a non-linear surface boundary condition). This calculated pressure field provides the only new boundary condition for the JH inner layer velocity field, all other boundary conditions are as described in JH.

\(\text{a)}\) The form of the solution

First the pressure field in layer 1 is calculated, assuming, for the moment, that the flow is inviscid. The linearized horizontal momentum equation is

\[
\frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial (\Delta p)}{\partial x} \quad . \quad . \quad . \quad (23)
\]

where \(u_0\) is the upstream velocity (uniform) and \(\Delta p\) is the pressure perturbation. Taking Fourier transforms and using \(u(x,z) = u_0(1 - \partial \zeta/\partial z)\) we obtain

\[
\rho u_0^2 \frac{\partial \zeta}{\partial z} = \tilde{\Delta} p \quad . \quad . \quad . \quad . \quad . \quad (24)
\]

It is known from JH that, over a hill, the magnitude of the pressure perturbation in turbulent airflow is to first order the same as that in an inviscid flow, when the upstream uniform velocity of the inviscid flow is approximately equal to the velocity in the upstream turbulent flow at height \(L\). Hence the pressure calculated in Eq. (24) above is to first order equal to that developed in a turbulent boundary layer with parameters satisfying

\[
u_0 = \frac{u^*}{\kappa} \ln \left( \frac{L}{z_0} \right) \quad . \quad . \quad . \quad . \quad (25)
\]

where it is now assumed that \(u(z) = (u^*/x) \ln(z/z_0)\) is the upwind profile in the boundary layer. \(u^*\) is the upwind surface friction velocity, \(\kappa\) von Karman's constant and \(z_0\) the surface roughness length. Then
\[ \tilde{\Delta p} = \frac{\rho u^*}{k^2} \left\{ \ln \left( \frac{L}{z_0} \right) \right\}^2 \tilde{\mathbf{f}}(x,z) \]  
(26)

where \( P(x,z) \) is as in JH. Hence from Eqs. (24), (25), (26)

\[ \tilde{P} = \frac{\partial \tilde{\mathbf{f}}}{\partial z} \]  
(27)

and from Eqs. (6), (3), (27)

\[ \tilde{P} = k(A_1 \left[ e^{kx} + e^{i\left[ k(x) - z \right]} \right] - \tilde{f} e^{i\left[ k(x) - z \right]}), \]  
(28)

where \( A_1 \) is defined in Eq. (11). Finally, to obtain the JH inner region pressure transform \( \tilde{p}^T \) we again use the non-linear surface boundary condition. Thus

\[ \text{as } z \to f(x), \quad \tilde{P} \rightarrow \tilde{p}^T = k(2A_1 e^{k(x)} - \tilde{f}). \]  
(29)

This expression enables the inner region velocity field to be calculated. The dimensionless horizontal perturbation velocity is

\[ \hat{u} = \text{Re} \int_0^\infty k \{ \tilde{f} - 2A_1 e^{k(x)} \} \left\{ \frac{C_r^2 + C_i^2 - C_r B_r - C_i B_i}{C_r^2 + C_i^2} - i \left( \frac{C_r B_i - B_r C_i}{C_r^2 + C_i^2} \right) \right\} e^{i\alpha k} dk \]  
(30)

where \( C_r, C_i, B_r, B_i \) are defined in JH; the expression for \( \tilde{p}^T \) is similar. Note that for a neutral atmosphere (\( \mu_2 = \mu_3 = 0 \)) \( A_1 = 0 \) and the pressure transform is identical to that obtained using linear boundary conditions throughout the analysis for this special case.

(b) Conditions and assumptions

Before describing the results of the inner region calculations it is necessary to make the conditions on their applicability and the intrinsic assumptions.

We are restricted to atmospheric situations in which solution \( \tilde{\mathbf{A}} \) of the three-layer model is applicable; the integrand of Eq. (30) or similar cannot be rotated onto the imaginary \( k \) axis because \( C_r \) and \( C_i \) have singularities there. This requires either that no lee wave singularities occur or that any singularities which are present occur at unimportant (large) values of \( k \) where the value of \( \tilde{p}^T \) is negligible.

There is clearly a minimum value of the lower interface \( I \) for which the solution is realistic. Following JH we assume that the inner region does not extend above the constant stress region of the atmospheric boundary layer. Assuming this region has height \( l_i \) and that \( I \leq 5l_i \), then

\[ I \geq 5l_i \]  
(31)

where \( l_i \) is the depth of the inner layer. We do not consider cases in which \( I < 5l_i \).

A further condition is that the solution should not be employed at or downwind of separation. When this occurs the estimate of \( I \) becomes inaccurate, the actual depth of the inner region varying rapidly downstream of a separation point (Hunt 1980). This point is mentioned here since the bowing up of the interface described in Section 4 makes upwind separation likely for a wide range of initial conditions.

An important assumption is that the inclusion of shear in the boundary layer (layer \( 1, z - f(x) < l(x) \)) does not affect the flow in the bulk of the atmosphere. The analysis of JH shows that the inner region \( (z - f(x) < l) \) velocity perturbations are driven by the pressure perturbation in the higher flow but that this is only affected to second order at
Figure 8. (a) Interface heights, (b) speed-up $\Delta S$($dz = 1.0$ m) and
A $\mu_s = 0.7 \times 10^{-3}$ m$^{-1}$
B $\mu_s = 1.0 \times 10^{-3}$ m$^{-1}$
C $\mu_s = 1.2 \times 10^{-3}$ m$^{-1}$
N neutral atmosphere ($\mu_s = \mu_a = 0$)
most by the inner region perturbations. JH shows further that the shear in the outer region 
\((l < 2 - f(x)) < I(x))\) has only a second order effect on the perturbations to the flow in that region.

\[(c) \text{ Results for the inner region}\]

The hill profile used is \(f(x) = h/(1 + (x/L)^2)\); with \(h = 665\) m and \(L = 2000\) m; this is a good representation of Great Dun Fell especially upwind of the summit. These lengths give the depth of the inner region \(l = 80\) m and we use \(z_0 = 0.2\) m (the observed value at Great Dun Fell). Figures 8(a), (b), (c) show different features of the same flows. In each of the flows \(A, B, C\) the upwind interface heights are \(I(\infty) = 400\) m, \(J(\infty) = 800\) m and \(\mu_2 = 1.5 \times 10^{-3}\) m\(^{-1}\) (typical of many three-layer situations). The effect of varying the upper layer stability is illustrated. Figure 8(a) shows the positions of the \(I, J\) interfaces and the corresponding streamlines for neutral flow \((N)\) are also shown for comparison. In Fig. 8(b) the speed-up defined as \(\Delta S = \Delta y(x, \Delta z)/\gamma(\Delta z)\) is plotted as a function of \(x\), \(\Delta y\) is the velocity perturbation at height \(\Delta z\) above the ground and \(\gamma(\Delta z)\) is the upwind velocity. We use \(\Delta z = 1\) m but \(\Delta S\) is insensitive to the value employed. As shown in JH, introducing the strongly sheared inner region approximately doubles the speed-up obtained using inviscid theory. It is seen that the speed-up increases with increasing \(\mu_2\) for all \(x\) although the increases are most marked above the summit and downwind. Also noticeable is the upwind lull which
develops even for case C ($\mu_2 = 1.2 \times 10^{-3} \text{m}^{-1}$). This is clearly caused by the positive pressure gradients (Fig. 8(c)) which occur well upwind of the hill before the very high negative pressure gradients which in the case of C and B cause the wind to speed-up much more rapidly than in the neutral case (N) as it approaches the summit.

Figure 9 shows the variation of the speed-up with mid-layer stability ($\mu_2$) for constant $\mu_3$. The upwind interface heights in all cases are $I(-\infty) = 400 \text{m}$, $J(-\infty) = 800 \text{m}$. Increasing the mid-layer stability decreases the speed-up except far downwind. In all cases the speed-up is less than that obtained for a totally neutral atmosphere.

Figure 10 shows the effect on the speed-up of varying the height of the inversion above ground. $\Delta S$ increases rapidly for $I \leq 1200 \text{m} (\sim 2h)$ as the inversion flow, initially supercritical becomes subcritical as $I$ increases. For $I \geq 1200 \text{m}$ raising the interface only slowly alters the flow; the speed-up is still greater than that for a neutral atmosphere when $I \sim 2400 \text{m} (\sim 4h)$.

Finally we attempt to generalize the results. Figs. 11(a), (b) show trends of the scaled speed-up at the summit $\Delta S/(h/L)$ with $\mu_2 L$ and $\mu_3 L$ for two different inversion heights. Since exact similarity requires that the six dimensionless numbers $\mu_2 L$, $\mu_3 L$, $h/L$, $h/I$, $h/J$, $z_0/L$ remain constant no exact relationships are to be expected; however the computations did show that $\Delta S/(h/L)$ is very sensitive to both $\mu_2 L$, $\mu_3 L$. The vertical bars show the error in the curves: these were obtained by computing the variability in $\Delta S/(h/L)$ for different values of $\mu_2$, $\mu_3$, $L$ whilst keeping $\mu_2 L$, $\mu_3 L$ constant. The allowed ranges of the variables are $5 \times 10^3 \text{m} < L < 10^2 \text{m}$, $h/L < 0.33$, $\mu_2 \leq 2.0 \times 10^{-3} \text{m}^{-1}$, $\mu_3 \leq 1.2 \times 10^{-3} \text{m}^{-1}$ and
$z_0 \approx 2 \text{ cm}$. The lee criterion Eq. (15) must not be satisfied. Figure 11(a) shows the trends for a low inversion. For low values of $\mu_3L$ increasing $\mu_2L$ rapidly decreases the speed-up whilst for higher values of $\mu_3L$ the reverse is true. Trends are not shown for $\mu_3L > 2.0$ as the errors become prohibitively large. In Fig. 11(b) the inversion is higher. (The error bars are not shown but are similar to those of Fig. 11(a).) The speed-ups are also greater, but less sensitive to $\mu_3L$. The two graphs have been calculated for specific values of $I, J$. However requiring that $(\mu_2 - \mu_3)(J - I), I$ and $\mu_3$ remain constant ensures that the vertical wavelength of the system remains constant and gives approximate similarity; and this enables us to estimate the speed-ups for different inversion thicknesses using Figs 11(a), (b). Computations showed that this is useful in the range $200 \text{ m} < (J - I) < 600 \text{ m}$.

6. Comparison with Observations

The predicted variability in the speed-up $\Delta S$ as the atmospheric conditions vary is large and therefore should be observable for hills of moderate size. In accordance with the theory Bradley (1980) observed that the presence of an inversion could both increase or decrease the speed-up. Crude observations made on the Great Dun Fell ridge (GDF) give
Figure 11. The variation of the scaled speed-up $L \Delta S/h$ at the hill summit with $\mu_3 L$ and $\mu_3 L$.

(a) $H(-\infty) = 400 \text{ m}, J(-\infty) = 800 \text{ m}$
(b) $H(-\infty) = 1200 \text{ m}, J(-\infty) = 1600 \text{ m}$.

The values of $\mu_3 L$ are marked on the curves. The vertical lines are error bars and the crosses denote $L dS/h$ for a neutral atmosphere.
TABLE 1(a)

<table>
<thead>
<tr>
<th>Date</th>
<th>( I(\infty) ) (m)</th>
<th>( J(\infty) ) (m)</th>
<th>( \mu_2 \times 10^4 ) (m/s)</th>
<th>( \mu_3 \times 10^4 ) (m/s)</th>
<th>10 m wind (m/s)</th>
<th>( \Delta S ) (z = 1000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3/6/80</td>
<td>500</td>
<td>900</td>
<td>2.0</td>
<td>1.0</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td>12/12/80</td>
<td>2310</td>
<td>2810</td>
<td>1.0</td>
<td>0.47</td>
<td>24</td>
</tr>
<tr>
<td>C</td>
<td>3/6/81</td>
<td>1840</td>
<td>3680</td>
<td>0.8</td>
<td>0.5</td>
<td>16</td>
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</tbody>
</table>

TABLE 1(b)

<table>
<thead>
<tr>
<th>Date</th>
<th>( u(x = 0)/u(x = 1000) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Observed</td>
</tr>
<tr>
<td>A</td>
<td>3/6/80</td>
</tr>
<tr>
<td>B</td>
<td>12/12/80</td>
</tr>
<tr>
<td>C</td>
<td>3/6/81</td>
</tr>
</tbody>
</table>

Comparison of observed and predicted wind velocities and speed-ups \( \Delta S \) on Great Dun Fell. (a) 10 m wind at summit. \( \Delta S = 0.76 \) for a neutral atmosphere. (b) Ratio of 2 m winds at the summit and 1000 m upwind. For a neutral atmosphere \( u(x = 0)/u(x = -1000) = 1.23 \).

encouraging results (Tables 1(a), (b)). The 10 m winds were averaged values of the anemograph record measured on the tower at the summit. Winds at 2 m were estimated using a hand-held anemometer to give three minute averages. Care was taken to choose well exposed representative sites; several three minute samples were taken at different locations roughly 100 m apart to check for any local effects. Further, the anemograph at the summit was used to check for any mesoscale or larger scale windspeed variations when anemometer measurements were being made. For the cases presented, hand held anemometer averages were probably accurate to about 10–15%. Input parameters used in the model and the geostrophic wind were estimated from radiosonde ascents made at Aughton and Long Kesh respectively 150 km SSW and 200 km W of GDF. The surface geostrophic wind speed was taken to be the wind speed measured at the 850 mb level. Discretion was used in the choice of ascent when there was a marked difference between them. However there can be considerable error in taking these distant soundings as representative of the atmospheric conditions in the vicinity of GDF. The hill profile used was \( f(x) = h/[1+(x/L)^2] \), \( L = 2000 \) m and \( h = 665 \) m, \( z_0 = 0.02 \) m, \( l = 80 \) m.

Table 1(a) presents the observed and calculated 10 m winds and speed-ups \( \Delta S \) for three days when the atmospheric conditions were very different. On each day the wind was from the SW (perpendicular to the GDF ridge). In case A the inversion was low and intense; the supercritical flow reduced the speed-up (for a neutral atmosphere \( \Delta S = 0.76 \) at the summit). In cases B, C deep neutral boundary layers were capped by slightly stable layers; in both cases the speed-up was increased and in C this is very marked, the flow being subcritical. Table 1(b) presents observed and calculated values of the ratios of 2 m windspeeds at the summit and at 1000 m (\( L/2 \)) upwind, for the same days as Table 1(a). In cases A, C agreement is good, but using the specified input data the model does not predict the large lull in case B. Nevertheless, considering Tables 1(a), (b) together the agreement between the calculations and observations is, in view of the crudity of the observations, surprisingly good, and gives support to the inner region theory presented.

7. Discussion

The theory predicts that an inversion capping a neutral boundary layer can influence the boundary layer flow markedly. For low level inversions (\( L \leq 2h \)) increasing the mid-
layer stability ($\mu_3$) generally increases the curvature of the inter-faces, effectively decreasing the speed of the boundary layer flow upwind and at the summit. Increasing the upper layer stability ($\mu_4$) increases the asymmetry of the flow causing an upwind lull, where separation may occur, and very strong winds downwind, where separation will be suppressed.

The model has a number of applications. As mentioned in the introduction it was originally conceived to provide a dynamical basis for microphysical calculations of cloud evolution at Great Dun Fell. However its ease of adaptability to hills of greatly differing size and shape will hopefully lead to its use at other locations.

Fraser et al. (1973) calculate that the effect of introducing cloud water into the flow is in most cases secondary to the effect of the hill. This suggests that our theory would give a good approximation to the airflow even when cloud is present, and points to the inadequacies of the present models of orographic rainfall. For example, Bader and Roach (1977) make the approximation that streamlines in the lower atmosphere ($Z \leq 1500 \text{ m}$) are parallel to the topography. While this may be a reasonable assumption in some cases of low stability, it is clearly very inaccurate for highly asymmetric flows. Further, our theory elucidates the failure of the orographic models to predict the correct sensitivity of the orographic enhancement of rainfall to the windspeed. The parameter $\mu = \left(\frac{g u_0^2 \partial \Phi}{\partial \varphi} \frac{\partial \Phi}{\partial z} \right)^{\frac{1}{3}}$ is most sensitive to windspeed and therefore, for a given potential temperature gradient, different windspeeds can give rise to markedly different flows. This is not allowed for in the orographic models.

Using the analysis of Crapper (1959) the theory has been extended to three dimensions. The surface boundary condition can still be employed when either $I \geq h$ or if $I < h$, the Froude number $F = 1/\mu_3(h - I) > 1$. Preliminary results show that the inversion interface is perturbed less in the three dimensional case with some flow round rather than over the hill. However, the speed-up $\Delta S$ is still sensitive to the changes in the higher flow.

ACKNOWLEDGMENT

Our thanks are due to Dr Marshall Baker for very helpful discussions on the mathematics and also to the referees who provided very constructive criticism of our work. One of us (DJC) acknowledges receipt of a research studentship from the Meteorological Office. This research was supported by the Natural Environment Research Council.

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