A two-dimensional numerical study of horizontal roll vortices in an inversion capped planetary boundary layer

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SUMMARY

The dynamics of large-scale horizontal roll vortices in an inversion-capped planetary boundary layer subject to surface heating are investigated by means of a numerical model. The rolls are assumed to be two-dimensional and calculated explicitly, while small scale turbulence is parametrized using a buoyancy-dependent mixing length hypothesis. The assumptions of the model are based upon atmospheric observations which show the occurrence of large eddies highly elongated in a direction close to that of the geostrophic wind and a partitioning of turbulence energy between these large eddies and smaller scales. The model produces realistic turbulence statistics and provides information on the sensitivity to various assumptions and parameters. Apart from the generation of internal gravity waves in the stable region above the boundary layer the results are insensitive to the details of the parametrization. The internal gravity wave generation is sensitive to most parameters and in particular to the roll orientation.

1. INTRODUCTION

In a previous paper (Mason and Sykes 1980, referred to hereafter as MS) the results of a two-dimensional numerical study of horizontal roll vortices in a neutral atmospheric boundary layer were presented. Here a similar study of an inversion-capped planetary boundary layer heated at the ground is described. In MS the eddies were due to a shear instability, whilst this paper is concerned with eddies driven mainly by buoyancy. The aim of the work is to understand the large-scale horizontal roll vortices observed in the planetary boundary layer. They are most commonly observed in near-neutral conditions with relatively strong winds over a uniform terrain. However, in spite of this association with near-neutral conditions it is evident from case studies (Le Moné 1973) that the observed rolls are driven mainly by buoyancy effects and are not really related to shear instabilities of the Ekman boundary layer. The wind shear is none the less important in providing the mechanism for organizing the convection into longitudinal rolls. Thus whilst the previous study was of value in reaching a broad understanding of turbulent boundary layers it did not correspond to observation. The present study, though still highly idealized, does bear fairly directly on atmospheric observations and on other simulations of the planetary boundary layer.

In the numerical simulations the large-scale motions are explicitly represented but assumed to be two-dimensional. The small-scale turbulence is presumed to be isotropic and represented by means of a mixing length hypothesis. The length scale in the mixing length has been chosen, on the basis of observations, to represent all small-scale three-dimensional motions. Apart from the important assumption of two-dimensionality and the observational basis for choosing the length scale, the method is similar to that employed in three-dimensional 'Large-Eddy' simulations (e.g. Deardorff, 1974). The external parameters are fixed and are typical of conditions under which large scale rolls are observed in the planetary boundary layer. Emphasis is placed on how the simulations depend upon various assumptions. An important benefit of understanding these large scale eddies is an indication of the space and time resolution required for their inclusion in more general three-dimensional 'Large-Eddy' simulations.

In §1 the equations and numerical techniques are outlined and then in §2 the details of the flow driving conditions and turbulence parametrization are presented. In §3 the linear stability of the system is considered and the properties of the full non-linear integrations given. The factors which are varied in these non-linear integrations are:


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1. the size of the computational domain
2. the angle of the computational domain to the basic geostrophic flow
3. the length scale used for the small scale turbulence parametrization
4. the magnitude of the static stability above the inversion.

Finally in §4 the main findings and conclusions are summarized.

2. NUMERICAL MODEL AND TURBULENCE CLOSURE

(a) Basic model

The equations considered are the two-dimensional ensemble-averaged equations for an incompressible Boussinesq fluid. Temperature is replaced as a variable by the buoyancy
\[ B = g(T - T_0)/T \]. In Cartesian \((x, y, z)\) co-ordinates rotating about the \(z\)-axis and with gravity parallel to the \(z\)-axis these are:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = - \frac{\partial p}{\partial x} - \frac{\partial P_o}{\partial x} + f v + \frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{13}}{\partial z}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + w \frac{\partial v}{\partial z} = - \frac{\partial p}{\partial y} - f u + \frac{\partial \tau_{12}}{\partial x} + \frac{\partial \tau_{23}}{\partial z}
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = - \frac{\partial p}{\partial z} + B + \frac{\partial \tau_{13}}{\partial x} + \frac{\partial \tau_{33}}{\partial z}
\]

\[
\frac{\partial B}{\partial t} + u \frac{\partial B}{\partial x} + w \frac{\partial B}{\partial z} = \gamma (B - B_0(x)) + \frac{\partial H'_1}{\partial x} + \frac{\partial H'_3}{\partial z}
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0
\]

There are no variations in the \(y\)-direction except for a background pressure gradient which, together with its \(x\)-component, provides the geostrophic flow making an angle \(\theta\) with the \(x\)-axis. \(\theta\) less than 90° implies that the geostrophic wind is to the right of the \(y\)-axis (see Fig. 1). \(P_o\) is a linearly varying background pressure so that \(\frac{\partial P_o}{\partial x}\) and \(\frac{\partial P_o}{\partial y}\) are constants related to the components of the geostrophic wind vector. \(p\) is the perturbation pressure, \(f\) is the Coriolis parameter and \(\tau_{ij}\) is a turbulent Reynolds stress tensor which is derived by the parametrization discussed below. The equation for buoyancy contains a term representing a damping to some prescribed state \(B_0(x)\) on a time scale \(\gamma^{-1}\). \(H'_i\) is a turbulent buoyancy flux vector determined in a similar manner to \(\tau_{ij}\). The equations are applied in a rectangular domain \([0, W] \times [0, D]\) periodic in the horizontal \(x\)-direction i.e. \(f(x + W) = f(x)\). The upper boundary conditions are zero stress and heat flux and the lower boundary conditions are turbulent flow over a rough boundary with an imposed negative buoyancy flux \(b\). The computational details of these boundary conditions are given below. A consequence of the prescribed buoyancy flux in the absence of a sink term would be a steady unbounded growth of the boundary layer. Such steady growth is of course characteristic of the daytime planetary boundary layer and the reasons for wishing to suppress it and achieve a steady state need to be considered.

The time dependent problem starts from arbitrary initial conditions and after some time an evolution independent of initial conditions should occur. It may then be assumed that the turbulence is in temporal equilibrium with the current mean state. As the turbulence time scales are somewhat faster than the rate of mean flow adjustment, in principle this is a fairly convincing approach. The problem is whether the time scales are sufficiently different for accurate turbulence statistics to be obtained before the mean flow changes. In the present study (see §3b below) it is found that the statistics have to be averaged over a period of a few hours to obtain an accuracy of \(\pm 5\%\). Over this period of time the mean flow would have changed significantly. The prime reason for seeking steady states is to avoid this difficulty and to be able to consider differences between simulations to an accuracy of \(5\%\) or better.
In three-dimensional large-eddy simulations the statistics at each height are averaged over a two-dimensional area and it may be easier to obtain such statistical significance.

Having accepted the need for an equilibrium solution the relationship between such solutions and atmospheric observations needs to be considered. For $B_0(z)$ a linear variation from zero at $z = 0$ to some value at $z = D$ has been chosen, i.e. uniform stable stratification with constant Brunt Vaisala frequency $N (N^2 = - \frac{\partial B}{\partial z})$. In a convective boundary layer, the buoyancy flux at the surface drives turbulence and mixes the boundary layer to a near constant buoyancy. This constant buoyancy is close to the value of the undisturbed buoyancy at the height of the top of the boundary layer. Thus the approximate form of the damping term $\gamma(B - B_0(z))$ is a maximum at the surface and near zero at the top of the boundary layer. In the equilibrium state the divergence of the buoyancy flux must equal the damping term. As seen below, the choice of $B_0(z)$ leads to the form of the heat flux variation with height being close to that seen in atmospheric observations. Owing to the effects of radiative heat transfer, direct short wave heating and advection, atmospheric observations are always slightly different from the results of idealized studies. It follows that, in spite of the arbitrary damping term, the present results are probably no less relevant than those obtained from other idealized studies of a basic stable stratification heated from below. It should be emphasized that only the mean structure is influenced directly by the damping term. In the results presented, $\gamma^{-1}$ is approximately one pendulum day and of no direct consequence to the turbulent eddies.

(b) Turbulence parametrization

As only small-scale eddies are parametrized, a relatively simple mixing length model closure hypothesis is considered. Indeed with the gross assumption of the large-scale eddies
being two-dimensional it would be hard to justify a more sophisticated approach. The usual method following Smagorinsky (1963) is to define

$$\tau_{ij} = \nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right); \quad H_i = \nu \frac{\partial B}{\partial x_i},$$

where $\nu = l^2(z) S$, $S = \left( \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right)^t$ and $l(z)$ is a prescribed mixing length scale. As in MS, $l(z)$ is limited to some value $l_0$ corresponding to the observed scale of the small three-dimensional eddies. To extend this approach to include buoyancy effects is not straight-forward as both the length scale $l(z)$ and implied velocity scale $l(z) S$ may be influenced. The formulation which is used varies according to the part of the flow being considered and it is convenient to discuss these regions separately.

First consider the region near the ground at heights $\leq l_0$. Here the fluxes are forced to correspond to those expected in observed Monin–Obukov similarity functions. A Richardson number $Ri$ is defined as $(\partial B/\partial z)/S^2$; this definition corresponds precisely to the ratio of the buoyancy and shear production terms occurring in the turbulence energy equation for the parametrized scale, i.e. ratio of $l^2(z)$ $S$ $\partial B/\partial z$ to $l^2(z) S^3$. Near the surface this reduces to the normal one-dimensional definition of $Ri$. The length scale $l(z)$ is then presumed to vary as $\kappa(z + z_0)^\phi^{-1}$ where $\phi$ is an empirically derived function of $Ri$ ($\kappa$ is the Von–Karman constant and $z_0$ the roughness length). Surface layer data (e.g. Busch 1973) suggest that $\phi = (1 - 4z Ri)^{-4}$ with $z = 3$ is a good fit for the momentum flux. $\phi$ is different for the heat flux but here the same length scale is used for both momentum and buoyancy. In the interior of the flow where only small scale relatively isotropic motions are parametrized, and observations lack detail, this simple approach is easy to justify. It is in the surface layer that observations show it to be inaccurate. However, because the buoyancy flux is specified in our simulations, the only effect of allowing for a difference confined to the surface region would be a trivial change in the surface buoyancy value.

The empirical function $\phi$ allows for variations in both velocity and length scales near the ground. Away from the ground where the flow has unstable buoyancy gradients, the scale $l(z)$ is limited to $l_0$, and it is appropriate to consider the equation for parametrized scale turbulence energy to determine the influence of buoyancy effects on the velocity scale. In homogeneous flows this velocity scale is $l_0 S$. Conventional modelling of the turbulence energy equation (e.g. Launder and Spalding 1972) relates this velocity scale (near the surface it is exactly $u_*$ the square root of the surface stress) to the turbulence energy, i.e. $E^t = C_E^{-4} l_0 S$ where $E$ is the turbulence energy and $C_E$ a constant determined by the observed stress energy ratio and used in diagnostics given below. It follows that in homogeneous flows the small-scale turbulence energy production is $l_0 E^t C_E^{-4} S^2$, $(u_*^2 \partial u/\partial z)^2$ in a surface layer) and equal to the dissipation $C_E^{-4} E^t/l_0, (u_*^2)/\kappa z$ in a surface layer). With the addition of buoyancy production and the assumption of the same scale for buoyancy and momentum the total small-scale turbulence energy production is $l_0 E^t C_E^{-4} (S^2 - \partial B/\partial z)$ i.e. $l_0 E^t C_E^{-4} (1 - Ri) S^2$, which with dissipation still equal to $C_E^{-4} E^t/l_0$ implies $E^t = C_E^{-4} l_0 S (1 - Ri)^t$ and thus a velocity scale $l_0 S (1 - Ri)^t$ (see also Busch et al. 1976). In the simulations in this paper the heat transfer away from the ground is largely on the resolved scale and the Richardson number $Ri$ generally small. The factor $(1 - Ri)^t$ is nevertheless included for completeness.

The remaining region of the flow requiring turbulence parametrization is the stable region at the top of the boundary layer. There, both the length and velocity scales are reduced. If second order closure techniques were used the velocity and length scales would reduce to give zero velocity scale at some critical Richardson number. There are uncertainties in formulating second order closure techniques and therefore we have adopted the simplest approach retaining this behaviour. For $0 < Ri < 1/\beta$ the length scale $l(z) \sim l_0 (1 - \beta Ri)$ and the velocity scale $\sim l(z) S$. In stable regions of the flow the factor
$\left(1 - \beta \text{Ri}\right)$ is the sole buoyancy factor allowing for changes in both length and velocity scale. For Richardson numbers greater than $1/\beta$, $l(z)$ is set to zero. The value of $\beta$ used is not based on observational data and has a value of 3 implying a critical Richardson number of 0.33. With $\beta > 3$ there are marked grid length instabilities at the base of the stable region (see §3b (iv) below). At the value $\beta = 3$ these modes are only just perceptible and of no consequence. It is tempting to suggest that with $\beta > 3$ the eddy viscosities are too small to damp out small-scale Kelvin Helmholtz instabilities. The algorithm used to match the formulations in the various regions of the flow is:

$$
\frac{1}{l(z)} = \frac{(1 - \text{Ri})^+ \phi}{\kappa(z + z_0)} + \frac{1}{l^*}
$$

\[
\begin{align*}
I^* &= I_0 \\
\phi &= (1 - 4\alpha \text{Ri})^+ \\
v &= I^2(z) S (1 - \text{Ri})^+ \\
I^* &= I_0 \psi \\
\psi &= (1 - \beta \text{Ri}) \\
v &= I^2(z) S
\end{align*}
\]

Ri < 0

Ri > 0

In the surface layer the factor $(1 - \text{Ri})^+$ in the expression for $l(z)$ cancels the $(1 - \text{Ri})^+$ occurring in the expression for $v$. Integrations were undertaken with this parametrization. Although the results seem satisfactory for short integration times horizontal grid length modes build up slowly at the surface and also to a lesser degree in the stable region of the fluid. Those in the stable region may not be important but those at the surface are significant. Increasing the horizontal resolution only slightly reduces them and suggests that they represent a failure of the parametrization.

In the atmospheric surface layer the stress $\tau_{11}$ is of comparable magnitude to but greater than $\tau_{13}$. Our parametrization gives $\tau_{13} \sim v \partial u/\partial z$ and $\tau_{11} \sim v \partial u/\partial x$ but $\partial u/\partial x$ is very much less than $\partial u/\partial z$ and the horizontal transport processes are thus underestimated. In the atmosphere the length scales of the horizontal velocity fluctuations near the surface do not vary much with height and are fairly typical of the main boundary layer. Also, observations in the stable region above the boundary layer show that horizontal motions persist and are of relatively large length scale. A full second order closure scheme would tend to allow for this anisotropy and therefore in our simple approach a different viscosity is used in conjunction with horizontal derivatives of horizontal velocity components. The horizontal viscosity $v_H$ is always based on the scale $l_0$ and is independent of the presence of boundaries or stable buoyancy gradients. The exact form is derived from consideration of an energy equation for horizontal energy with production confined to horizontal gradients of horizontal velocity (the two-dimensional form of $S$).

Thus

$$
v_H = v_H' = I_0^2 \left\{ \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 \right\}^+ \quad \text{for } v_H' > v
$$

and $v_H = v$ for $v_H' < v$.

Thus, in the interior unstable regions of the flow the isotropic viscosity is retained. With these anisotropic viscosities the integrations contain no significant grid-scale features. (In runs with $\beta = 4$ the grid-scale features at the base of the stable region are reduced but still quite pronounced.) The part played by these plausible horizontal viscosities is largely cosmetic but, as discussed below, the properties of internal gravity waves in the stable region of the flow are affected.

(c) Lower boundary condition

The lower boundary conditions consistent with the form of $l(z)$ are derived by assuming
that the flow between the surface and the first grid points is in local equilibrium and uninfluenced by pressure gradients or advection effects due to the large eddies. Calculation of these effects in actual integrations show that they are small but not negligible; however, the errors average to zero over the whole of the lower surface of the domain. A test integration with increased resolution indicated that these errors and other resolution errors, are not significant.

The boundary conditions are applied by assuming

\[ \frac{\phi(x + z_0)}{u_\ast} \frac{\partial V}{\partial z} = 1 \]

and

\[ \frac{\phi(x + z_0)}{B_\ast} \frac{\partial B}{\partial z} = 1 \]

where \( V \) is the total velocity at the lowest grid point, \( B_\ast \) is \( b/u_\ast \) where \( b \) is the specified buoyancy flux and \( \phi \) is defined as above with \( \text{Ri} = (\partial B/\partial z)/(\partial V/\partial z)^{-2} \). The stress boundary conditions are applied by resolving the stress \( u_\ast^2 \) according to the direction of \( u \) and \( v \) at the lowest grid point. The values of \( u_\ast \) are calculated by starting with a trial value of \( u_\ast \) and then using an iterative procedure involving the integration of the above equations.

(d) Numerical procedure

The equations of motion, subject to the above conditions, are solved on a Cartesian mesh uniformly spaced in the \( x \)-direction but stretched in the vertical direction. This allows a closer spacing of computational levels near the lower boundary and at the top of the boundary layer. The fine resolution at the top of the boundary layer is necessary to resolve the sharp gradients at the inversion. Variables are stored on a staggered mesh, as in Williams (1969) and all finite differences are effectively second order accurate in both space and time. Leap-frog time differencing is applied to the non-linear and Coriolis terms but the smaller damping and Reynolds stress terms are advanced by forward steps. The Poisson equation for pressure, which is based on the continuity equation, is solved by a direct method using a Fast Fourier transform in the horizontal and line inversion of the resulting tridiagonal matrices in the vertical.

The initial conditions for the integrations are steady state one-dimensional solutions of the above equations and boundary conditions. They are obtained by the iterative procedure mentioned in MS, except that with the addition of the equation for buoyancy, the procedure only converged with an under relaxation of the change in eddy viscosities.

3. Results

This paper concentrates on results obtained with parameters typical of those under which large scale rolls are seen in atmospheric observations. With a geostrophic wind \( u_\ast \) of \( 10 \text{ m s}^{-1} \) and a roughness length \( z_0 = 0.1 \text{ m} \) a buoyancy flux \( b = -10^{-3} \text{ m}^2 \text{ s}^{-3} \) is considered. The damping coefficient \( \gamma \) has a value chosen to ensure that the depth of the boundary layer \( z_1 \) is about \( 10^3 \text{ m} \). It follows that the buoyancy scaling velocity \( u_\ast = (bz_1)^{3/2} \approx 1.0 \text{ m s}^{-1} \). The most significant quantity in determining the dynamics of the boundary layer is the Monin–Obukov length \( L = u_\ast^3/b \). Since \( u_\ast \) is not specified the precise value of \( L \) in a particular instance depends on the flow, but taking \( u_\ast = 0.5 \text{ m s}^{-1} \) as typical (with \( u_\ast = 10 \text{ m s}^{-1} \) and \( z_0 = 0.1 \text{ m} \), it follows that \( L = 400 \text{ m} \). The resulting large eddies have scale \( \gtrsim L \) and are thus essentially buoyancy driven.

Atmospheric observations suggest a wavelength for the rolls of \( 1-2 \text{ km} \) (\( \sim 2 \times z_1 \)) and the size of the computational domain for most integrations is \( W = 4 \text{ km} \) and \( D = 10 \text{ km} \). Runs were also undertaken with \( W = 2 \) and \( 8 \text{ km} \); the results are discussed below. In most cases the computational mesh has 32 points in the horizontal and 60 in the vertical. The 10 km deep domain is chosen so that internal gravity wave disturbances can propagate away from the boundary layer and be dissipated at heights above 7 km without risk of reflection.
This dissipation is accomplished by means of a damping applied with increasing effect towards the upper boundary (as in Mason and Sykes 1978). The relatively large number of grid points in the vertical allows the vertical resolution to vary smoothly from 10 m at the surface to 80 m in the middle of the boundary layer to 30 m around the top of the boundary layer. Above 1700 m the mesh expands to about 400 m and remains at this value to 10 km. The horizontal resolution in most runs is 125 m but half this value in a test run. The time step in the numerical integrations is limited by the Courant–Friedrichs–Lewy condition for horizontal advection, and with a horizontal resolution of 125 m, 8 s is used. With a $32 \times 60$ mesh the arithmetical computation time per time step on an IBM 360/195 is 0.2 s.

A preliminary study of the stability of the basic states upon which eddies are allowed to grow is undertaken with a view to assisting the interpretation of the main results. In its attention is restricted to a few gross parameters. The basic undisturbed one-dimensional steady-state solution is perturbed by an arbitrary doubling of the surface stress over one quarter of the domain. The perturbations are then allowed to grow freely but each time the vertical velocities exceed $10^{-3} \text{ m s}^{-1}$ the perturbation from the initial state is reduced by a factor of 10 and the integration continued. In this way the most unstable wavenumber, within the quantization imposed by the domain, is selected and its growth rate determined. The amplitude $10^{-3} \text{ m s}^{-1}$ gives modes quite linear in behaviour yet much larger than the small vertical velocity truncation noise which occurs once the system is perturbed.

Figure 2a, b and c show a selection of results. In the first (Fig. 2a) $l_0$ is fixed at 40 m and other parameters are as noted above. The growth rates and wavenumbers are shown as a function of the angle $\theta$ between the domain and geostrophic wind. Maximum growth rate occurs for angles around 80° and 90° and minimum wavelength occurs at 70°. At maximum growth rate the wavelength is $\sim 0.6 z_i$ and the growth rate $\sim 0.9 w_*/z_i$. The maximum growth rates are very much larger than in the homogeneous Ekman layer flow studies in MS. The basis of the present instability is simple buoyant overturning and maximum growth rate occurs around $\theta \sim 80°$ where the transverse vertical shear across the convective rolls is a minimum. For the homogeneous Ekman layer the maximum growth rate occurs for angles slightly greater than 90° and is linked to the Ekman layer inflexion instability. Values of $\theta \sim 80°$ are in keeping with those observed in the atmosphere.

Figure 2b illustrates the behaviour with $\theta = 90°$ as the mixing length scale $l_0$ is varied. As $l_0$ is increased, the growth rates and the wavelength of the most unstable mode decrease. At $l_0 = 640$ m the flow is stable for all values of $\theta$. The choice of the most appropriate value of $l_0$ for our non-linear integrations is discussed below. It is of interest that the results shown here indicate that for these parameters, solutions obtained by conventional 'K-theory' are only stable if $l_0$, or its equivalent, is $\geq z_i/2$.

Figure 2c provides a link to the homogeneous work in MS. The buoyancy flux $b$ is varied with $l_0$ fixed at 40 m and $\theta$ at 90°. The scaled growth rates show a minimum at $b = -10^{-5}$. This is merely because growth rates reduce to a near constant value for $b \leq -10^{-3}$ and $z_i/w_*$ is then no longer an appropriate scale. For these negligible buoyancy fluxes it is surprising that in spite of the inversion at $z_i$, growth rates and wavenumbers are similar to those seen in the homogeneous study of MS. Stability integrations with a rigid upper boundary imposed at $z = 10^3$ m give no growth for $b \leq -10^{-4}$. This difference between the rigid lid and inversion-capped case is presumably related to the observation that in the latter the vertical motion is not completely confined to the boundary layer but extends into the stable flow above.

The stability study indicates growth of instabilities for a wide range of angles $\theta$ and scales $l_0$. This section proceeds to show how the fully developed flows depend on these parameters. In MS it was found that the fully developed states showed a variation on the time scale $f^{-1}$. Here the percentage variations on this long time scale are roughly halved whilst there are larger short time scale fluctuations. The long time scale variations in surface stress are similar to those shown in MS but, largely due to the increased mean value, the percentage changes on this time scale are 10% as against 20% in the homogeneous study. Here we also examine the statistics of the horizontally averaged vertical velocity variance.
\( \bar{w}^2 \). At most heights the standard deviation of the instantaneous value from a time mean over a two hour period is \( \sim 8\% \) of the mean. The standard deviation of means over \( 2\pi f^{-1} \) (17 h) compared with the mean over \( 8\pi f^{-1} \) is about 1% of the mean. These variations do not seem to be linked with the inertial oscillation and have a fairly random character. It is clear that averaging over a long period is necessary to achieve reliable statistics and, as in MS, integrations were conducted for a period of \( 6\pi f^{-1} \) and the averages taken from the last \( 4\pi f^{-1} \) period. Table 1 presents some important basic quantities taken from such time averages. \( C_x \) is the geostrophic drag coefficient defined by \( \tau_x = C_x u_f^2 \) where \( \tau_x \) is the
TABLE 1. THE EFFECT OF VARYING \( l_0 \) AND \( \theta \)

<table>
<thead>
<tr>
<th>( l_0 )</th>
<th>20 m</th>
<th>40 m</th>
<th>80 m</th>
<th>120 m</th>
<th>160 m</th>
<th>320 m</th>
<th>640 m</th>
<th>1280 m</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_x \times 10^8 )</td>
<td>1.2</td>
<td>1.5</td>
<td>1.7</td>
<td>1.9</td>
<td>2.1</td>
<td>2.3</td>
<td>2.4</td>
<td>2.5</td>
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<td>( a )</td>
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<td>23°</td>
<td>22°</td>
<td>21°</td>
<td>19°</td>
<td>18°</td>
<td>17°</td>
<td>17°</td>
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<td>( 2 - D ) solutions</td>
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</tr>
<tr>
<td>( \theta = 70^\circ )</td>
<td>( C_x \times 10^8 )</td>
<td>2.7</td>
<td>2.6</td>
<td>2.4</td>
<td>2.2</td>
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<td>( a )</td>
<td>26°</td>
<td>23°</td>
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<tr>
<td></td>
<td>( w_{\text{max}}^2 )</td>
<td>0.24</td>
<td>0.18</td>
<td>0.20</td>
<td>0.22</td>
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<tr>
<td>( \theta = 80^\circ )</td>
<td>( C_x \times 10^8 )</td>
<td>2.7</td>
<td>2.6</td>
<td>2.5</td>
<td>2.4</td>
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<td>0.33</td>
<td>0.22</td>
<td>0.22</td>
<td>0.22</td>
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<tr>
<td>( \theta = 90^\circ )</td>
<td>( C_x \times 10^8 )</td>
<td>2.8</td>
<td>3.0</td>
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<td></td>
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<td>9°</td>
<td>12°</td>
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<tr>
<td></td>
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<td>0.31</td>
<td>0.22</td>
<td>0.22</td>
<td>0.22</td>
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<tr>
<td>( \theta = 100^\circ )</td>
<td>( C_x \times 10^8 )</td>
<td>2.5</td>
<td>2.7</td>
<td>2.7</td>
<td>2.7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( a )</td>
<td>7°</td>
<td>11°</td>
<td>17°</td>
<td>18°</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( w_{\text{max}}^2 )</td>
<td>0.09</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

x-component of surface stress. \( C_x \) is the dissipative drag due to the boundary layer and a maximum of \( C_x \) implies maximum energy dissipation. \( \delta \) is the angle of the total surface stress vector to the geostrophic wind. \( w_{\text{max}}^2 \) is the maximum value occurring in the vertical profile of the horizontally and temporally averaged resolved scale \( w^2 \). \( w_{\text{max}}^2 \) is a measure of the large scale resolved energy present in the finite amplitude rolls. The variations of these quantities with \( l_0 \) for fixed \( \theta = 80^\circ \) are not monotonic. For large \( l_0 \) the flows are stable one-dimensional solutions and \( C_x \) decreases as \( l_0 \) is reduced. However, between \( l_0 = 640 \) and 320 m an instability occurs and \( C_x \) increases dramatically. Further reduction in \( l_0 \) increases the energy on the resolved scale but produces very little change in the stress until \( l_0 \) is less than 40 m. For \( l_0 \) less than 40 m the drag coefficients decrease in spite of \( w_{\text{max}}^2 \) increasing. The dependence upon \( \theta \) is considered for \( l_0 = 40 \) and 120 m. In both these cases the values of \( C_x \) and \( w_{\text{max}}^2 \) are greatest at \( \theta = 80^\circ \).

In the homogeneous flow study of MS \( l_0 \) was taken as 40 m on the basis of observations but it was convincing that this value of \( l_0 \) actually gave a maximum of energy dissipation in the boundary layer. In MS the angle \( \theta = 100^\circ \) was selected entirely on the basis of maximum dissipation. Here the angle \( \theta = 80^\circ \) is chosen on the basis of both maximum dissipation and also maximum \( w_{\text{max}}^2 \). Maximum \( C_x \) occurs for \( l_0 \sim 320 \) m but the variations are not very significant for \( l_0 \sim 40 \) m. The approach being considered here is dependent upon observational data for the existence of the large scale two-dimensional structures and it is appropriate to use the observations to determine the dominant scale of the small eddies. As discussed in MS the observations of Lemone (1973) suggest that \( l_0 \) should be \( \sim 40 \) m and this value is used in most of the integrations. It is reassuring that the results are not unduly sensitive to this choice and below, some results with \( l_0 = 120 \) m are illustrated. In what follows the choice of domain length is considered and then more details of the \( l_0 = 40 \) m and \( \theta = 80^\circ \) case are presented. After that the dependence of the results upon the choice of parameters is examined.

Length of domain

Here we simply consider whether the length of the domain is materially affecting the results. Table 2 shows how the values of \( C_x \) and \( w_{\text{max}}^2 \) depend on domain length. Domains of

<table>
<thead>
<tr>
<th>Box length</th>
<th>( w_{\text{max}}^2 )</th>
<th>( C_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 \times 10^4 m</td>
<td>0.42</td>
<td>0.287</td>
</tr>
<tr>
<td>4 \times 10^4 m</td>
<td>0.32</td>
<td>0.289</td>
</tr>
<tr>
<td>8 \times 10^4 m</td>
<td>0.31</td>
<td>0.292</td>
</tr>
</tbody>
</table>
length 2, 4 and 8 km with 16, 32 and 64 grid points respectively are considered. The 2 km domain contains a single roll, the 4 km domain usually has 2 rolls and the 8 km domain has 4 rolls. In all cases the rolls change slowly with time and drift steadily. However, only in the 4 and 8 km domains do individual rolls occasionally appear to decay and new ones grow from disturbances starting at the lower boundary. It is evident from the results that the 4 and 8 km domains give very similar, and thus presumably domain length independent, results. The 2 km box gives a larger \( w_{max}^2 \) but smaller \( C_x \) illustrating that the domain is affecting the convection. In all subsequent results a 4 km domain is used.

**Results with \( l_0 = 40 \text{ m} \); \( \theta = 80^\circ \)**

In order to discuss these results some notation concerning various co-ordinate frames and averaging is introduced. In the preceding section the basic 'roll co-ordinate' equations are presented in terms of \( u, v, w \) and \( B \). In the remainder of this paper the results are presented in a co-ordinate frame based on the geostrophic wind direction, i.e. geostrophic wind velocity \( (u_g, 0, 0) \). The variables are divided into mean and fluctuating parts. For the mean part a horizontal space and time average is taken, and for notational simplicity we denote the mean quantities in geostrophic co-ordinates by \( \bar{u}, \bar{v} \) and \( \bar{B} \). In time, fluctuations occur on two distinct scales, a shorter scale due to the turbulent eddies and a long time scale due to inertial variations in the mean flow with period of order a pendulum day. It is not appropriate to consider the latter slow variations as turbulence. The horizontal average of \( w \) in each realization is zero and the moments of \( w \) and moments involving an odd power of \( w \) (e.g. \( uw, wB, wu^2 \)) contain little low frequency variance. However \( u^2 \) and \( v^2 \) contain significant low frequency variance especially at the base of the stable region. In order to filter out such slow variations the fluctuations are derived relative to instantaneous horizontal means. The final moments are temporal averages of these quantities and are denoted by \( uw, u^2, wB \) etc.

![Figure 3. \( l_0 = 40 \text{ m}, \theta = 20^\circ \) results: mean velocity component profiles. The solid curve is the \( \tilde{u} \) profile and the dashed curve is the \( \tilde{v} \) profile.](image)

The vertical profiles of various temporally and horizontally averaged quantities are now considered. Figure 3 shows the profiles of \( \tilde{u} \) and \( \tilde{v} \). The \( \tilde{u} \) profile is logarithmic near the lower surface with a small gradient in the bulk of the boundary layer. The \( \tilde{v} \) profile shows a maximum in the middle of the boundary layer. At the level of the inversion the profiles show a vertical oscillation in direction and shear. As discussed below, this oscillation is characteristic of an Ekman layer and is due to the Coriolis terms; the various terms involving
production and diffusion in the equations for the Reynolds stress are quite significant in magnitude but largely in balance giving little dissipation. These Ekman layer oscillations are likely to be absent from observed planetary boundary flows unless conditions remain steady for long enough to allow geostrophic adjustment to occur, i.e. greater than one pendulum day.

Figure 4. \( I_s = 40 \text{ m}, \theta = 80^\circ \) results: vertical momentum flux profiles \( \bar{uw} \) and \( \bar{vw} \). The solid curves are the resolved scale fluxes and the dashed curves the total resolved and parametrized fluxes.

Figure 4 shows the shear stress profiles; the explicitly resolved and total (resolved plus parametrized) stresses are presented. The parametrized small-scale stresses are dominant at the surface but unimportant above a few hundred metres. Both \( \bar{uw} \) and \( \bar{vw} \) are mainly in the sense of down gradient diffusion in the velocity profiles. The main exception occurs around the inversion where the stresses show no evidence of the Coriolis force induced oscillations. Above the inversion the shear stresses do not go to zero but remain significant. This contribution is due to gravity waves that are trapped near the inversion and also radiated upwards. (These wave radiation stresses can only occur in the plane of a two-dimensional computation and the values of \( \bar{uw} \) at these heights would be zero with \( \theta = 90^\circ \).) They form a significant part of the flow under consideration but as seen below, they are sensitive to changes in the parameters. In order to understand these wave motions the heuristic model of boundary layer eddies perturbing the inversion level and these perturbations then acting like low hills generating internal gravity waves is explored. Although this model is difficult to apply quantitatively it allows a qualitative appreciation of the important parameters. If the energy of the boundary layer rolls could be converted entirely into potential energy then the height \( h_r \) of the perturbations in the basic stable density field would be given by \( h_r^2 = \frac{1}{2} w_r^2 / N^2 \) where \( w_r \) is a velocity scale representing the roll motions and \( N^2 = - \frac{\partial B/\partial z}{\partial z} \) is the Brunt–Vaisala frequency. Linear theory of inviscid flow over small corrugations (Queney 1947) shows that the Froude number \( F_r = u_r k / N \) (where \( u_r \) is the velocity difference between the corrugations and the interior and \( k \) is the wavenumber of the corrugations) is the crucial parameter determining how these corrugations give rise to internal gravity waves. The average wave stress due to such uniform flow over periodic sinusoidal corrugations with amplitude \( h_r \) is \( A = \frac{1}{2} (F_r^{-2} - 1) h_r^2 k^2 u_r^2 \) for \( F_r < 1 \) and zero for \( F_r > 1 \). For \( F_r > 1 \) the excitation frequency is greater than \( N \) and the disturbance is evanescent in character. Substitution of the above
expression for $h^2$ gives $A = \frac{1}{4}(F_r^{-2} - 1)^2 F_r^2 w^2$ which has a maximum of $w^2/8$ for $F_r = 1/\sqrt{2}$. In the present example $k \approx 2\pi/2 \times 10^{-3}$ m$^{-1}$, $N \approx 10^{-2}$ s$^{-1}$ and $u_r \approx 1$ m s$^{-1}$ giving $F_r \approx 0.3$. This suggests that the wave radiation is close to optimum and $F_r \sim 0.3$ gives $A \sim 0.08 w^2$. The observed value of wave radiation stress $\sim 0.025$ m$^2$ s$^{-2}$ which is thus consistent with $w^2 \sim 0.3$ and $h_r \sim 40$ m. This value of displacement is consistent with results from our integration but there is no real check on $w^2$ other than to be certain that it is less than the values of $w^2$ which occur in the rolls. In fact, peak values of $w^2$ are around $1.5$ and, (see 3b above), $w_{max}^2$ is $0.3$. The utility of this heuristic model will be explored further when we consider the effect of variations in the angle $\theta$.

Figure 5. $l_0 = 40$ m, $\theta = 80^\circ$ results: mean buoyancy profile and vertical buoyancy flux. The resolved scale flux is a solid curve and the total resolved and parametrized flux is a dashed curve.

Figure 5 illustrates the buoyancy $B$ and buoyancy flux $\overline{wB}$. The buoyancy profile has a minimum in the middle of the boundary layer, high values close to the surface and an increase above the undisturbed gradients at the inversion. The flux is similar to atmospheric observations and is different in form from gradient diffusion of the buoyancy profile. Although the buoyancy has a minimum at $z \sim 0.4 z_i$ the flux does not change sign until $z \sim 0.8 z_i$, and above this the downward flux is large even though by definition of $z_i$ the buoyancy gradient does not increase much until $z = z_i$. Above the boundary layer some resolved heat flux seems to be associated with the gravity waves. Above 3 km this heat flux declines even though as Fig. 4 shows, this is not so for the momentum stresses.

Steady inviscid small amplitude internal gravity waves do not produce a heat flux. In order to consider the origins of this heat flux it is instructive to examine the equation for resolved scale buoyancy variance. In our near steady state the production term $\overline{wB} (\partial B/\partial z)$ is not balanced by the vertical advection divergence $\overline{\partial(wB^2)/\partial z}$ but by dissipation arising from the use of horizontal viscosities. It is not clear whether this dissipation in the stable fluid above the boundary layer has an atmospheric counterpart. To confirm this role of the horizontal viscosities a test integration was conducted with $v_h = k(x)^2 S(1 - R_i)^2$ for $R_i < 1$ and $v_h = 0$ otherwise. This eliminates the heat flux above the boundary layer but other results are unaltered. The vertical velocity variance in the stable region is greater and nearly independent of height. It is, however, in part due to unrealistic grid length scale disturbances.
At the top of the boundary layer the downward buoyancy flux remains similar to that found with the usual horizontal viscosities. In this part of the flow the Richardson numbers are generally greater than unity and the resolved heat flux is non-linearly driven with the temperature variance production term \( \bar{w}B \partial B/\partial z \approx \partial(\bar{w}B^2)/\partial z \) as found by Deardorff (1969).

![Figure 6.](image)

Figure 6. \( l_s = 40 \text{ m}, \theta = 80^\circ \) results: profiles of the components of the turbulence energy \( \bar{u}^2, \bar{v}^2 \) and \( \bar{w}^2 \). The solid curves are the resolved energy and the dashed curves the parametrized energy appropriate for all three components.

Vertical profiles of the resolved scale \( \bar{u}^2, \bar{v}^2 \) and \( \bar{w}^2 \) are presented in Fig. 6 along with a diagnostic estimate of two thirds of the parametrized scale energy. The latter is calculated by assuming a fixed ratio \( C_E \) of parametrized stress to parametrized energy i.e. \( C_E = 0.3 \) (consistent with observations of stress to energy ratios in shear driven turbulence see e.g. Launder and Spalding 1972). The estimate of total variance in each component, is obtained by adding the parametrized value to each of the resolved components (see Figs. 12, 13 and 14). Near the surface, on scales less than the Monin-Obukov length \( L \), the parametrized component is large enough to ensure that the energy components increase towards the surface. This is consistent with the turbulence in the surface layer being produced mainly by shear. Away from this surface region \( \bar{w}^2 \) is fairly typical of observations of convective boundary layers. The maximum value near the middle of the boundary layer is about 0.4 \( \bar{w}^2 \). The small maximum in \( \bar{u}^2 \) at about 1.3 \( z_i \) is an interesting feature related to the trapping of gravity waves above the boundary layer. The other component of the roll motion \( \bar{u}^2 \), has maxima at the surface and at the inversion. The \( \bar{u}^2 \) profile is quite different; this is determined by the roll motion producing a vertical interchange of \( \bar{u} \). \( \bar{u}^2 \) is thus closely related to the local value of \( \partial \bar{u}/\partial z \).

Figure 7 shows the resolved and parametrized buoyancy variance. The data have been non-dimensionalized in terms of \( B^* = -b/w_s = 10^{-3} \). In a dry atmosphere the numerical values are thus comparable with observations of temperature \( \theta \) scaled in terms of \( \theta^* = w\theta/w_s \). The small scale parametrized component has been diagnostically calculated by equating the production and dissipation terms of a second-order turbulence closure equation for temperature variances to give

\[
B^*_r = \frac{H^2}{q} \frac{2}{D_s C_E^2}
\]
where $H$ is the parametrized buoyancy flux magnitude, $q$ the diagnostic estimate of parametrized turbulence energy, $C_q$ the energy stress ratio noted above and $D_\theta$ a constant equal to 1.25 deduced from the ratio of the decay rates of energy and temperature in isotropic non-buoyant turbulence. The form of the buoyancy variance profiles is in accord with expectations with large values near the surface and around the inversion. The complex structure at the top of the boundary layer is probably the result of wave energy trapped there.

Figure 7. $l_o = 40\, \text{m}, \theta = 80^\circ$ results: profiles of normalized buoyancy variance $\bar{B}^2/B_o^2$. The solid curve is the resolved scale and the dashed curve the total resolved and parametrized.

Figure 8. $l_o = 40\, \text{m}, \theta = 80^\circ$ results: profiles of the resolved scale production and diffusion of turbulence energy.
BOUNDARY LAYER ROLL VORTICES

To complete the presentation of averaged statistics for this case some information on the balance of terms in the equation for resolved turbulence energy is presented in Fig. 8, these being the large scale advection of turbulence energy i.e. \(- \partial \bar{\omega} / \partial z \{ \bar{w}^2 + \frac{1}{4}(\bar{w}^4 + \bar{w}^2 + \bar{w}^4) \}\) and the large scale production of turbulence energy i.e. \(- \bar{u} \bar{w} \frac{\partial \bar{u}}{\partial z} - \bar{w} \bar{w} \frac{\partial \bar{w}}{\partial z} + \bar{w} \bar{B}\). The separate parts of the latter term \(- \bar{u} \bar{w} \frac{\partial \bar{u}}{\partial z} - \bar{w} \bar{w} \frac{\partial \bar{w}}{\partial z} + \bar{w} \bar{B}\) are shown in Fig. 9. Near the surface shear production dominates with buoyancy production becoming comparable at 200 m. Most production occurs in the lower half of the boundary layer from which advection terms remove energy so that there is a net gain in the upper half of the boundary layer. At the inversion the shear production is large; there is a small negative contribution above the larger positive contribution. The vertical advection term nearly balances this production and the dissipation is slight; this is an inevitable consequence of the small vertical eddy viscosities in the stable region. At the inversion the mean flow acceleration and deceleration, which the shear production term implies, is balanced through the Coriolis term acting on the mean flow. The advection of turbulence energy arises from the pressure and flow advection terms. In the boundary layer the pressure term has magnitude of roughly half the flow advection term, but at the inversion the pressure term dominates.

Figure 10 shows a few realizations of the vertical velocity field. These are spaced 10 min of time apart and are shown in a four kilometre square box. The results above 4 km have been excluded. The flow is dominated by 2 rolls and these move by about 1 km in the 20 min period. This wavelength is greater than the wavelengths of the fastest growing linear modes, but similar to atmospheric observations. The rolls are not steady in form or highly regular but show signs of growth and decay on various scales. Above the rolls the gravity wave disturbances can be seen. The phase of the waves in relation to the rolls varies more than that shown in these realizations; on average the waves propagate across the domain more slowly than the rolls.

For the case considered in this paper the comparison is limited by the available data. The most useful comparisons are with the over sea data of Penel and Lemone (1974) and Nicholls and Readings (1979). Since no attempt has been made to simulate the situations in detail comparison is naturally limited to the gross features. An exact comparison would involve not only the parameters so far considered but also effects due to water vapour, clouds, radiation and advection.
Figure 10. $L_0 = 40 \text{ m}$, $\theta = 80^\circ$ results: realizations of the vertical velocity field. The realizations are spaced 600s apart and presented with the domain height cut at 4 km. The contour intervals in Fig. 10a, b, c are 0.108 m s$^{-1}$, 0.106 m s$^{-1}$ and 0.128 m s$^{-1}$ respectively and negative values are denoted by dashed lines.
Nicholls and Readings considered data averaged into two categories, one with a mean \( z_i/L \sim 3 \) and one with \( z_i/L \sim 0.9 \). Here \( z_i/L \) is \( 2.5 \) and the drag coefficient (see Table 1), corresponding to the land value of \( z_0 = 0.1 \) m is roughly twice that found over the sea. The stresses \( \bar{\nu} \bar{w} \), \( \bar{v} \bar{w} \) and \( \omega \bar{B} \) agree in form with the observations. There are differences in \( \bar{w} \), but \( \bar{w} \) is sensitive to the precise co-ordinate frame chosen. The variances presented by Nicholls and Readings were scaled by \( u^* \). Near the surface they found \( \sigma_{\bar{w}/u^*} \sim 2.4 \), \( \sigma_{\bar{w}/u^*} \sim 1.1 \) and independent of stability. \( \sigma_{\bar{w}/u^*} \) was \( 1.6 \) for \( z_i/L \sim 0.9 \) and \( \sigma_{\bar{w}/u^*} \sim 2.7 \) for \( z_i/L \sim 3.9 \). Here \( u^* = 0.54 \) m s\(^{-1} \) and a comparison requires appropriate scaling of the data shown in the figures (Figs. 12, 13 and 14 below show the estimated total variances). Near the surface (at \( z = 40 \) m) we find \( \sigma_{\bar{w}/u^*} = 2.1 \), \( \sigma_{\bar{w}/u^*} = 1.8 \), and \( \sigma_{\bar{w}/u^*} = 1.2 \); these values are encouragingly similar to the observations. Above the surface the results show \( \bar{u} \) and \( \bar{v} \) to fall off more rapidly than the observations. \( \bar{w} \) is more difficult to compare because of the marked changes which Nicholls and Readings observed between their stability classes.

Pennell and Le Mone gave data for \( z_i/L \sim 1.5 \). Near the surface they obtained \( \sigma_{\bar{u}/u^*} \sim 2.4 \), \( \sigma_{\bar{u}/u^*} \sim 1.7 \) and \( \sigma_{\bar{w}/u^*} \sim 1.2 \) which again shows fair agreement. In the bulk of the boundary layer \( \bar{u} \) and \( \bar{v} \) behaved much as in Nicholls and Readings data but \( \bar{w} \) compares favourably with the present results. Pennell and Le Mone's data extended above the boundary layer and it is encouraging that significant variance, presumably due to gravity wave activity, was found. The magnitude is roughly twice that found here.

Nicholls and Readings also measured temperature fluctuations and found that near the surface \( \sigma_{\bar{T}/T^*} \) where \( T^* = \bar{T}/u^* \) was \( 2.5 \) for \( z_i/L \sim 3.9 \) and \( 4.5 \) for \( z_i/L \sim 0.9 \). Here this quantity is found to be 2.7. This is in reasonable agreement but the vertical variation of \( \bar{E} \) is more rapid than in the observations and more typical of the vertical variation in a convective boundary layer over land.

The only significant discrepancy between the model results and the observations is the more rapid decrease with height of \( \bar{u} \), \( \bar{v} \), and \( \bar{E} \) in the model. The cause is difficult to ascertain. It does not lie in the choice of a land value of \( z_0 \). If \( z_0 \) were reduced to the sea value but \( u^* \) increased so as to retain the same value of \( z_i/L \), the results, above the lowest grid points, would be essentially identical. Below the lowest grid point the velocity and temperature gradients would be greater than in the present integration but would, given the smaller \( z_0 \), support the same fluxes. As far as the data are concerned there are many possible extra effects which might influence these statistics which have not been included – for example – clouds, baroclinicity and mesoscale fluctuations in both the atmosphere and ocean. The latter effect may be the most serious. Nicholls and Readings presented spectra of their statistics and their results included length scales up to 60 km. It is evident that in some cases about half the variance was on scales greater than the 4 km box length used here, and perhaps not due to boundary layer turbulence.

Overall the above comparison presents no worrying contradictions but on the other hand no reason for declaring a particular success. Adaptation of the model towards realistic conditions and also the provision of more extensive data would be needed to provide a conclusive verification.

**Dependence of results on the angle of domain \( \theta \)**

In Table 1 the effects of change in roll angle on surface stress and maximum vertical velocity variance are presented. The largest effects of changing roll angle are seen in \( \bar{u} \bar{w} \), and are a consequence of changes in both the ‘roll co-ordinate’ velocity profile and the gravity wave effects. The ‘roll co-ordinate’ velocity profile is defined as the profile of the x-component of velocity in the computational domain. The resolved \( \bar{u} \bar{w} \) profiles for \( \theta = 100, 90, 85, 80, 75 \) and 70° are shown in Fig. 11. The parametrized component is significant only near the surface and gives values of surface stress \( \tau_{x3} \) varying from 0.03 at 100° through 0.08 at \( \theta = 80° \) to 0.14 at \( \theta = 70° \). The gravity wave component is zero for \( \theta = 70° \) and small for
\( \theta = 100^\circ \); thus the variation between \( \tau_{23} = 0.03 \) and 0.14 must be a consequence of the roll eddies acting in different velocity profiles.

For \( \theta = 90^\circ \) this "roll co-ordinate" velocity profile is the \( \bar{u} \) profile in Fig. 4. Ignoring the small oscillation at the inversion \( \bar{v} \) is positive everywhere with a maximum in the middle of the boundary layer. As \( \theta \) is varied from 90° a component of the \( \bar{u} \) profile is added. To a first approximation the \( \bar{u} \) profile is constant so for \( \theta < 90^\circ \) a nearly constant negative component is added and for \( \theta > 90^\circ \) a positive component. It follows that for \( \theta = 70^\circ \) the roll coordinate velocity profile is entirely negative and for \( \theta = 90^\circ \) it is entirely positive. In these cases the resolved \( \bar{w} \) profiles are dominated by transfer from zero at the surface to the value above it seen in Fig. 11 that they take the expected signs. Between \( \theta = 70^\circ \) and 90° the roll coordinate velocity profile takes different signs. At \( \theta = 80^\circ \) the profile is negative near the surface and inversion but positive in between. The resolved stresses reflect these changes but were also affected by a varying gravity wave contribution.

To understand the changes in the gravity wave component the heuristic model discussed above is considered. It gives a wave drag \( \sim \frac{1}{2} (F_r^{-2} - 1) F_r^2 w_2^2 \) (where \( F_r = u_r k/N \)) for \( F_r < 1 \) and zero for \( F_r > 1 \). Of the various parts of this expression, \( w_2^2 \) is simplest; it is roughly proportional to \( w_{max}^2 \) and as seen in Table 1 \( w_{max}^2 \) is a maximum for \( \theta = 80^\circ \). The Froude number variations depend on the relative changes in wavelength and \( u_r \). The velocity fields show that the wavelengths increase monotonically with \( \theta \) in the range 70° to 100°. \( u_r \) is roughly equal to the difference between the mean boundary layer velocity and the interior velocity. It follows from the discussion of velocity profiles given above that \( u_r \) varies from a negative value at \( \theta = 70^\circ \) to a positive value at \( \theta > 100^\circ \). At \( \theta \) about 100°, \( u_r \) is close to zero. Overall in the range \( \theta = 70^\circ \) to 100° the Froude number is greatest for \( \theta = 70^\circ \) and least for \( \theta = 100^\circ \).

At \( \theta = 70^\circ \), \( F_r \) is greater than unity and no gravity waves occur. The complete disappearance of wave drag is probably related to the size of the domain; a longer domain might have some energy on scales with \( F_r < 1 \). For \( \theta = 75^\circ \) the gravity waves are a maximum. Here \( w_{max}^2 \) is less than at \( \theta = 80^\circ \) and the Froude number must be close to the optimum value 1/\( \sqrt{2} \). For \( \theta > 80^\circ \), \( w_{max}^2 \) and \( F_r \) decrease away from the optimum.

**Dependence of results on \( l_0 \)**

Table 1 shows how the surface stress depends upon \( l_0 \). The choice of \( l_0 = 40 \text{ m} \) is made on the basis of atmospheric observations and above it is noted that the surface stresses did not alter for somewhat larger \( l_0 \). Here the effect of varying \( l_0 \) is examined in more detail by
Figure 12. Illustrating the alteration of the $\overline{w^2}$ profile with $l_0$ changed from 40 m to 120 m. The solid curves are the total (resolved and parametrized) and resolved $\overline{w^2}$ for $l_0 = 120$ m. The solid curve of smaller magnitude is the resolved part. The dashed curve is the total $\overline{w^2}$ with $l_0 = 40$ m.

Considering the case with $l_0 = 120$ m. The variations from the $l_0 = 40$ m case are slight in the total $\overline{w}$, $\overline{w}$, and $\overline{wB}$ stresses but more significant in the components of turbulence energy. Figure 12 shows total and resolved $\overline{w^2}$ for $l_0 = 120$ m, and the total $\overline{w^2}$ for $l_0 = 40$ m. In the upper part of the boundary layer the total $\overline{w^2}$ is unchanged by the variation of $l_0$. In the lower part there are significant changes in the shape of the profile. For $l_0 = 40$ m there is a

Figure 13. Illustrating the alteration of the $\overline{v^2}$ profile with $l_0$ changed from 40 m to 120 m. The solid curves are the total (resolved and parametrized) and resolved $\overline{v^2}$ for $l_0 = 120$ m. The solid curve of smaller magnitude is the resolved part. The dashed curve is the total $\overline{v^2}$ with $l_0 = 40$ m.
minimum between the surface layer and the middle of the boundary layer whilst for \( l_0 = 120 \text{ m} \) a monotonic decrease with height occurs. An appreciation of which of these profiles accords best with observations would be of value but suitable data is not available. Figure 13 shows the \( v^2 \) profiles. In the lower part of the boundary layer these agree well but near the inversion \( v^2 \) is reduced in the \( l_0 = 120 \text{ m} \) case. This corresponds to a substantial reduction in the gravity waves above the boundary layer.

From a consideration of continuity, the ratio of \( v^2/w^2 \) in the gravity waves should depend on the ratio of the vertical to horizontal wavenumbers i.e. \( v^2/w^2 \propto m^2/k^2 \) where \( m \) and \( k \) are the vertical and horizontal wavenumbers respectively. The linear theory of internal gravity wave generation in flow over shallow hills (Queney 1947) gives \( m^2/k^2 = (F_r^{-2} - 1) \) and as \( F_r \) tends to unity the vertical wavenumber becomes small and \( v^2/w^2 \) small. In the example with \( l_0 = 120 \text{ m} \) the surface stress in \( \gamma \)-direction increases slightly over that in the \( l_0 = 40 \text{ m} \) case. This changes the velocity profile giving an increase in \( u_* \) above the \( l_0 = 40 \text{ m} \) value. Since the gravity wave component is roughly half that in the \( l_0 = 40 \text{ m} \) case, it seems that the increase in \( F_r \) may have been greater than that needed to give optimum response. This being so \( (F_r^{-2} - 1) \) would be expected to be small and \( m^2/k^2 \) and \( v^2/w^2 \) small. This is the case, and the wave energy is mainly in the \( w^2 \) component with only a third as much in \( v^2 \). Finally Fig. 14 illustrates the profiles of \( \overline{u^2} \). These are remarkably unaffected by the change to \( l_0 \).

Figure 14. Illustrating the alteration of the \( \overline{u^2} \) profile with \( l_0 \) changed from 40 m to 120 m. The solid curves are the total (resolved and parametrized) and resolved \( \overline{u^2} \) for \( l_0 = 120 \text{ m} \). The solid curve of smaller magnitude is the resolved part. The dashed curve is the total \( \overline{u^2} \) with \( l_0 = 40 \text{ m} \).

Whilst considering the consequences of varying \( l_0 \) it is appropriate to note the consequences of varying the constant \( \alpha \) used near the surface and the constant \( \beta \) used near the inversion (see §2). The constant \( \alpha \) makes the values of mixing length increase more rapidly with height away from the surface. The values away from the surface are still limited to \( l_0 \). A variation in \( \alpha \) between zero and 5 causes little change to the flows presented here. In the \( l_0 = 40 \text{ m} \) case these changes are very small and confined to heights less than about 80 m. With \( l_0 = 120 \text{ m} \) the changes are significant but much less than those produced by changing \( l_0 \) from 40 m to 120 m. With much smaller magnitudes of Monin–Obukhov length than
considered here, $\alpha$ would be more important but its effects would be confined to the lowest grid spacings and felt as a change in the surface drag coefficient.

The consequences of varying $\beta$ are also fairly minor. As already noted $\beta > 3$ gives grid length instabilities at the base of the inversion. Although these induce unpleasant looking grid length scale features in the vertical profiles the other changes are very slight. $\beta < 3$ gives increased parametrized diffusion near the inversion but even with $\beta = 1$ the changes are slight and less than those seen when $l_0$ was varied between 40 and 120 m. However, with $\beta$ equal to zero large effects occur. No account of the effect of stability on the mixing length is taken and around the inversion very much larger parametrized transports are produced.

**Dependence of results on static stability $\partial B_0/\partial z$**

The basic static stability $\partial B_0/\partial z$ in the integrations presented above is $10^{-4}$ s$^{-2}$ corresponding to a Brunt–Vaisala frequency of $10^{-2}$ s$^{-1}$. The usual non-dimensional measure of this static stability is $w_*^{-1} \partial z_d/\partial t$ where $z_d$ grows with time in time dependent evolution. On the assumption that the entire surface buoyancy flux is used to heat the fluid in the boundary layer of depth $z_i$ to a uniform value equal to the undisturbed buoyancy at a height $z_d$, it follows that $w_*^{-1} \partial z_d/\partial t = b^{2/3} z_i^{-4/3} (\partial B_0/\partial z)^{-1}$. This equals $10^{-2}$ in the cases considered above. In this section the values $\partial B_0/\partial z = 0.3 \times 10^{-4}$ and $3 \times 10^{-4}$ are considered, i.e. $w_*^{-1} \partial z_d/\partial t = 3 \times 10^{-2}$ and $0.3 \times 10^{-2}$.

![Figure 15. Illustrating the alteration of the resolved scale $w^*$ profile with the basic stable stratification $\partial B/\partial z$ changed from $0.3 \times 10^{-4}$ s$^{-2}$ to $3.0 \times 10^{-4}$ s$^{-2}$. The solid curve is with the greater value.](image)

This variation of the basic static stability does not produce any dramatic changes and the consequences are well illustrated by the changes seen in the resolved $w^*$ profiles given in Fig. 15. With the larger static stability the gravity waves are reduced and the roll motion is more sharply confined to the boundary layer. The lower static stability gives greater wave energy and less confined roll motion. Consideration of the heuristic model suggests that the smaller static stability increases the Froude number towards the optimum value (as did varying $\theta$ to 75°). If the static stability were further reduced a sudden loss of wave energy would be expected (as occurred for $\theta \leq 70°$). Increasing the static stability simply decreases the Froude number with a consequent reduction in wave energy (as with increasing $\theta$ above 75°).
4. CONCLUSIONS

Mason and Sykes' (1980) model of horizontal roll vortices in a neutral planetary boundary layer has been extended to include a surface heat flux and an inversion cap. In the neutral model the eddies were driven by a shear flow instability. The results were sensitive to the roll orientation and showed large variations on a time scale of days. These time variations were related to geostrophic mean flow adjustments.

In the present study buoyancy forces produce more vigorous eddies, and shear energy production, though significant, is not dominant. The preferred orientation of the rolls is deduced to be 10° to the left of the geostrophic wind and corresponds to a minimum of shear across the rolls. This orientation also corresponds with observations and contrasts with the 10° to the right of the geostrophic wind found in the previous study.

There are still adjustments of the mean velocities on the geostrophic time scale but the turbulence statistics are not so strongly affected. The sensitivity of the structures to the roll orientation and the small scale mixing length parametrization is also less marked.

The results compare quite favourably with observations but a more detailed comparison will require careful modelling of particular situations and examination of the data for complicating influences. Although the main features of the boundary layer are found insensitive to roll orientation two features are significantly changed. The lateral momentum flux $\overline{v^2}$ is sensitive to the velocity profile across the rolls and changes quite markedly with roll orientation. Changes in the amplitude of internal gravity waves generated in the stable fluid above the boundary layer are also evident in the variation of the $\overline{v^2}$ profiles with roll orientation. These internal gravity waves have momentum stresses with magnitude up to about 10% of the total surface stress and are very sensitive to roll orientation and other parameters. A heuristic model of the boundary layer rolls perturbing the inversion and these perturbations of the inversion generating the gravity waves is discussed. The model provides an explanation of the observed changes and is verified by the wave energy being found to be a maximum when a suitably defined Froude number is close to unity.

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