An internal symmetric computational instability

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SUMMARY

A rapidly growing instability in energy–enstrophy conserving finite difference forms of the primitive equations is described. The instability is unusual in that it is purely internal. It arises because the linearized forms of the equations do not conserve momentum. The modifications necessary to control the instability are discussed.

1. INTRODUCTION

A rapidly growing instability in a multilevel primitive equation model which uses an energy–enstrophy conserving scheme is described. Most computational instabilities can be studied using a one-level model; in the present case the instability is purely internal and cannot occur in a one-level model. Much of the work was done some years ago (Hollingsworth and Källberg 1979, hereafter called HK) and was thought to be of only local interest.

Recent work (Arakawa and Lamb 1981; Mesinger 1981) suggests that it may be of somewhat wider interest. These latter developments on finite difference representations for the primitive equations have used the vector invariant form of the equations

\[
\frac{\partial u}{\partial t} - (f + \xi)v = -\frac{\partial (\phi + K)}{\partial x}
\]

\[
\frac{\partial v}{\partial t} + (f + \xi)u = -\frac{\partial (\phi + K)}{\partial y}
\]

\[
\frac{\partial \phi}{\partial t} + \frac{\partial (u\phi)}{\partial x} + \frac{\partial (v\phi)}{\partial y} = 0
\]

(1)

where \(\xi = v_x - u_y\), \(K = \frac{1}{2}(u^2 + v^2)\) and \(\phi = gh\).

To recover the usual advective form of the equations one must cancel common terms between \(\xi u, \xi v\) on the one hand and \(K_x, K_y\) on the other. The scheme studied here used the form of Eq. (1) and the cancellation does not occur even in the linearized form of the finite difference equations. As a result a spurious momentum source is present. The strong energy/enstrophy constraints on the model result in the occurrence of spurious energy conversions. Non-conservation of momentum is a common feature of advective schemes. It will not cause this instability provided momentum is conserved for the linearized finite difference equations.

In section 2 the finite difference scheme used is described. Section 3 describes the symptoms of the instability in a forecast model. A linear analysis of the scheme is presented in section 4. These results are reproduced in a much simpler analysis in section 5 and verified in nonlinear integrations in section 6. Section 7 shows that the same instability is present in more severe form in the scheme developed by Arakawa and Lamb. Section 8 discusses a modification of the scheme, due to Renner (1981), which eliminates the instability.

2. FINITE DIFFERENCE SCHEME

The finite difference scheme under consideration was developed by Sadourny (Burridge and Haseler 1977) on the C-grid of Arakawa and Lamb (1977).

The scheme is defined as follows:

\[
\frac{\partial u}{\partial t} - [ZV] + \delta_x(\phi + K) = 0
\]

\[
\frac{\partial v}{\partial t} + [ZU] + \delta_y(\phi + K) = 0
\]
\[ \frac{\partial h}{\partial t} + \delta_x U + \delta_y V = 0 \]

where \( U = \vec{h} u \), \( V = \vec{h} v \), \( K = \frac{3}{2}(u^2_x + v^2_y) \) and

\[
Z = (f + \delta_x v - \delta_y u)\vec{h}, \quad [ZV] = \frac{3}{2}V^{xy}\vec{Z}^x + \frac{3}{2}V^{xy}\vec{Z}^y - \frac{1}{3}V^{xy}\vec{Z}^x - \frac{1}{3}V^{xy}\vec{Z}^y,
\]

\[
[ZU] = \frac{3}{2}U^{xy}\vec{Z}^x + \frac{3}{2}U^{xy}\vec{Z}^y - \frac{1}{3}U^{xy}\vec{Z}^x - \frac{1}{3}U^{xy}\vec{Z}^y.
\]

For a fluid with \( \vec{h} \) constant and no divergence the scheme reduces to the \( J_7 \) Jacobian of Arakawa (1966). This scheme (which we shall call the EE scheme) conserves energy and, provided there is no divergence in the flow, conserves potential enstrophy, defined as \( \vec{h}^{xy}Z^2 \). We also refer to a simpler scheme developed by Sadourny (1975) where the rotation terms \( ZU, ZV \) are written as \( ZV = \vec{V}^{xy}\vec{Z}^x \) and \( ZU = \vec{U}^{xy}\vec{Z}^y \). This latter scheme, which conserves potential enstrophy but not energy, is referred to as the E scheme. More recently Arakawa and Lamb (1981) developed a finite difference treatment of these equations which conserves energy and enstrophy for a general flow. Their scheme is more complicated than the EE scheme but, as shown below, it, too, permits the instability.

3. Symptoms of the Instability in a High-Resolution Model

Initial work on the ECMWF gridpoint model used the EE scheme with a horizontal resolution of 3.75°. Integrations showed that the jets tended to weaken after a few days; this was attributed to the coarse resolution. The first integrations at higher resolution (1.875°) exhibited a catastrophic weakening of the jets after two or three days.

Figure 1 shows the 500 mb height field at days 2 and 3 in a nine-level integration from real data using the EE scheme and with no physical parametrizations. The plotted field is truncated at zonal wave-number twenty. Between days 2 and 3 there has been a dramatic loss of intensity in the jets over the Pacific and western Europe. If the integration is continued, the fields degenerate into small-scale noise by day 7. Integrations either with the dry convective adjustment used by Smagorinsky et al. (1965) or with a complete set of physical parametrizations (for radiation, the planetary boundary layer, dry convection, moist convection, large-scale rain and internal diffusion) showed the same collapse of the jets between days 2 and 3. In the adiabatic integration the total kinetic energy increased steadily with time as the long-wave kinetic energy decreased. In the run with just dry convection, the total kinetic energy stayed more or less constant as the long-wave energy decreased. In the third run the kinetic energy decreased steadily. This suggests that the model was generating small-scale noise which was exacerbated by convective overturning in the absence of dry convection. When internal dissipation was included, as well as dry convection parametrization, this small-scale noise was damped but the loss of energy in the longer waves still occurred. Corresponding runs with the E scheme showed no indication of pathological behaviour.

4. Analysis for a Multilevel Model

In this section the results of a linear perturbation analysis of the EE scheme in a multilevel model with a simple basic state on an \( f \)-plane are presented. We consider the primitive equations in sigma coordinates and we suppose that in the basic state the temperature is isothermal, with temperature \( T_0 \), and the uniform zonal flow \( \vec{u} \) is balanced by a meridional pressure gradient \( \vec{p}_a(y) \). We linearize about this basic state and consider perturbations for which \( \partial / \partial x = 0 \). The finite difference equations may be found in Burridge and Haseler (1977). The derivation of the perturbation equations is straightforward (see HK for details). As presented by Burridge and Haseler the finite difference scheme for the rotation terms applies to \([\vec{p}_a^* u(Z/\vec{p}_a^*)]\) and \([\vec{p}_a^* v(Z/\vec{p}_a^*)]\). It was found (in the nonlin-
Figure 1. 500 mb fields at day 2 (left) and day 3 (right) in a run with the EE formulation and no physical parametrizations. The fields have been truncated at zonal wave-number twenty.
ear model discussed later) that the instability was not affected by these terms in $p_*$ and so we present here the results for the slightly simpler scheme for $[uZ]$ and $[vZ]$.

To specify boundary conditions for the analytical calculation, the solution has to be periodic with period $N \Delta y$. Thus we have $(3k + 1)N$ equations for $(3k + 1)N$ unknowns, where $k$ is the number of levels. The problem could be much simplified if it were separable. This is not the case because of the variation of the mean pressure field with latitude. We therefore write the problem in the form $dX/dt = AX$, where $X$ is a column vector of all the perturbation quantities and the matrix $A$ contains the dependencies of the parameters of the problem. In all there are $(3k + 1)N$ eigensolutions of this equation. Of these eigensolutions, $N$ have zero frequency, corresponding to steady-state solutions. A further $kN$ also have zero frequency corresponding to the Rossby-wave solutions. The remaining $2kN$ solutions will correspond to northward and southward moving gravity waves.

To identify the vertical mode to which each of these frequencies corresponds one uses the ratios of the free gravity wave speeds to each other in an inspection of the eigenfrequencies. The eigenvalues and the fastest growing eigenfunction have been calculated for a range of models with between two and nine levels and periodicities of between $3 \Delta y$ and $25 \Delta y$. In each of these cases the same general features were found. The solutions for $N = 3$ or 4 with five equally spaced levels were quite typical of the unstable modes. For every internal mode there were exponentially growing solutions, but the external mode was always neutral. Figure 2 shows the variation of the largest growth rate corresponding to each internal mode for $N = 3$, as $\bar{u}$ is varied from 0 to 40 m s$^{-1}$, in the five-level model with $\Delta y = 100$ km. The highest internal mode is the most unstable and there is an inverse dependence of growth rate on the gravity wave speed, $c$. For example, at $\bar{u} = 40$ m s$^{-1}$ in Fig. 2 the values for the growth rate $\sigma_\nu$ are approximately $0.5 \times 10^{-4}$, $0.2 \times 10^{-4}$, $0.1 \times 10^{-4}$ s$^{-1}$ for the three highest internal modes. The corresponding phase speeds are

![Figure 2](image)

Figure 2. Growth rate as a function of $\bar{u}$ for the four internal modes when $N = 3$ (i.e. the north–south per $3 \Delta y$). The curves are labelled II, III, IV, V in order of decreasing gravity wave speed. ($f = 1.1 \times $

$\Delta y = 100$ km, 5 equally spaced levels.)
10.98, 26.07 and 58.85 m s\(^{-1}\), so that the inverse relationship between \(\sigma_v\) and \(c\) is very marked. The growth rates have an almost exactly linear dependence on the strength of the basic flow; departures from linearity were within the accuracy of the drawing. Finally the growth rates are very large. For the largest growth rates shown on Fig. 2, \(\sigma_v/f \sim 0.5\), so that the e-folding times are six to seven hours.

We next consider how the growth rate varies with resolution. Fig. 3 shows the variation of growth rate with \(\Delta y\) for the fastest growing mode (with \(N = 4\)) in two models, one with five equally spaced levels and one with nine levels as in the GFDL model referred to earlier. In both cases we find a marked variation of growth rate with horizontal grid size. The nine-level model has larger growth rates and more sensitivity to \(\Delta y\). The slowest moving gravity wave in the nine-level model has a phase speed of 251 m s\(^{-1}\) as

![Figure 3. Variation of growth rate with \(\Delta y\) for \(N = 4\) and \(u = 40\) m s\(^{-1}\): (a) nine-level model with GFDL distribution of levels; (b) model with 5 equally spaced levels.](image-url)
compared with 10.98 m s\(^{-1}\) in the five-level model. Note that the e-folding time decreases by a factor of two, from 12 to 6 h in the nine-level model as \(\Delta y\) is changed from 400 to 200 km. This is consistent with the rapid decay of the jets in our high-resolution forecast model (1.875°) compared with the much slower decay in the same model with a resolution of 3.75°, on which our initial work was done. In the latter case the decay of the jets over a period of several days had been ascribed to the coarse horizontal resolution. In Fig. 4 we

![Diagram](image-url)

Figure 4. Structure, as a function of height and latitude, of \(T', u', v'\) for the most unstable mode in a five-level model with \(\tilde{u} = 40\) m s\(^{-1}\), \(\Delta y = 100\) km, \(f = 1.1 \times 10^{-4}\) s\(^{-1}\) and \(N = 4\). The e-folding time is 6.6 h.

show the structure of the most unstable mode for \(N = 4\) in the 5-level model with \(\tilde{u} = 40\) m s\(^{-1}\). There are four zero crossings in the \(v\) field so that we do indeed have the highest vertical mode.

In summary, the multilevel analytic calculations show that instability is possible. There is a linear dependence of the growth rate on \(\tilde{u}\); an inverse dependence on the equivalent depth and a marked sensitivity to resolution for the highest internal mode. In the next section we present a simple theory which explains these results both qualitatively and even quantitatively.

5. A SIMPLIFIED ANALYSIS FOR THE INTERNAL MODES

Linearized analysis of the EE scheme in a one-level model shows that the scheme is
stable in all circumstances, so long as the mean height of the fluid is positive. This has been confirmed by exhaustive numerical experimentation with analytic and real data over a wide range of mean depths of the fluid. The linearized analysis of the last section shows that in a multilevel model the external mode is never unstable but that instability occurs for all internal modes. In this section we discuss the nature of the instability in an approximate manner using the shallow water equations. The approximation that we make is incorrect for an external mode but the results clearly justify it for the internal modes.

Let us linearize the shallow water equations (1) about the following basic state: \( \bar{u} = \text{constant}, f = \text{constant}, \bar{v} = 0, \phi = \phi_0 + \phi_1(y) \), where \( f\bar{u} = -\partial \phi_1/\partial y \), \( Z = f \) and \( \phi_0 \) is constant. The linearized equations for \( x \)-independent perturbations are

\[
\begin{align*}
\partial u'/\partial t & = Zu' \\
\partial v'/\partial t & = -Z\bar{u}' - \partial(\bar{u}u')/\partial y - \partial \phi'/\partial y \\
\partial \phi'/\partial t & = -\partial(\phi_0 v')/\partial y - \partial(\phi_1 v')/\partial y.
\end{align*}
\tag{2a}
\tag{2b}
\tag{2c}
\]

In the \( v \) equation, (2b), the terms \(-Z\bar{u}'\) and \(-\partial(\bar{u}u')/\partial y\) cancel since \(-Z' = \partial u'/\partial y\).

We now depart from an exact analysis by neglecting the term \( \partial(\phi_1 v')/\partial y \) in (2c). This term prevents the instability in a 1-level model and so its neglect is improper if we are studying an external mode. However, we are interested in the internal modes. For such modes we feel justified in neglecting this term on physical grounds; in the analysis of the last section the basic state was isothermal and so the mean pressure gradient had no projection on the internal modes. The approximation is justified \textit{a posteriori} by the fact that many features of the exact linear analysis are reproduced in the approximate analysis of this section. With this approximation we have the following simple system

\[
\begin{align*}
\partial u'/\partial t & = fu' \\
\partial v'/\partial t & = -f\bar{u}' - \phi'_y \\
\partial \phi'/\partial t & = -\partial(\phi_0 v')/\partial y.
\end{align*}
\tag{3a}
\tag{3b}
\tag{3c}
\]

We now seek the analogue for Eq. (3a, b, c) using the EE finite difference scheme. If we assume the same basic state as before and again neglect the term in \( \phi_1 \), then the linearized forms of the finite difference equations for \( x \)-independent perturbations are

\[
\begin{align*}
\partial u'/\partial t & = f\bar{u}' \\
\partial v'/\partial t & = -f\bar{u}' - \frac{1}{2}i\bar{u}(Z_{j-1}^* + 4Z_j^* + Z_{j+1}^*) + i\bar{u}Z_j^* - \partial \phi'/\partial y \\
\partial \phi'/\partial t & = -\phi_0 \partial \phi'/\partial y.
\end{align*}
\tag{4a}
\tag{4b}
\tag{4c}
\]

Here \( j - 1, j, j + 1 \) refer to northern, central, and southern latitudes. The terms corresponding to \(-\frac{1}{2}i\bar{u}(Z_{j-1}^* + 4Z_j^* + Z_{j+1}^*) + i\bar{u}Z_j^*\) in the \( v \) equation cancel in the continuous case. In the finite difference case they give

\[-(i\bar{u}/6\Delta y)(-u_{j-2} - 3u_{j-1} + 3u_j + u_{j+1}) - (i\bar{u}/\Delta y)(u_{j-1}' - u_j').\]

These terms do not cancel but rather (in the special case of a three grid wave, when \( u_{j-2} = u_{j+1}' \)) they combine to give \( \frac{1}{2}i\bar{u}Z_j^* \). For the rest of this qualitative analysis we concentrate on the special case of the three grid wave.

If we assume solutions proportional to \( \exp(\pm il\Delta y) \), where \( l = \pm 2\pi/(3\Delta y) \) then Eqs. (4) become, for a three grid wave,

\[
\begin{align*}
\partial u'/\partial t & = \Gamma f\bar{u}' \\
\partial v'/\partial t & = -\Gamma f\bar{u}' - \frac{1}{2}i\bar{u}\Lambda u' - i\Lambda \phi' \\
\partial \phi'/\partial t & = -c^2i\Lambda \phi
\end{align*}
\tag{5a}
\tag{5b}
\tag{5c}
where \( \Gamma = \cos(\frac{1}{2}l\Delta y) \), \( \Lambda = -\sin(\frac{1}{2}l\Delta y)/(\frac{1}{2}\Delta y) \), and \( c^2 = \phi_0 \). Equations (5) are the finite difference analogue of Eqs. (2). We note the spurious momentum source in the \( v \) equation. The dispersion relationship for the frequency, \( \sigma \), is given by

\[
\sigma^2 = -(f^2\Gamma^2 + \Lambda^2 c^2) - \frac{1}{2}i\bar{f}\bar{u}\Gamma\Lambda.
\]

(6)

For typical parameter values \( f \sim 10^{-4}\text{ s}^{-1}, c \sim 10\text{ m s}^{-1}, \Delta y \sim 200\text{ km}, \bar{u} \sim 40\text{ m s}^{-1} \) it can be shown that the growth rate, \( \sigma_r \), is given by

\[
\sigma_r \sim f\bar{u}/8c.
\]

(7)

This relationship summarizes in a very simple form two of the most important results of the last section. These are the linear dependence of growth rate on \( \bar{u} \) and the inverse dependence on \( c \). Quantitatively Eq. (7) is of the right order of magnitude. For \( \bar{u} = 40\text{ m s}^{-1} \) and \( c = 10\text{ m s}^{-1} \), \( \sigma_r \sim \frac{1}{2}f \) so that the e-folding time is about 6 h. However Eq. (6) says nothing about the third important aspect: the dependence of growth rate on resolution.

To examine this we show the dependence of \( \sigma_r \) on \( c \) and \( \Delta y \) for fixed values of \( f(=1.1 \times 10^{-4}\text{ s}^{-1}) \) and \( \bar{u}(=40\text{ m s}^{-1}) \) in Fig. 5. It is seen that, for \( c \) large, the growth rate is almost independent of \( \Delta y \) while for \( c \) small there is a significant variation of growth rate with \( \Delta y \). A quantitative comparison between the exact and approximate calculations shows agreement to three significant figures when \( c \sim 2.51\text{ m s}^{-1} \).

To summarize, the approximate treatment of this section gives results which: (i) provide a simple explanation of the qualitative features of the instability; and (ii) give very exact quantitative results. The physical interpretation of this agreement between the full
analytic calculation and the simplest possible analogue for the instability is that the non-conservation of momentum dominates the full analytic calculation. At this point we may recall that in the E scheme (Sadourny 1975) the term \(-Z \ddot{u}\) is written \(-Z \ddot{u}'\). Hence the term \(-Z \ddot{u}\) has the finite difference expression \(-\ddot{u}Z_j\), when \(\partial / \partial x = 0\), and so it cancels exactly the term \(-\ddot{u}Z_0\). Thus the linearized equations with the E scheme have neither spurious momentum nor spurious energy sources and so will be stable.

6. NONLINEAR INTEGRATIONS

In this section we present some results from integrations with a nonlinear model for the simple initial states discussed earlier. The model is an adiabatic \(f\)-plane channel version of the ECMWF model and has been described by Källberg (1977). Variations in the east–west direction were suppressed. The presence of rigid boundaries at the north and south might be thought to weaken the relevance of the analysis of section 4, which assumed periodicity in the north–south; this proves not to be the case. The integrations were made with a model which had five equally spaced levels in the vertical and 32 points in the north–south with a grid size of 100 km. The initial state consisted of an isothermal atmosphere with a uniform zonal flow of 40 m s\(^{-1}\) at all levels. The zonal flow was balanced by a meridional pressure gradient. To this was added an \(x\)-independent perturbation of 1 m s\(^{-1}\) in the \(v\) field. The integration proceeded without any significant features for two days. On day three we began to see a small but significant distortion of the zonal flow. There were bands of convergence and divergence at each level and a close inspection showed that the convergence and divergence had opposite signs between layers. The dominant meridional wave-numbers were 4 and 8. This presumably is related to the number of points in the grid. Figure 6 shows the wind fields at day 4 at levels 4 and 5 (700 mb and 900 mb). We see that the zonal flow has been completely disrupted. The integration, which should have preserved a zonal flow of 40 m s\(^{-1}\) with some small perturbations, has gone completely awry with easterlies and westerlies in excess of 60 m s\(^{-1}\). A plot of the evolution of the r.m.s. divergence of the wind field in the period from day three to day four shows that the divergence is growing exponentially in time with an estimated e-folding time of 6-46 h. The calculated e-folding time for a disturbance on this flow is 6-64 h, if the north–south periodicity is 8 grid points. The agreement in the structure and growth rate of the disturbance with the analytical calculation is sufficiently close to suggest that the calculation has captured the main features of the disturbance in the model. Finally we mention that energy conservation is satisfied by a redistribution of mass through changes in the surface pressure field.
7. The Arakawa and Lamb scheme

Recently Arakawa and Lamb (1981) proposed a finite difference scheme (the AL scheme) for the shallow water equations which conserves energy and enstrophy for a general flow. The AL scheme suffers from the same weakness as the EE scheme. As we saw earlier the problem in the EE scheme was that in the linearized $v$ equation the term

$$-\frac{1}{2}u(Z'_{j-1} + 4Z'_j + Z'_{j+1})$$

coming from the $Zu$ term did not cancel the term $\tilde{u}Z'_j$ coming from the term $\partial K/\partial y$. In the AL scheme the term coming from $\partial K/\partial y$ is exactly the same while the rotation term it should cancel is

$$-\frac{1}{2}u(Z'_{j-1} + 2Z'_j + Z'_{j+1}).$$

For a three grid wave in the EE scheme the non-zero residual is $\frac{1}{2}\tilde{u}Z'_p$, while for the AL scheme the non-zero residual is $\frac{3}{4}\tilde{u}Z'_p$. Thus the non-cancellation is more severe in the AL scheme than in the EE scheme and so the instability is likely to be more rapid.

![Figure 7. Variation of growth rate with $\Delta y$ for the AL scheme (continuous line) and the EE scheme (dashed line). The model uses the nine-level GFDL distribution; $\bar{u} = 40$ m s$^{-1}$, $f = 1.1 \times 10^{-4}$ s$^{-1}$ and $N = 3$.](image)
This is demonstrated in Fig. 7 which compares the e-folding times of the most unstable modes in the AL and EE scheme for a 9-level model as used by Smagorinsky et al. (1965). The growth rates are significantly higher for the AL scheme.

8. Modification of the VK Term

Clearly the instability can only be controlled by a redefinition of the finite difference form for VK. The conservation properties of the EE scheme can be retained and the instability removed by making the following redefinitions of the mass flux and kinetic energy (Renner 1981).

\[
U = (\frac{2}{3} \hat{h}^{xy} + \frac{1}{3} \hat{h}^y)u \\
V = (\frac{2}{3} \hat{h}^{xy} + \frac{1}{3} \hat{h}^x)v \\
K = \frac{1}{2} \left\{ \frac{1}{3} (u^{xy} + \overline{v^{xy}}) + \frac{1}{3} (u^x + \overline{v^y}) \right\}.
\]

The linear analysis of this scheme shows no potential for instability. We have made nonlinear integrations on this scheme in the \( f \)-plane channel and it behaves well with no indication of trouble. We have not implemented it in a global model because of the expense and because experience indicates that the simpler E scheme performs quite satisfactorily for forecasting purposes.

Arakawa and Lamb (1981) indicate that they have found the instability in their model also and that a modification of the mass fluxes and kinetic energy along the above lines eliminates the instability. Mesinger (1981) has implemented a version of the above modification in his model and has also found satisfactory performance.

8. Conclusion

We have outlined the symptoms and cure for a rapidly growing instability in certain types of finite difference formulations of the primitive equations. The instability is unusual in that it is purely internal and cannot occur in a one-level model. The instability arises because of a non-cancellation of certain terms in the finite difference formulation of the momentum equations.

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