Mountain torque and angular momentum in barotropic planetary flows: equilibrium solutions

By B. L. SAWFORD and J. S. FREDERIKSEN
CSIRO Division of Atmospheric Physics, Station Street, Aspendale 3195, Australia

(Received 16 March 1982; revised 26 October 1982)

SUMMARY

The climatic-mean-value of the total zonal relative angular momentum of the atmosphere in topographically forced inviscid barotropic models on a sphere is examined using the methods of equilibrium statistical mechanics. Models incorporating wave–wave interactions as well as simplified systems in which only wave–zonal-flow interactions are allowed are considered. It is shown analytically that for a given truncation the equilibrium state of the system does not depend on the presence of wave–wave interactions although the detailed evolution towards it does.

We interpret our results in terms of a generalization of the properties of the equilibrium solutions for unforced barotropic flow. For realistic resolution (with truncation wavenumber $\approx 15$) most flows evolve towards an equilibrium with westward angular momentum, and the solutions are strongly dependent on the initial conditions. An important exception is the case when the flow initially has westward angular momentum, when the equilibrium state remains close to the initial state regardless of resolution. An upper bound on the climatic-mean-value of the solid-body rotation component of the flow is derived. In general, highly truncated systems are insensitive to the initial conditions and evolve towards a state in which the angular momentum is eastward and close to the upper bound.

Reasonable rescaling of the global topography used has little influence on the climatic-mean angular momentum, but the case of no topography represents a singular limit in which angular momentum is conserved.

1. INTRODUCTION

The earth and atmosphere exchange angular momentum through both surface drag and mountain torque. Recently Egger and Metz (1981) investigated the effect of the latter on the long-term balance of the total zonal relative angular momentum, $M$, in the atmosphere. In order to isolate the mountain torque mechanism and to reduce the problem to tractable proportions they used an inviscid barotropic model for their analysis which involved two separate approaches. First they treated analytically a highly truncated 'one-mode' system consisting of a solid-body-rotation zonal flow and one wave component over topography having a single spectral component. They then carried out numerical integrations on a $\beta$ plane and on a sphere with less (but still rather severely) truncated systems containing up to five zonal wavenumbers. They found in the latter case that for integrations of several weeks or more, $M$ has a climatic-mean-value which can be understood reasonably well from the results of their one-mode analysis.

The results obtained from such simple systems may be useful for understanding aspects of more realistic cases including viscosity and forcing due to differential heating. However, the conclusions reached by Egger and Metz are limited by the severe truncation used. The fact that they study severely truncated systems is understandable since the method of numerical integration of the barotropic equations to obtain the equilibrium solutions is rather expensive for less severe truncations. Unfortunately, as will be shown, the results for more reasonable resolution including zonal wavenumbers up to 15, say, are quite different from those with severe truncation.

The equilibrium solutions for the inviscid barotropic model may in fact be obtained much more simply by using the methods of statistical mechanics. We (Frederiksen and Sawford 1980, 1981; Frederiksen 1982) have recently derived such solutions for flow on a sphere and have used them to treat a number of problems in both unforced and topographically forced barotropic models. In this paper, we apply these statistical mechanical methods to the problem posed by Egger and Metz, which may be stated explicitly as follows: When does a statistical equilibrium (or climatic-mean) state exist in the topographically forced inviscid barotropic model and what is the mean angular momentum of that state? We will be particularly interested in the effect of resolution, the nature and
The strength of the initial flow field and the height of the topography on the climatic-mean angular momentum. Because the statistical mechanical solutions are semi-analytic we are able to undertake a much more comprehensive treatment than Egger and Metz and in particular we treat systems with more realistic resolution. Furthermore, the relative simplicity of the solutions makes interpretation and extrapolation of the results much more straightforward than is the case with numerical integration.

The plan of the paper is as follows. In section 2 we give the details of the model, introduce briefly the equilibrium solutions and derive some analytical results. In section 3 we present and analyse numerical results for the equilibrium angular momentum (we actually present the equilibrium solid-body rotation component of the flow), while our conclusions are summarized in section 4.

2. Theory

(a) Model details

Taking \( a \) (earth’s radius) and \( \Omega^{-1} \) (earth’s angular velocity)\(^{-1} \) as length and time scales, the non-dimensional equations defining our spherical equivalent barotropic model are

\[
\partial \nabla^2 \psi / \partial t + J(\psi, \nabla^2 \psi + 2\mu + h) = 0
\]

where \( h = 2\mu g A H / R T_0 \) and \( J(f, g) = (\partial f / \partial \lambda)(\partial g / \partial \mu) - (\partial f / \partial \mu)(\partial g / \partial \lambda) \).

In Eq. (1), \( \psi \) is the streamfunction, \( \mu \sin(latitude) \), \( \lambda \) longitude, \( t \) time, \( H \) the height of the topography, \( R \) the gas constant for air, \( T_0 \) the horizontally averaged global surface temperature, \( g \) the acceleration due to gravity and \( A \) the value of the vertical profile factor at 1000 mb (see, e.g., Holton (1972) p. 129). Throughout this paper we use the values \( A = 0.8 \), \( R = 287 \) J kg\(^{-1} \) K\(^{-1} \) and \( T_0 = 273 \) K. These are equivalent to the scale height of 10 km used by Egger and Metz. Note that throughout this paper all equations are written in non-dimensional form. However, some quantities are given in dimensional form in order to convey their magnitude better. The same symbol is used for both non-dimensional and dimensional forms of the same quantity, the latter being distinguished by the accompanying units.

In order to write Eq. (1) in spectral form, both the streamfunction and the forcing function, \( \hat{h} = 2\mu + h \), are expanded in spherical harmonics:

\[
\psi = \sum_{m = -J}^{J} \sum_{n = |m|}^{|m| + J} \psi_{mn}(\mu)e^{im\lambda},
\]

\[
\hat{h} = \sum_{m = -J}^{J} \sum_{n = |m|}^{|m| + J} \hat{h}_{mn}(\mu)e^{im\lambda}.
\]

Here \( \psi_{mn} = \psi^*_{-m} \) and \( \hat{h}_{mn} = \hat{h}^*_{-m} \), since we take \( P_n^{-m}(\mu) = P_n^m(\mu) \).

The functions \( P_n^m(\mu) \) are orthonormal associated Legendre functions and \( \psi_{mn} \) and \( \hat{h}_{mn} \) are spectral amplitudes, \( m \) is the zonal wavenumber, \( n \) the total wavenumber and \( J \) is the rhomboidal truncation wavenumber. The rhomboidal truncation scheme is used throughout this article.

Insertion of Eq. (2) into Eq. (1) yields the truncated spectral equations

\[-n(n+1) \partial \psi_{mn}(t) / \partial t = i \sum_{p, q, r, s} K^r_{pq} \psi_{rs}(t) \hat{h}_{pq} +
\]

\[+ \frac{1}{2} i \sum_{p, q, r, s} \{ (s(s+1) - q(q+1)) K^r_{pq} \psi_{rs}(t) \psi_{pq}(t), \]

where the interaction coefficient \( K^r_{pq} \) and the selection rules which must be satisfied are given in Eq. (2.3) of Frederiksen and Sawford (1980).
Equation (1) and more particularly the truncated Eqs. (3) conserve kinetic energy

\[ E = \frac{1}{4} \sum_{m=-J}^{J} \sum_{n=|m|}^{|m|+J} n(n + 1) |\psi_{mn}|^2. \]  

(4)

and potential enstrophy

\[ F = \frac{1}{4} \sum_{m=-J}^{J} \sum_{n=|m|}^{|m|+J} \left( n(n + 1) |\psi_{mn}|^2 + \hat{h}_{mn} \right)^2. \]  

(5)

The component of mean relative angular momentum parallel to the earth's axis of rotation is

\[ M = \frac{1}{4\pi} \int (1 - \mu^2)(\partial \psi / \partial \mu) ds' = -\sqrt{\bar{\psi}} \psi_{01} = \frac{2}{3} u_{01}, \]  

(6)

where the integration is over the surface of the sphere. Thus the mean angular momentum is determined entirely by the (0, 1) mode of the streamfunction field or equivalently by the solid-body rotation velocity coefficient \( u_{01} = -\sqrt{\bar{\psi}} \psi_{01} \), and it is these quantities which are of direct interest here.

(b) Equilibrium solutions

Full details of the derivation of the equilibrium solutions are given in Frederiksen and Sawford (1980, 1981) and are not repeated. The essential requirements are the constraint equations (4) and (5), and that the equations of motion (3) satisfy Liouville's equation. The solutions consist of ensemble averages (over a large number of realizations of the system, all with the same \( E \) and \( F \)) which are equivalent to long-time averages over a single realization (Frederiksen and Sawford 1980 and references therein). For the forced system considered here, the expectations of the spectral coefficients are non-zero,

\[ \langle \psi_{mn} \rangle = \beta \hat{h}_{mn}/(\alpha + \beta n(n + 1)). \]  

(7)

The expectations for the energy, \( E_{mn} \), and potential enstrophy, \( F_{mn} \), in the \((m, n)\) and \((-m, n)\) modes are

\[ E_{mn} = E_{-mn} = \frac{1}{2}(\alpha + \beta n(n + 1))^{-1} + \frac{1}{2} n(n + 1) |\langle \psi_{mn} \rangle|^2 \]  

(8)

\[ F_{mn} = F_{-mn} = \frac{1}{2} n(n + 1)/(\alpha + \beta n(n + 1)) + \frac{1}{2} \alpha^2 \beta^{-2} |\langle \psi_{mn} \rangle|^2. \]  

(9)

The parameters \( \alpha, \beta \) are determined by equating the expectations for the total energy and potential enstrophy to the invariant values \( E, F \) determined by the initial conditions. The statistical mechanical equilibrium solutions given here are rigorous for sufficiently large number of degrees of freedom as discussed in Frederiksen and Sawford (1980, 1981). In practice, as few as ten degrees of freedom may be sufficient for barotropic flows to be ergodic and for statistical mechanical solutions to be equivalent to long-time averages (see, for example, Basdevant and Sadourny 1975; Kells and Orszag 1978; Bell 1980). In particular, we find that for severe truncations our statistical mechanical solutions agree quite well with the numerical results of Egger and Metz.

Since the ensemble averages are equivalent to long-time averages, Eq. (7) clearly represents the stationary part of the streamfunction field. While the expectation of the transient part of the streamfunction is zero, the energy of the transient field is not and we see that the energy expectation Eq. (8) consists of two parts. The second term on the right clearly represents the energy of the stationary or climatological-mean flow, while the first term represents the energy of the transient flow.

The derivation of the statistical mechanical solutions involves the usual axiom of equal \textit{a priori} probability for all accessible points in phase space. It should be noted, however, that the same equilibrium solutions may also be obtained through the turbulence closure approach using the direct interaction approximation, the test-field model or
the eddy-damped quasi-normal Markovian equation. In fact, Carnevale (1979) and Carnevale et al. (1981) have proved a Boltzmann-type H-Theorem for inviscid barotropic flow over topography described by the eddy-damped quasi-normal Markovian equation which is a statistical closure derived from the basic field equations. This theorem states that the entropy of the flow will increase until the statistical mechanical equilibrium solution is reached.

In the absence of topographic forcing it can be shown (Salmon et al. 1976) that there is a unique equilibrium solution. In the topographically forced case we know of no such proof; however, there is no numerical evidence that any other statistical mechanical equilibrium solution (which survives under the addition of small random perturbations to the initial conditions) exists for Eq. (3) (Frederiksen and Sawford 1981; Egger and Metz 1981; Frederiksen 1982; see also Bretherton and Haidvogel 1976; Herring 1977; Holloway 1978).

Finally, we emphasize that throughout this article the equilibrium solutions are obtained not by numerical integration of the primitive equations (1) or (3) but by solving directly Eqs. (7) to (9).

(c) Neglect of eddy interactions

In many model studies (for example of wave–zonal-flow interactions or as in the present case of topographic forcing) only nonlinear interactions between the zonal flow \((m = 0\) terms) and the eddy flow \((m \neq 0\) terms) are included in the spectral evolution equations and interactions among the eddy terms are ignored (Matsumo 1971; Holton 1976; Lordi et al. 1980). Such simplified models are often considered to incorporate the essential features of the nonlinear interactions and are of course simpler to analyse than the fully nonlinear equations. Egger and Metz, for example, use the simplified evolution equations

\[
n(n + 1) \frac{\partial \psi_{mn}}{\partial t} = n(n + 1) \frac{\partial \psi_{mn}}{\partial t} = im\sqrt{2} \psi_{01}[\psi_{mn}(n(n + 1) - 2) - h_{mn}] + 2im\psi_{mn} \tag{10}
\]

and

\[
\frac{\partial \psi_{01}}{\partial t} = \sqrt{n} \sum_{m} \psi_{mn} h_{mn}^{*} \tag{11}
\]

in which only eddy terms of the topography have been retained, the zonal flow is restricted to the solid-body rotation term, \(\psi_{01}\), and interactions between eddy flow terms have been ignored.

We show in the appendix that Eqs. (10) and (11) conserve energy and potential enstrophy and satisfy Liouville’s equation. Thus these simplified equations have the same equilibrium solution as the fully nonlinear system with the same energy and potential enstrophy. That is, for a given initial flow field and topography, the climatic-mean states are identical for the fully nonlinear system Eq. (3) and the simpler system Eqs. (10) and (11), despite the fact that the detailed evolution of the systems will be quite different. We can thus now understand why Egger and Metz found their results to be insensitive to the inclusion of eddy interactions. This rather surprising result holds even if only a single topographic mode (necessarily with both \(\pm m\) components in order to ensure reality) is retained.

(d) Limits on the climatic-mean angular momentum

The equilibrium expectation of \(\psi_{01}\) is, from Eq. (7),

\[
\langle \psi_{01} \rangle = h_{01}(\alpha/\beta + 2). \tag{12}
\]

Since the topographic contribution to \(h_{01}\) is much smaller than that due to the Coriolis parameter, so that \(h_{01} \approx 2\sqrt{3}\), we find that approximately

\[
\langle u_{01} \rangle = -2(\alpha/\beta + 2). \tag{13}
\]
Thus only the ratio \( \alpha/\beta \) influences these expectations and hence the mean angular momentum.

Now since the transient component of the kinetic energy is necessarily positive for all modes, we have the constraint
\[
\alpha + \beta \eta > 0 \quad n_{\min} \leq n \leq n_{\max}
\]
(14a)

where
\[
\eta = n(n + 1),
\]
(14b)

and \( n_{\min} \) and \( n_{\max} = 2J \) are the minimum and maximum total wavenumbers retained in the truncated system of equations. For all cases considered here \( n_{\min} = 1 \).

For positive \( \beta \), Eqs. (14) require \( -\alpha/\beta < \eta_{\min} = 2 \) and so from Eq. (13) \( \langle u_{01} \rangle \) is always negative, i.e. the solid-body rotation is always westward and the mean angular momentum negative.

For negative \( \beta \), Eqs. (14) require \( -\alpha/\beta > \eta_{\max} \) and hence from Eq. (13) the solid-body rotation is eastward and bounded above,
\[
0 < \langle u_{01} \rangle < 2/(\eta_{\max} - 2) = u_m.
\]
(15)

The upper limit \( u_m \)† is of course always valid, but is non-trivial only when the equilibrium flow is eastward.

This upper limit was obtained by Egger and Metz by an heuristic generalization of their one-mode results. It has been put on a rigorous foundation for multi-mode systems (for which the equilibrium expectation is given by Eq. (12)) by the present analysis. Note that \( \eta_{\max} \) entering in Eq. (15) is determined by the resolution of the integration and not by the resolution of the topography if the latter happens to be less.

If the topographic term is also included in the \( h_{01} \) term in Eq. (12) it is possible for \( u_m \) to be exceeded, but since the rotational part is usually much larger than the topographic part (see section 3(c)), \( u_m \) as given by Eq. (15) is still an excellent approximation to the upper bound on \( \langle u_{01} \rangle \).

(e) The form of the solutions

Clearly, as the resolution of the integration is increased, \( u_m \) decreases and \( \langle u_{01} \rangle \) might also be expected to decrease. However, the precise dependence of \( \alpha/\beta \) and \( \langle u_{01} \rangle \) on the integral invariants and resolution is complicated by the forcing coefficients (particularly the rotation contribution which is usually dominant). In order to simplify the situation a little let us first consider the way in which \( \alpha/\beta \) behaves in the unforced case. Then, in order for the flow to be realizable (Kraichnan 1975; Frederiksen and Sawford 1980 and references therein)
\[
\eta_{\min} \leq F/E \leq \eta_{\max}.
\]
(16)

A brief inspection of Eqs. (8) and (9) shows that in this unforced case \( \alpha/\beta \) depends only on the ratio \( F/E \). In particular, for \( F/E \) close to the lower limit \( \eta_{\min} \), \( \alpha < 0 \), \( \beta > 0 \), \( \alpha/\beta \approx -\eta_{\min} \) and the energy spectrum is strongly peaked at small wavenumbers. At the other extreme \( \alpha > 0 \), \( \beta < 0 \), \( \alpha/\beta \approx -\eta_{\max} \) and the energy spectrum is strongly peaked at large wavenumbers. As \( F/E \) increases from \( \eta_{\min} \) to \( \eta_{\max} \) first \( \alpha \) then \( \beta \) changes sign. The situation is summarized schematically in Fig. 1(a) where \( \alpha/\beta \) is shown as a function of \( F/E \) for fixed \( \eta_{\min} \) and \( \eta_{\max} \) and in Fig. 1(b) where \( \alpha/\beta \) is shown as a function of \( \eta_{\max} \) (i.e. of resolution) for fixed \( F/E > \eta_{\min} \).

We will be most interested in varying the resolution for given initial conditions; that is with diagrams similar to Fig. 1(b) in which the way in which \( \alpha/\beta \) varies with resolution

† In the notation of Egger and Metz \( u_m \approx u_{\min} \); although the reason for their choice of notation is clear from their paper, we feel it is incongruous to denote an upper limit by the subscript 'min'.
depends on the ratio $F/E$. In particular, the resolution at which $\beta$ changes sign (or where $\alpha/\beta$ is singular), $\eta_\infty$, increases (decreases) with $F/E$.

With the inclusion of forcing, we see from Eqs. (8) and (9) that the ratio $\alpha/\beta$ is no longer determined solely by $F/E$, but depends on $F$ and $E$ separately, as well as the forcing coefficients. However, for $\beta^2 \ll \alpha$, which is the case, near where $\beta$ changes sign, Eqs. (8) and (9) reduce to

$$E_{mn} \approx \frac{1}{2}(\alpha + \beta n(n + 1))^{-1}$$  

(17)

and

$$F'_{mn} - \frac{1}{2}|H_{mn}|^2 \approx \frac{1}{2}(n(n + 1))^{1/\alpha + \beta n(n + 1)}$$  

(18)

and the system behaves approximately like the unforced case but with $F' \approx$
\[ F - \frac{1}{4} \sum_{m,n} |\hat{h}_{mn}|^2 \] replacing the enstrophy. While this approximation can only hold when \( \beta \) is suitably small, i.e. near \( \eta_0 \) in Figs. 1(a and b), we anticipate that with some modifications, diagrams such as these will provide a basis for understanding and interpreting more general situations. Because of the variety of cases to be studied, we will defer further development of these ideas until the next section where the different situations will be analysed individually as they arise.

3. Equilibrium Calculations

(a) Solid-body rotation initial flow field

Many simple model studies are based on solid-body rotation zonal flows in order to capture the essential physics of zonal-mean flow and topographic interactions while retaining the advantages of partial analytical tractability (Frederiksen and Sawford 1981; Frederiksen 1982 and references therein). Here we follow Egger and Metz and consider two eastward initial solid-body rotations with zonal velocities at the equator, \( u_i = 17 \text{ m s}^{-1} \) and \( 34 \text{ m s}^{-1} \), and an initial westward flow, \( u_i = -17 \text{ m s}^{-1} \) over global topography. Table 1 gives non-dimensional velocities and streamfunctions corresponding to these flows. We have used the smoothed global topography which is shown in Fig. 1 of Frederiksen and Sawford (1981). The solutions are obtained by solving Eqs. (7) to (9) directly as in Frederiksen and Sawford.

<table>
<thead>
<tr>
<th>Dimensional</th>
<th>Non-dimensional</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_i (\text{m s}^{-1}) )</td>
<td>( \psi_f (\text{km}^2 \text{s}^{-1}) )</td>
</tr>
<tr>
<td>(+17)</td>
<td>(+88)</td>
</tr>
<tr>
<td>(+34)</td>
<td>(+177)</td>
</tr>
</tbody>
</table>

Figure 2(a) shows the equilibrium solid-body rotation velocity in dimensional form as a function of resolution. Also shown is the limiting velocity \( u_m \). Clearly in all cases \( \langle u_{01} \rangle \leq u_m \). (Note that since we include the \((0, 1)\) topographic component in our calculations it is possible in principle for \(\langle u_{01} \rangle\) to slightly exceed \( u_m \).)

We discuss the two eastward flow cases first. At the largest resolution \( (J = 15) \), \( u_m \) is nearly zero \((\approx 0.5 \text{ m s}^{-1})\) and for both cases \( \langle u_{01} \rangle \) is in fact westward and considerably less than \( u_m \). Here \( \langle u_{01} \rangle \) also depends strongly on the initial flow. Although Egger and Metz did not carry out integrations at this resolution, Edelmann (1972) observed in his work that an initial eastward solid-body rotation ultimately became westward. With decreasing resolution we see from Fig. 2(a) that the two curves converge and approach the \( u_m \) curve. The \( 17 \text{ m s}^{-1} \) case essentially reaches the \( u_m \) curve at \( J = 6 \), while the \( 34 \text{ m s}^{-1} \) case does not do so until about \( J = 4 \). At \( J = 3 \) we see that \( \langle u_{01} \rangle \) for the \( 17 \text{ m s}^{-1} \) case falls away from the \( u_m \) curve. As concluded by Egger and Metz, if \( u_i < u_m \) then \( \langle u_{01} \rangle \approx u_i \) - clearly the solid-body rotation cannot have more energy than it did initially. We note that our statistical mechanical equilibrium results are in excellent agreement with those obtained by numerical integration by Egger and Metz which are also shown for comparison as filled symbols in Fig. 2(a).

For westward initial flow, Fig. 2(a) shows that \( \langle u_{01} \rangle \approx u_i \) independent of resolution. This result is again consistent with the more limited conclusions of Egger and Metz.

As anticipated earlier, the results shown in Fig. 2(a) are easily interpreted in terms of modifications by the forcing coefficients to Figs. 1(a and b). Since we deal mostly with changes in resolution for fixed \( F \) and \( E \), Fig. 1(b) and its extension is most relevant. Then, for example, the location of the singularity in \( \alpha/\beta \) is determined by the ratio \( F'/E \) (see Eqs. (17) and (18)).
Figure 2. (a) Equilibrium solid-body rotation velocity, and (b) $|\alpha/\beta|$, as a function of resolution for solid-body rotation initial flows. The filled symbols in Fig. 2(a) are the results of the numerical integrations of Egger and Metz (1981). The solid line in Fig. 2(b) is for $\alpha/\beta > 0$ and the dotted line for $\alpha/\beta < 0$. The solid line in Fig. 2(b) is for $\alpha/\beta > 0$ and the dotted line for $\alpha/\beta < 0$. 
Figure 2(b) shows the ratio $\alpha/\beta$ as a function of resolution for the three initial solid-body rotation flows under discussion. Because of the range of values encountered we have used a log scale and plotted $|\alpha/\beta|$. The positive and negative cases are clearly marked. We have also plotted the truncation wavenumber $J$ rather than $n_{\text{max}}$ on the abscissa in order to avoid compression of the small wavenumber region. Also shown on the figure is the line $\alpha/\beta = -n_{\text{max}}$ and values of $\alpha/\beta$ corresponding to $\langle u_{01} \rangle = \pm |u_i|$ (see Eqs. (19) and (20) below).

Both eastward initial flow cases qualitatively confirm the analysis at the end of section 2. Let us examine the 17 m s$^{-1}$ case in some detail. We see that $|\alpha/\beta|$ becomes very large at around $J = 9$ and 10. With decreasing resolution (where $\alpha > 0$, $\beta < 0$) $|\alpha/\beta|$ decreases and approaches the $n_{\text{max}}$ line as in the unforced case shown in Fig. 1(b). However, whereas in the unforced case $\alpha/\beta$ follows this line until $F'/E$ is reached after which solutions cease to exist, in the present case the solid-body rotation takes up an increasing proportion of the energy. Ultimately, almost all the energy is in the solid-body rotation and is limited by the initial energy. Then we must have, from Eq. (13) and recalling that the topographic contribution to $h_{01}$ has been ignored there,

$$\frac{\alpha/\beta}{1 + |u_i|} / |u_i| \quad (\alpha > 0; \beta < 0). \quad (19)$$

As the resolution is increased beyond $J = 10$, where $\alpha$, $\beta$ are both positive, $\alpha/\beta$ decreases again as in the unforced case shown schematically in Fig. 1(b). Now the equilibrium solid-body rotation is westward and the solution is ultimately limited by the state in which the solid-body rotation contains all the energy. Then we have again from Eq. (13)

$$2(1 - |u_i|)/|u_i| \leq \alpha/\beta. \quad (20)$$

Note that this limiting state corresponds to $\alpha$, $\beta > 0$ unless $|u_i| > 1$. Thus negative $\alpha$ states are not possible unless the energy of the initial field is larger than $7.19 \times 10^4$ m$^2$ s$^{-2}$. Frederiksen and Sawford (1980) have observed negative $\alpha$ states in the unforced case.

To a reasonable approximation (within 10%)

$$F'/E = 4/u_i. \quad (21)$$

This result (which is in terms of non-dimensional variables) is valid provided $u_i \ll 1$ and $h_{m} \ll 2\sqrt{2}$. Thus we have $F'/E \approx 109$ and $F'/E \approx 55$ for the 17 m s$^{-1}$ and 34 m s$^{-1}$ cases respectively, and we see from Fig. 2(b) that, as anticipated, decreased $F'/E$ results in a shift to smaller wavenumber.

We are now in a position to explain the behaviour of the westward initial flow case when forcing is present. Since $u_i$ is now negative, we see from Eq. (21) that $F'/E$ is actually negative and the singularity in $\alpha/\beta$ is displaced so far to low wavenumbers (we may regard it as occurring at 'negative resolution') that the large wavenumber limit is valid for all $J$; that is, the solid-body rotation remains westward and very close to its initial value. Differences between westward and eastward flow over topography are discussed in Frederiksen (1982).

(b) The effect of initial eddy energy

Addition of eddy energy to the initial flow field changes the ratio $F'/E$ and hence, as already discussed, shifts the singularity in $\alpha/\beta$ to larger or smaller scales depending on whether $F'/E$ is increased or decreased. Thus the effect of adding eddy energy depends on the spectrum of the eddy energy. For example, from Eq. (21) we see that, to a first approximation, if the eddy and solid-body rotation energies are equal in magnitude, then $F'/E$ will be increased (decreased) if $F'_{\text{eddy}}/E_{\text{eddy}}$ is greater than (less than) $4/u_i$.

Figure 3 shows the effect of adding eddy energy with different spectral distributions to a 17 m s$^{-1}$ eastward solid-body rotation. The actual values used are shown in Table 2.
Table 2: Initial Flows with Eddy Energy Added to 17 m s\(^{-1}\)  

<table>
<thead>
<tr>
<th>Run</th>
<th>(E_{\text{eddy}})</th>
<th>(F'_{\text{eddy}})</th>
<th>(F'<em>{\text{eddy}}/E</em>{\text{eddy}})</th>
<th>(E_{\text{eddy}}) (m(^2) s(^{-2}))</th>
<th>(F'_{\text{eddy}}) (s(^{-2}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2.78 \times 10^{-4})</td>
<td>(6.99 \times 10^{-2})</td>
<td>251</td>
<td>60</td>
<td>72 \times 10^{-10}</td>
</tr>
<tr>
<td>2</td>
<td>(3.48 \times 10^{-4})</td>
<td>(3.55 \times 10^{-2})</td>
<td>102</td>
<td>75</td>
<td>89 \times 10^{-10}</td>
</tr>
<tr>
<td>3</td>
<td>(6.28 \times 10^{-4})</td>
<td>(1.29 \times 10^{-2})</td>
<td>21</td>
<td>136</td>
<td>86 \times 10^{-10}</td>
</tr>
</tbody>
</table>

\[E_{\alpha_1} = 96.3; \quad F'_{\alpha_1} = 2.73 \times 10^{-10} \text{ (dimensional)}\]
\[E_{\alpha_1} = 4.46 \times 10^{-4}; \quad F'_{\alpha_1} = 5.13 \times 10^{-2}; \quad F'_{\alpha_1}/E_{\alpha_1} = 115 \text{ (non-dimensional)}\]

Figure 3(a) shows \(\langle u_{\alpha_1} \rangle\) as a function of resolution. As before we find the solutions most sensitive for large resolution and relatively insensitive for the more highly truncated systems studied by Egger and Metz. It is thus not surprising that they found the addition of eddy energy to have little effect in their experiments. We see that as \(F'/E\) increases, \(\langle u_{\alpha_1} \rangle\) remains close to \(u_m\) up to larger wavenumbers and that transition to westward flow also occurs at larger wavenumbers. This is quite consistent with the trend already noted with the superrotation initial flows. However, the lowest wavenumber for which solutions exist also increases with increasing \(F'/E\) and (compare run 2 with the solid-body rotation case) when eddy energy is added without altering the \(F'/E\) ratio.  

![Figure 3](image-url)  

Figure 3. (a) Equilibrium solid-body rotation velocity.
In order to explain further and to analyze some of these points we turn to Fig. 3(b) where $|\alpha/\beta|$ is plotted as a function of resolution. The notation is analogous to that in Fig. 2(b). We first compare run 2 with the solid-body rotation case. $F'/E$ is approximately the same in both cases and so the singular wavenumber is about the same. However, because run 2 has more energy available, solutions are not as strongly limited by the magnitude of $\langle u_{01} \rangle$. Thus with increasing resolution $\alpha/\beta$ becomes smaller and $\langle u_{01} \rangle$ larger than for the solid-body rotation case. With decreasing resolution both cases approach the $\eta_{\text{max}}$ line at very similar rates, but run 2 is little influenced by the weaker energy constraint and so solutions cease to exist when $\eta_{\text{max}} \approx F'/E$, which is similar to the case when forcing is absent altogether.

The other cases shown conform to the principles which have already emerged. Thus, for run 1, which has the largest $F'/E$, the diagram is displaced to higher wavenumbers, and again because of the larger amount of energy available with decreasing resolution the existence of solutions is limited by the condition $\eta_{\text{max}} = F'/E$. In contrast, for run 3 where the ratio $F'/E$ has been reduced considerably, the diagram is shifted towards low wavenumbers, and the smaller value of $F'/E$ permits the solution to follow the $\eta_{\text{max}}$ line to wavenumbers small enough for the energy constraint to again become important. None of the cases studied here have sufficient initial energy to permit negative $\alpha$ solutions.

In the case of an initial westward solid-body rotation we have already noted that $F'/E$ is negative and it is thus clear that a large amount of eddy energy must be added in order to move the singularity in $\alpha/\beta$ to sufficiently high wavenumbers to permit an

![Figure 3](image-url)  

Figure 3. (b) $|\alpha/\beta|$, as a function of resolution for initial flows containing eddy energy. The solid line is for $\alpha/\beta > 0$ and the dotted line for $\alpha/\beta < 0$. 
eastward equilibrium solid-body rotation. This conclusion confirms and explains the observations of Egger and Metz.

(c) The effect of topography

It can be seen from Eq. (3) that the only mechanism for changing $\psi_{01}$ (and hence $u_{01}$) is through interaction of the eddy flow and forcing terms of the same wavenumber. The $\hat{h}_{01}$ forcing term (that is, variable rotation and the $h_{01}$ topography component) or indeed any other zonal forcing, cannot on its own alter the magnitude of $u_{01}$. Thus the $\psi_{01}$ term does not equilibrate and the climatic-mean state is the trivial one $u_{01} = u_i$ unless there is some topographic forcing in the eddy modes.

Egger and Metz are incorrect in supposing that there should be a gradual change in $\langle u_{01} \rangle$ as the height of the topography is increased from zero. Rather, the $h = 0$ case is a singular limit and no matter how small the eddy topographic forcing terms, $\psi_{01}$ will eventually evolve to an equilibrium state. This state will only depend on the topography through the expression for the total enstrophy in Eq. (5). Furthermore, for the global topography used here, the topographic contribution to $\hat{h}_{01}$ is about 1/30 of that due to rotation. We thus expect that the solutions will not be very sensitive to changes in the topographic forcing coefficients. This is confirmed by Fig. 4 where we plot $\langle u_{01} \rangle$ as a function of resolution for various rescaled versions of the global topography. For each case the initial flow is solid-body rotation with $u_i = 17 \text{ m s}^{-1}$.

While the equilibrium state is not very sensitive to the magnitude of the forcing coefficients, the rate of change of $\psi_{01}$ and hence presumably the time taken to reach equilibrium are. We thus suggest that the quite strong dependence of $\langle u_{01} \rangle$ on topography reported by Egger and Metz may be a reflection of the fact that as the topo-

![Figure 4. Equilibrium solid-body rotation zonal velocity as a function of resolution for rescaled global topography. For each case the initial flow is solid-body rotation with $u_i = 17 \text{ m s}^{-1}$.](image-url)
graphy is reduced, equilibration takes longer and the values reported overestimate the true equilibrium values due to insufficient integration time.

4. Conclusions

We have used the methods of statistical mechanics to calculate the climatic-mean angular momentum in a topographically forced inviscid barotropic model on a sphere. Two interesting analytical results have been derived. The first of these is that, provided wave–zonal-flow interactions are retained, the presence or absence of wave–wave interactions has no influence on the equilibrium state corresponding to any given initial flow; however, the detailed evolution of the system towards the equilibrium state does depend on such interactions. We have thus proved the assertion, made by Egger and Metz on the basis of limited numerical integrations, that the climatic-mean value of the angular momentum is independent of such interactions. Secondly, we have derived an upper bound to the equilibrium angular momentum again explaining the numerical observation of Egger and Metz.

Our other results follow from evaluation of the equilibrium solutions for a wide range of parameters. We have interpreted these results in terms of a generalization of the well-known properties of equilibrium solutions for unforced flow. Whereas in the unforced case the critical parameter determining the nature of the solutions is the ratio of enstrophy to energy, in the forced model the enstrophy is replaced in this ratio by an adjusted potential enstrophy (see Eq. (18)). In addition, further constraints are imposed on the system by the rather obvious requirement that the solid-body rotation stationary energy be not greater than the initial energy.

We find that for realistic resolution ($J = 15$) most flows have a westward equilibrium solid-body rotation, thus explaining the numerical results of Edelmann (1972). Flows which consist initially entirely of a westward solid-body rotation are very stable and at equilibrium remain very close to the initial condition, regardless of resolution. This result is consistent with the potential vorticity conservation arguments of Kasahara (1966) and is also in agreement with the numerical work of Egger and Metz.

Highly truncated systems are insensitive to the initial conditions which is why Egger and Metz were able to explain successfully many of their limited resolution results in terms of analytical 'one-mode' results. However, at realistic resolutions the solutions are strongly dependent on the initial conditions and the 'one-mode' results no longer apply.

In treating systems with initial eddy energy we find the results for realistic resolution depend on the spectrum of the eddy energy, whereas for the low resolution used by Egger and Metz, and in agreement with their work, our results are insensitive to the addition of initial eddy energy.

We find the mean angular momentum depends only weakly on any reasonable (factors of 0.5 to 5.0) rescaling of the topography. The results of Egger and Metz are incorrect in this regard, probably because insufficient time was allowed for the integrations to equilibrate with the weaker forcing. In fact, contrary to their supposition that there should be a gradual change as $h$ increases from zero, the case for $h = 0$ is a singular limit. No matter how small the eddy topographic terms are, the system will eventually evolve to a statistical equilibrium in which $\langle u_{01} \rangle$ is in general quite different from the initial value $u_i$.

Acknowledgements

It is a pleasure to thank S. M. Kepert for his able assistance in this study. We wish to thank J. Evans for typing the manuscript.
APPENDIX

(a) Liouville’s theorem

In order to prove Liouville’s theorem for the system Eqs. (10) and (11) we need to show that (Frederiksen and Sawford 1980)

\[ \sum_{m \geq 0} \left( \frac{\partial \psi_{mn}}{\partial a_{mn}} + \frac{\partial h_{mn}}{\partial b_{mn}} \right) = 0 \]  

(A.1)

where dot denotes the derivative with respect to time and where

\[ \psi_{mn} = 2 \{ (2 - \delta_{m0}) (n+1) \}^{-1/2} (a_{mn} + ib_{mn}) \quad m \geq 0 \]  

(A.2)

with \( b_{mn} = 0 \). From Eq. (10)

\[ -n(n+1)a_{mn} = |m| \sqrt{\frac{2}{3}} \cdot \sum_{n(n+1)} -2 \{ (2 - \delta_{m0}) (n+1) \}^{-1/2} a_{01} \left[ b_{mn} \{ (n+1) - 2 \} - \right. \]

\[ \left. - \text{Im}(h_{mn}) \{ (2 - \delta_{m0}) (n+1) \}^{1/2} \right] + 2 |m| b_{mn} \]  

(A.3)

and

\[ n(n+1)b_{mn} = |m| \sqrt{\frac{2}{3}} \cdot \sum_{n(n+1)} \frac{1}{2} \right] \times \]

\[ \times a_{01} \{ a_{mn} \{ (n+1) + 2 \} - \text{Re}(h_{mn}) \{ (2 - \delta_{m0}) (n+1) \}^{1/2} \} + 2 |m| a_{mn} \]  

(A.4)

The required result follows immediately from (A.3), (A.4) and Eq. (11). It is readily seen that it also holds for the \( \psi_{01} \) component.

(b) Energy constraint

From Eqs. (10) and (11) we have

\[ n(n+1)\psi_{mn}^* \frac{\partial \psi_{mn}}{\partial t} = i m \sqrt{\frac{2}{3}} \psi_{01} \psi_{mn} \psi_{mn}^* \{ n(n+1) - 2 \} - \]

\[ -i m \sqrt{\frac{2}{3}} h_{mn} \psi_{mn}^* \psi_{01} + 2im \psi_{mn} \psi_{mn}^* \]  

(A.5)

and

\[ 2\psi_{01} \frac{\partial \psi_{01}}{\partial t} = 2 \sqrt{\frac{2}{3}} \psi_{01} \sum_{m > 0 ; n} m \text{Im}(\psi_{mn}^* h_{mn}^*) \]  

(A.6)

On summing (A.5) over all \( m \neq 0, n \) the first and third terms on the right-hand-side vanish and the second cancels when added to (A.6) so that the kinetic energy Eq. (4), is conserved.

(c) Potential enstrophy constraint

Again from Eqs. (10) and (11) we have

\[ \{ n(n+1)\psi_{mn}^* - h_{mn}^* \} n(n+1) \frac{\partial \psi_{mn}}{\partial t} = n(n+1)im \sqrt{\frac{2}{3}} \psi_{01} \psi_{mn} \psi_{mn}^* \{ n(n+1) - 2 \} - \]

\[ -n(n+1)im \sqrt{\frac{2}{3}} \psi_{mn} h_{mn}^* - im \sqrt{\frac{2}{3}} \psi_{01} h_{mn}^* \psi_{mn} \{ n(n+1) - 2 \} + \]

\[ + im \sqrt{\frac{2}{3}} \psi_{01} h_{mn}^* h_{mn}^* + 2im(n+1)\psi_{mn}^* \psi_{mn} - 2imh_{mn}^* \psi_{mn} \]  

(A.7)

and

\[ 2\{ 2\psi_{01} - h_{01} \} \frac{\partial \psi_{01}}{\partial t} = 2 \sqrt{\frac{2}{3}} \psi_{01} \sum_{m > 0 ; n} m \text{Im}(\psi_{mn}^* h_{mn}^*) - 2 \sqrt{\frac{2}{3}} h_{01} \sum_{m > 0 ; n} m \text{Im}(\psi_{mn}^* h_{mn}^*) \]  

(A.8)

On summing (A.7) over all \( m \neq 0, n \), again only the cross terms in \( \psi \) and \( h \) are
nonzero and these cancel when added to (A.8) (recalling that $\hat{h}_{01} = 2\sqrt{2}$ since no topographic forcing is included in this mode) so that the potential enstrophy Eq. (5) is conserved.

REFERENCES


