Aircraft measurements of mountain waves and their associated momentum flux over the British Isles

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SUMMARY
The Hercules and Canberra aircraft of the Meteorological Research Flight have made a number of flights to investigate mountain waves over different mountainous areas of the British Isles. Results from five flight days are presented in this study. For each flight day the observed values of the dominant wavelength and the wave amplitude as a function of height were compared with the values predicted by two-dimensional linear lee wave theory. Good agreement was obtained in four of the five cases. Measurements of the vertical flux of horizontal momentum show that the largest values were obtained when the wavelength was 20.9 km and the waves were only weakly trapped. On this occasion, the flux was observed to be constant with height in the range over which measurements were made, supporting a theoretical result of Eliassen and Palm. The value \(-0.35 \text{ N m}^{-2}\) is comparable with some aircraft measurements made over the Rocky Mountains.

1. INTRODUCTION
When a stably stratified airstream passes over a mountain, vertically and horizontally propagating gravity waves are generated which have the ability to exchange momentum between the atmosphere and the underlying surface. Eliassen and Palm (1961) showed that this momentum flux is constant with height provided that the background mean wind, \(U(z)\), is positive at all levels. Bretherton (1969) showed that momentum is removed from the atmosphere at levels where the waves break down into turbulence, either through their amplitude becoming large or at a critical level where \(U(z)\) falls to zero.

The availability of research aircraft equipped with gust probes and inertial navigation systems and therefore able to measure accurately the three components of wind has made possible direct measurements of the wave momentum flux. A number of such measurements have been made over the Rocky Mountains in the U.S.A. Lilly and Kennedy (1973) report an occasion on which a momentum flux of \(-0.7 \text{ N m}^{-2}\) was observed between the surface and a turbulent layer at 16 km, at which level the momentum flux fell to zero. Lilly (1978) describes observations of a large amplitude, long wavelength disturbance in which a momentum flux of \(-1.2 \text{ N m}^{-2}\) was measured. Lilly (1972), using a general circulation model with a simple parametrization of the wave momentum flux, has shown that such momentum transports may be significant on the global scale.

The Hercules and Canberra aircraft of the Meteorological Research Flight (MRF) have made a number of flights over mountainous regions of the British Isles to investigate mountain waves set up by mountains which are small by comparison with the Rockies. The type of underlying terrain was generally similar for each of the flights described in this study, consisting of systems of ridges and valleys typically 10 km or less in width with an amplitude of a few hundred metres and extending over areas with horizontal dimensions of a few hundred kilometres.

Satellite observations of these areas (Fig. 1) frequently show wave activity with wavelengths commonly in the range 10 to 20 km. Theoretical studies have shown that such wavelengths correspond to a resonance property of the atmosphere (Queney et al. 1960; Smith 1979). Wave energy is trapped in the lower troposphere and may thus be advected large distances downstream of the mountains over which the waves were formed, leading to their common name of 'lee waves'. Waves longer than about 30 km are often referred to as hydrostatic mountain waves since their perturbation pressure and density fields are almost in balance. They are able to propagate freely in the vertical and wave energy tends to be confined to the region over the mountains. Such hydrostatic waves are thought to
be responsible for severe downslope winds and clear air turbulence observed over the Rocky Mountains (Klemp and Lilly 1975; Lilly 1978). One object of the present study has been to compare aircraft observations of the wavelength and amplitude of lee waves with predictions of two-dimensional linear theory. Vergeiner (1971) and Smith (1976) have performed similar studies and have found that linear theory gives a good estimate of wavelength although underestimating amplitude. In this study, it has not been possible to predict absolute values of the wave amplitude owing to the difficulty of modelling the spectral distribution of topographic forcing by the complex terrain. It was, however, possible to predict and observe relative amplitudes at different heights. In addition, this study has attempted to relate the measured momentum fluxes to the wavelength and degree of trapping of the lee waves.

2. THE AIRCRAFT INSTRUMENTATION

The instrumentation of the MRF aircraft relevant to this study has been fully described elsewhere; see for example Nicholls (1978) for the Hercules and Axford (1968) for the Canberra. As a brief summary, however, the main items of equipment used in this study and common to both aircraft are:

(i) a noseboom-mounted pitot-static probe with angle-of-attack and angle-of-sideslip vanes, for measuring the airflow relative to the aircraft;
(ii) an inertial navigation system (INS) to measure aircraft velocity relative to the ground; and
(iii) fast-response platinum resistance thermometers.

INS-derived horizontal velocities are subject to drifts and long period oscillatory
errors. These are corrected by software procedures which compare INS velocity components with those derived from Doppler radar or, on the Hercules only, from hyperbolic navigation systems (DECCA and LORAN). When these corrections have been applied, it is possible to specify the horizontal wind components with an absolute accuracy of about 0.5 m s\(^{-1}\) for the Hercules and 1.0 m s\(^{-1}\) for the Canberra. INS-derived vertical velocity is corrected using pressure altitude as a height reference. For the Canberra, this correction procedure is performed entirely by software at the data-processing stage. The Hercules INS, however, contains its own aneroid device the output of which is compared with the doubly-integrated vertical acceleration in a loop with a 10-minute time constant. Thus vertical velocity output from the INS is inertially derived for periods shorter than ten minutes, and given by the rate of change of pressure height for longer times. Some small residual drifts may still remain, however, and these are again removed by software procedures.

There is, however, a problem associated with the use of pressure height as a reference, in that vertical accelerations due to the waves themselves may cause significant horizontal pressure variations on the wave scale. This was observed in particular in strong lee waves on flight H117. Such non-hydrostatic pressure variations should not be interpreted as variations in aircraft altitude as this would introduce an oscillatory error into the measurements of the vertical wind component, \(w\). Whilst this has only a very small effect (<1%) on estimates of the mean wave amplitude along a run, the effect on the measurement of the vertical flux of horizontal momentum may be 20% or more. The procedure adopted on all flights has been to examine the values of vertical velocity corrections, and if they show any significant correlation with \(w\), they are recomputed using a longer averaging period.

3. THE EXPERIMENTAL FLIGHTS

(a) Flight patterns

Data from five separate days have been used, the Hercules flying on all five days and being joined by the Canberra on one. Basic information for each day is given in Table 1. On each flight a generally similar flight pattern was used. This commenced with a 'profile descent', a slow descent to measure the wind and temperature structure of the atmosphere from an altitude of about 6 km down to 500 m above sea level. Ideally, this descent should be made in undisturbed conditions representative of the flow incident on the mountains under investigation, thus it was normally carried out over a sea area immediately upwind.

A reference ground track was then defined running parallel to the wind direction measured at low levels on the profile descent between geographical coordinates about 200 km apart. A number of straight and level runs were then flown along this track at altitudes between 2 and 7 km, although on the H117/C453 flight day the Canberra made measurements up to 13-6 km.

This flight pattern is designed for a wave field which is two-dimensional, that is, having no variation in the direction perpendicular to the background wind. Figure 1 shows that although this is not strictly true, in the sense that the wave crests are not always perpendicular to the background wind direction, there do exist regions of wave activity where the component of wavelength measured along this direction remains constant for large distances, of the order of 200 km in the acrosswind direction. This suggests that for the purpose of measuring perturbation wind components parallel to the background wind direction, the positioning of the pattern in the acrosswind direction is not critical.

(b) Observational data

For each individual run, the recorded data were reduced to 0.5 s averages (a spatial resolution of 50-75 m) of \(u\), \(v\) and \(w\), the three components of the wind vector, \(\theta\) the
<table>
<thead>
<tr>
<th>Flight Date</th>
<th>Location End points</th>
<th>Conditions</th>
<th>Altitudes (km)</th>
<th>Wind at 1 km (deg)</th>
<th>Wind at 1 km (m s⁻¹)</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>H117/C453</td>
<td>N. England</td>
<td>Stable WNW flow, Anticyclone over Bay of Biscay with frontal system lying over N. Scotland.</td>
<td>1.6, 2.6, 3.5, 5.7, 7.6, 9.5, 10.7*, 11.6*, 13.0*, 13.6*</td>
<td>290</td>
<td>25.1</td>
<td>Both Hercules and Canberra flown on this occasion.</td>
</tr>
<tr>
<td>12 Jan. 1976</td>
<td>54°35'N 3°38'W to 54°03'N 0°30'W</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>H389</td>
<td>N. Wales</td>
<td>Cold WNW flow. Low pressure to the north of the British Isles.</td>
<td>2.2, 3.1, 4.6, 6.1, 7.7</td>
<td>280</td>
<td>15.5</td>
<td>Lee waves observed in stratocumulus throughout flight. Some medium level wave cloud.</td>
</tr>
<tr>
<td>20 June 1980</td>
<td>52°54'N 4°30'W to 52°18'N 2°5'W</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>H399</td>
<td>N. Wales</td>
<td>In cold air on the northern side of a waving cold front lying over S. England.</td>
<td>2.1, 2.4, 3.1, 3.7, 4.6</td>
<td>280</td>
<td>12.9</td>
<td>Frontal surface encountered at about 4km. Lee waves observed in cloud below this level.</td>
</tr>
<tr>
<td>18 July 1980</td>
<td>53°13'N 4°28'W to 52°32'N 2°11'W</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>H421</td>
<td>N. Scotland</td>
<td>Cold northerly flow with anticyclone to the west.</td>
<td>2.4, 3.1, 3.7, 6.1</td>
<td>355</td>
<td>12.8</td>
<td>Wave activity not as extensive as anticipated.</td>
</tr>
<tr>
<td>3 Dec. 1980</td>
<td>58°45'N 4°30'W to 57°00'N 4°30'W</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>H428</td>
<td>N. Scotland</td>
<td>Warm sector conditions. Cold front approaching operational area from the NW.</td>
<td>1.8, 2.1, 3.1, 3.7, 4.6, 6.1</td>
<td>245</td>
<td>30.5</td>
<td>Lee waves observed in stratocumulus throughout flight, extending over a wide area. Little medium and high cloud due to very dry air above inversion.</td>
</tr>
<tr>
<td>8 Jan. 1981</td>
<td>57°23'N 6°47'W to 58°7'N 3°59'W</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
potential temperature, $p$ the static pressure, and measurements of the aircraft position and velocity. All velocities are given in a right-Cartesian system of axes with the $x$ axis parallel to the reference wind direction.

Figure 2 shows measurements of $w$, the vertical wind component, for six alongwind runs from flight H428. There is a well-developed train of waves with a wavelength of about 21 km which remained stationary with respect to the ground for a period of about three hours. Figure 3 shows similar measurements for flight H389, which qualitatively resemble those from the other three flights H399, H421 and H117/C453. There is noticeably less coherence between successive runs at different altitudes than on H428, although at present it is not clear whether this is due to temporal variability in the background wind and temperature fields or to the type of acrosswind spatial variability shown in Fig. 1. In the present study it has been assumed that the overall wave field remains stationary with respect to the ground for the duration of the experimental part of the flight, a reasonable assumption in the case of flight H428 although there is some doubt about the other cases.

Figures 4(a) and (b) show cross-sections of $\theta$ and $u$ in the alongwind vertical plane for the two-aircraft flight H117/C453. In producing these cross-sections, variability due to the lee waves and smaller scales has been subjectively removed to show more clearly any larger scale features. The disturbance revealed by this procedure resembles those reported by Reid (1975) over South Island, New Zealand, and by Lilly and Kennedy (1973) over the Rocky Mountains. Klemp and Lilly (1978) demonstrate that such disturbances are due to the hydrostatic part of the mountain wave spectrum in which the horizontal wavenumber $k$ is small compared with the Scorer parameter $l$. For typical atmospheric values of $l$, this implies a wavelength greater than about 30 km. Such waves propagate
almost vertically and are non-dispersive, thus the hydrostatic part of the disturbance is confined to the region directly above the mountain. The forward-steepening of the isentropes evident at a height of about 10 km in Fig. 4(a) has been ascribed to nonlinear effects of finite mountain height and asymmetric cross-section (Smith 1977; Klemp and Lilly 1978).

If the flow is assumed to be two-dimensional, stationary and adiabatic, the isentropes indicate that vertical displacements of up to 1.5 km will occur as a result of the hydrostatic disturbance, whilst the maximum displacement due to lee waves at about 10 km altitude is of the order of a few hundred metres. The observer in the Canberra aircraft noted slight turbulence and the beginning of a sheet of cirrus cloud in the approximate position of the long-wave crest at 10.5 km altitude. Satellite pictures often show cirrus cloud streaming downwind of areas of lee wave activity, a particularly good example over the same topography as the H117/C453 flight being shown in Fig. 5. From the measurements on this flight, it seems that such cirrus is associated with the presence of the hydrostatic disturbance rather than the lee waves.

(c) Measurements of momentum flux

The vertical flux of alongwind horizontal momentum, averaged along a run, is given by

\[
\overline{F} = \overline{\rho \, u'w'} = \left( \overline{\rho/L} \right) \int u'w' \, dx
\]

(1)

the integration being from \( x_1 \) to \( x_2 \), where \( \overline{\rho} \) is the mean density, \( L (= x_2 - x_1) \) is the run length, and the primed quantities are residuals after the removal of a mean and linear trend from the data for a particular run. The product \( u'w' \) oscillates with an amplitude which may be a factor of 10 to 100 larger than its mean value, thus it is possible that if the
Figure 5. TIROS N images for 1350 z, 11 September 1979, (a) visible; (b) infrared. These show clearly an area of lee waves over N. England with a plume of cirrus cloud extending for several hundred miles downstream. (Reproduced by Courtesy of the Dept. of El. Eng., University of Dundee.)
run samples unequal numbers of updraughts and downdraughts, Eq. (1) may give a biased result. To guard against this possibility, the running integral \( G(x^*) \), given by

\[
G(x^*) = \int_{x_1}^{x^*} u' w' \, dx
\]

the integral being from \( x_1 \) to \( x^* \), was plotted for each run. Examples from two different flights are shown in Figs. 6(a) and (b). The mean momentum flux, \( \bar{F} \), may then be defined by

\[
\bar{F} = \bar{\rho} (\bar{G}(x_2) - \bar{G}(x_1))/(x_2 - x_1)
\]

where \( \bar{G}(x) \) is a smooth curve fitted through the values of \( G(x) \) to remove variability on the lee wave and shorter scales. Figure 6(a) suggests that for flight H428, the major

![Figure 6. Plots of the quantity \( G(x^*) = \int_{x_1}^{x^*} u' w' \, dx \) for (a) H428 run 4, and (b) H389 run 4.](image)

contribution to the momentum flux was from the 20-9 km lee waves. This is confirmed by cospectral analysis of \( u' \) and \( w' \), which shows a well-defined band of negative values centred on the lee-wavelength. Cospectra from flight H389, whilst still showing contributions from the lee wave scale, suggest that there may also be significant contributions from larger scales of motion. Values of \( \bar{F} \) defined as in Eq. (3) were calculated for each experimental run and are shown in Fig. 7.

The momentum fluxes measured on flight H428 have a mean of \(-0.35 \text{ N m}^{-2}\), which is comparable with measurements made over the Rocky Mountains by Lilly and Kennedy (1973). However, the association of the momentum flux with the lee wave scale of motion on this flight is in contrast to other Rocky Mountain results in which the flux is due primarily to long wavelength, hydrostatic waves (Vergeiner and Lilly 1970; Lilly 1978).

4. **Linear lee wave theory**

There have been numerous theoretical investigations of the airflow over mountains. Reviews of many of these, together with full derivations of the relevant equations, are given by Queney *et al.* (1960) and Smith (1979), and a brief summary only will be given
here. The flow is assumed to be two-dimensional and steady, and to be composed of mean and perturbation quantities such that the equations may be linearized in the conventional manner. The equations of momentum, thermodynamics and continuity may then be reduced to a single equation involving the vertical velocity, \( w(x, z) \) in a density-scaled form:

\[
\frac{\partial^2 \hat{w}(x, z)}{\partial x^2} + \frac{\partial^2 \hat{w}(x, z)}{\partial z^2} + l^2(z) \hat{w}(x, z) = 0
\]  

where

\[
\hat{w}(x, z) = (\rho(z)/\rho(0))^{1/2} w(x, z)
\]

\[
l^2(z) = \frac{N^2}{U(z)^2} - \left(\frac{1}{U(z)}\right)\frac{\partial^2 U(z)}{\partial z^2}
\]

\[
N^2 = \left(\frac{g}{\theta(z)}\right)\frac{\partial \theta(z)}{\partial z}
\]  

\( U(z) \) and \( \theta(z) \) represent the wind and potential temperature structure of the undisturbed background flow. If the vertical velocity field is then represented by a Fourier integral,

\[
\hat{w}(x, z) = \int_0^\infty w^*(k, z) \exp(ikx) \, dk,
\]

(4) may be rewritten as

\[
\frac{\partial^2 w^*(k, z)}{\partial z^2} + \left(l^2(z) - k^2\right) w^*(k, z) = 0.
\]  

Thus it is clear that the behaviour of a wave in the vertical is highly dependent on its
horizontal scale, given by the wavenumber $k$, and also the wind and stability structure of
the background flow, which are combined in the definition of $l^2(z)$, the Scorer parameter.
$2\pi/l$ defines an atmospheric length scale, the distance travelled by an air parcel during one
small amplitude buoyancy oscillation. Two extreme cases are obvious. If $k^2 > l^2$, the
solutions to (5) are exponential and the wave amplitude decays with height. If $k^2 < l^2$, the
solutions to (5) are sinusoidal and the waves are able to propagate vertically as well as
horizontally. In the long wave limit $k^2$ is negligible compared with $l^2$, and the vertical
behaviour of the waves becomes independent of their horizontal wavelength. Given
typical atmospheric profiles of wind and potential temperature, $l^2(z)$ is normally a strong
function of height (see Fig. 8), large values being produced by high stability, in particular
inversion layers and in the stratosphere. Minima of $l^2(z)$ are generally found at levels
where the background wind $U(z)$ has a sharp minimum and thus $\partial^2 U/\partial z^2$ is large and
positive. Lee waves are essentially a resonance property of the atmosphere whose presence
depsends on $l^2(z)$ decreasing with height at some level (Smith 1979).

![Graph showing profiles of background values of wind $U(z)$, potential temperature $\theta(z)$, and the Scorer parameter $l^2(z)$ for flight H389.]

It can be shown (Smith 1979) that at large distances downstream from a mountain,
the only non-vanishing disturbance is contributed by those wavenumbers, $k_1$, which
satisfy (5), with the lower boundary condition $w_1^0(k_1, 0) = 0$ and an appropriate upper
boundary condition, which depends on the value of $|l^2(z) - k_1^2|$ at the upper boundary.
The wave is able to propagate vertically in the lower troposphere where $k_1^2 < l^2$, but at
the level where $k_1^2 = l^2$ wave energy is reflected back downwards and above this level the
wave amplitude decays exponentially. If the wave was perfectly trapped, wave energy
would be confined to the high-$l^2$ layer in the lower troposphere and carried downstream in a manner analogous to the propagation of electromagnetic radiation in a waveguide. However, large values of $l^2$ in the stratosphere may be such that the vertical propagation of wave energy is again possible in this region. Thus the upper boundary condition is applied at a level in the stratosphere where $l^2(z) = l_2^2$, and allows vertical propagation if $k_n^2 < l_2^2$, but exponential decay otherwise. Mathematically, $k_n$ and $w^*(k_n, z)$ represent an eigenvalue and the corresponding eigenfunction of (5), and for realistic profiles of $l^2(z)$ they must be determined numerically. If the upper boundary condition is such that vertical propagation of wave energy is possible then the solution, $k_n$, will be found to be complex, with a small positive imaginary part, $\text{Im}(k_n)$. This represents the downstream decay of the wave amplitude due to the leakage of wave energy through the low-$l^2$ barrier in the upper troposphere.

For each of the five flights, wind and potential temperature measurements from the profile descent were supplemented by appropriate radiosonde data to produce a profile of $l^2(z)$ from the surface to a level above any significant structure in the stratosphere; typically about 18 km. For each profile, the eigenvalues, $k_n$, and eigenfunctions of (5) were determined numerically using boundary conditions as described above. $2\pi/\text{Re}(k_n)$ gives the wavelength of the predicted lee waves and $1/\text{Im}(k_n)$ the downstream distance over which the waves forced by a single obstacle would decay in amplitude by a factor of $1/e$. For four of the flights, only one solution with a wavelength of between 5 and 60 km was found in each case. For flight H399 a second solution was also obtained, however, no evidence of this appeared in the observations and it has been disregarded. The values of predicted lee-wavelength and decay length are shown in Table 2 together with estimates of the observed wavelength for each flight.

<table>
<thead>
<tr>
<th>Flight</th>
<th>Observed wavelength: mean value from all runs (km)</th>
<th>Eigensolution from upstream profiles</th>
<th>Momentum flux, $F$: mean value of all runs (N m$^{-2}$)</th>
<th>Peak wave amplitude (m s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>H399</td>
<td>9.6</td>
<td>7.8</td>
<td>$&gt;10^4$</td>
<td>0.9</td>
</tr>
<tr>
<td>H421</td>
<td>10.9</td>
<td>8.4</td>
<td>$&gt;10^4$</td>
<td>2.0</td>
</tr>
<tr>
<td>H389</td>
<td>11.6</td>
<td>9.9</td>
<td>1560</td>
<td>1.7</td>
</tr>
<tr>
<td>H117/C453</td>
<td>18.3</td>
<td>12.7</td>
<td>1650</td>
<td>2.2</td>
</tr>
<tr>
<td>H428</td>
<td>20.9</td>
<td>21.9</td>
<td>101</td>
<td>2.4</td>
</tr>
</tbody>
</table>

Due to the difficulty of adequately modelling the spectral distribution of topography, it is not possible to use the eigenfunction, $w^*(k_n, z)$, to predict absolute values of wave amplitude, although it is possible to make some qualitative comparison with the observations. Spectral analysis of $w$ confirms that for the majority of runs from the five flights, the variance of $w$ was dominated by the lee waves. Thus, assuming that all the variance is due to a single sine wave of constant amplitude, it is possible to define a run-mean wave amplitude by $\tilde{w} = (2\sigma_w^2)^{1/2}$, where $\sigma_w^2$ is the measured variance of $w$. The eigenfunction $w^*(k_n, z)$ after correction for density effects (see Eq. (4a)), may be scaled by an appropriate constant to compare its behaviour with the measurements of $\tilde{w}$. The results of this procedure are shown in Fig. 9.

The results in Table 2 and Fig. 9 show that two distinct categories of lee wave motion may be identified. Flights H389, H399 and H421 represent the first type, with wavelengths of around 10 km which are strongly trapped, as shown by the large $1/e$ decay lengths. In each of these three cases the eigenfunction has a maximum at around 2-5 km and decays to relatively small values at high altitudes, which is supported by the observed
Figure 9. Comparison of measured wave amplitude with linear theory. The points represent the measured run-mean wave amplitudes $\tilde{w}$, whilst the continuous curves show the density-corrected eigenfunction, $(\rho(z)/\rho(z))^{1/2}w^*(k_x, z)$, derived from upstream soundings. The broken curve in (e) shows the density-corrected eigenfunction obtained for flight H117/C453 using the modified wind profile shown in Fig. 10 and described in the text.

wave amplitudes, $\tilde{w}$ generally decreasing with height. Flight H428 represents the second type with a rather longer lee-wavelength of around 20 km, with which the theoretical value is in good agreement, and a decay length of 101 km indicating relatively weak trapping. The eigenfunction has a peak at a higher altitude of about 5 km and values in the stratosphere which are of similar magnitude to those in the lower troposphere. In the height range 2 to 6 km the available observations of $\tilde{w}$ are in good agreement, showing generally increasing values in this region.
5. DISCUSSION

(a) *Comparison of observed wavelength and amplitude variation with 2D linear theory*

As shown in the preceding section, 2D linear theory is able to identify correctly the two broad categories of lee wave motion seen during the five flights in this study, with particularly good agreement between theory and observation being obtained for flight H428. In the case of flight H117/C453, however, linear theory gives a rather poor estimate of the wavelength and fails to predict the presence at high altitudes (12 to 13 km) of wave amplitudes comparable with those observed at lower levels. This suggests that on this day, the \( l^2(z) \) profile obtained from the upstream descent was in some way unrepresentative of the air actually passing over the mountains. Vergeiner (1971) and Berkshire and Pickersgill (1978) have calculated linear lee wave eigenvalues, the former using a multilayer approximation to measured \( l^2(z) \) profiles whilst the latter used a simple model of three layers. Both show that in some cases the lee wave properties can be sensitive to small changes in \( l^2 \). For any particular flight day, there are a number of reasons why the \( l^2(z) \) profiles used in the above calculations may be unrepresentative. On some flights, it is clear that the wind and temperature measurements made on the profile descent were contaminated by wave effects due to topography further upstream. Smoothing was applied subjectively to remove these effects whilst maintaining the sharpness of the temperature inversion at low level. The experimental part of the flights necessarily occupied about 3 to 4 hours and it is possible that changes in the background wind and temperature fields sufficient to alter the lee wave characteristics could occur in this time. Such evidence as is available from repeated runs at certain altitudes indicates that this possibility is rather unlikely for the flights in this study. It is notable, however, that the flight where the waves were obviously stationary over a long period, namely H428, was also that on which the best agreement between observation and theory was obtained.

As shown in section 3, there is clear evidence for the presence of a long wavelength disturbance on the H117/C453 flight day. No similar features were observed on the other flights, although it should be noted that they did not reach the altitudes at which the observed long wavelength feature is particularly prominent. It may be seen from Fig. 4(b) that the long wavelength hydrostatic disturbance causes the winds measured directly over the mountains to differ substantially from their values upstream. To show this more clearly, a wind profile was derived from the cross-section at the position 100 km on the alongwind axis. This is shown in Fig. 10 together with the upstream wind profile and the \( l^2(z) \) profiles calculated from each. A new lee wave eigensolution was calculated using this derived \( l^2(z) \) profile, which gave a weakly trapped wave of 17.8 km wavelength and a decay length of 103 km. This wavelength is in better agreement with the observed values, and the new eigenfunction, whilst still differing from observation in the region of the upper troposphere, does at least suggest the existence of relatively high wave amplitudes in the lower stratosphere.

The calculation of lee wave eigensolutions using simulated \( l^2(z) \) profiles has given some insight into their general behaviour. Short wavelength (about 10 km), strongly trapped waves, as observed on flights H389, H399 and H421, tend to result from profiles with high values of \( l^2 \) in a layer in the lower troposphere with a deep layer of low \( l^2 \) above. The profile for flight H389 (Fig. 8) is a good example of this. Long wavelength (about 20 km), weakly trapped lee waves tend to result when the values of \( l^2 \) are more uniform throughout the depth of the troposphere. To obtain values of \( l^2 \) in the lower troposphere which are closer to the typical value of \( 10^{-7} \text{ m}^{-2} \) in the upper troposphere requires either a weak inversion or strong winds at or near the inversion level, as is the case on flight H428 (see Fig. 11). Similar conditions of higher windspeeds at low levels apparently result from the presence of the long wave hydrostatic disturbance on flight H117/C453, and it is possible that this modification of the wind field could account for some of the differences between the observed lee wave properties and those predicted from the upstream \( l^2(z) \) profile.
Figure 10. Flight H117/C453. Profiles of wind, $U(z)$, and Scorer parameter, $P(z)$, from the upstream soundings (continuous curves); and derived for the 100 km alongwind position in Fig. 4(b) as described in the text (broken curves).

Figure 11. Profiles of the background values of wind, $U(z)$, potential temperature, $\theta(z)$, and Scorer parameter, $P(z)$, from upstream soundings on flight H428.
Smoothed cross-sections of the wind component \( u \) were also produced for flights H389 and H428. Whilst showing some differences between upstream values and those directly over the mountains, these were smaller than for the H117/C453 case, and produced only small changes in the predicted wavelengths and in the form of the eigenfunction, \( w^*(k_u, z) \).

(b) \textit{Momentum flux in leaky lee waves}

Eliassen and Palm (1961) derived important relationships between the vertical fluxes of energy and momentum in stationary, two-dimensional, linear mountain waves. They showed that in a disturbance whose amplitude tends to zero at large distances from the mountain and where \( U(z) > 0 \) at all heights, then

\[
\overline{p'w'} = -\overline{U(z)}\overline{\rho' u'w'} \tag{6}
\]

and

\[
\partial(\overline{\rho' u'w'})/\partial z = 0 \tag{7}
\]

where \( \overline{p'w'} \) is the vertical flux of wave energy and the overbar implies an average over the whole disturbance. Thus upward-propagating waves are associated with positive (upward) energy fluxes and negative (downward) momentum fluxes. As shown in section 4, vertical propagation is only possible when \( k^2 < \lambda^2(z) \) so wavelengths for which \( k^2 > \lambda^2(z) \) at all heights will have no associated energy or momentum fluxes.

In general, the variation with height of the Scorer parameter \( \lambda^2(z) \) gives rise to the reflection of wave energy at some levels and hence to the possibility of resonant lee wave modes. In the particular case of the perfectly trapped mode described in section 4, the wave is completely reflected at the level at which \( k^2 = \lambda^2(z) \) and there is no net vertical flux of energy. However, Bretherton (1969) has shown that such modes are still capable of producing a net drag on the mountain as, although the Reynolds stress associated with a perfectly trapped mode is zero, the absence of downstream decay of the disturbance amplitude means that the total wave drag, \( \int_\infty^{\infty} \rho' u'w' \, dx \), is non-zero. Bretherton also shows that if some dissipative process acts on the waves such that its amplitude does eventually decay at some distance downstream, the momentum flux is then zero above the level of dissipation and negative and constant below it. The wave thus removes momentum from the atmosphere at the level at which it is dissipated.

The stability of the stratosphere is generally such that waves with horizontal wavelengths greater than about 6 km are able to propagate vertically so, with typical conditions of wind and stability found over the British Isles, this means that perfectly trapped modes will not be found. Any resonant modes which do exist will be damped, their amplitude decaying downstream due to the leakage of energy into the stratosphere. This positive energy flux into the stratosphere is, from (6), associated with a downward momentum flux which, because of the downstream decay of the wave, will be constant with height.

The difference in behaviour between perfectly trapped and leaky lee wave modes has implications for the actual measurement of wave momentum flux. As the wave becomes more strongly trapped so the disturbance will extend over larger distances downstream. Under these circumstances, the product \( F_L \) obtained from fixed length runs may underestimate the total drag on the mountain. Also the measurements would be expected to be more noisy as the mean momentum flux becomes a smaller fraction of the amplitude of oscillation of \( u'w' \).

The non-divergence of the momentum flux is supported by the results from flight H428 shown in Fig. 7(a), and to a lesser extent by flight H389 shown in Fig. 7(c). Table 2 gives mean values of momentum flux measured over all the runs on each flight. It may be seen that negative values are obtained on flights H399 and H421 when the individual measurements were more scattered. On both these occasions, the linear lee wave solutions indicate that the waves were very strongly trapped. In general terms, large values of momentum flux appear to be associated with long, weakly trapped lee wave modes,
although to define this relationship more closely requires measurements made in a wider variety of $I^2(z)$ profiles.

6. Conclusion

Resonance lee waves have been observed by the MRF aircraft over three different mountainous regions of the British Isles. Two-dimensional linear theory has been used to estimate the lee wavelength and the distribution of wave amplitude with height from the profiles of wind and potential temperature measured upstream of the mountains. In four out of five cases examined here, there is good general agreement between theory and observation on both these factors. In particular, linear theory correctly differentiates between three cases when the lee waves were strongly trapped and another case when the trapping was much weaker. In a fifth case, it is suggested that the rather poor comparison between observed and theoretically predicted lee waves was due to the simultaneous presence of a disturbance of much longer wavelength, which caused sufficient alteration of the background wind field to modify the lee wave properties.

Measurements of the vertical flux of horizontal momentum show that partially trapped lee waves may give rise to values which are comparable with those obtained over the Rocky Mountains. Of the five flights examined here, the largest downward flux was obtained on the flight with the longest lee-wavelength and the weakest trapping. Such long, weakly trapped lee waves tend to occur when there is no sharp peak at low levels in the profile of $I^2$, the Scorer parameter. This might be due to a weak inversion or, as in the case of flight H428 in this study, strong winds occurring at and immediately above the inversion level.

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Appendix

List of symbols

$x, y$ horizontal right-Cartesian coordinates, with the $x$ axis parallel to the reference wind direction
$U(z)$ $x$ component of wind in the background undisturbed flow
$N$ Brunt–Väisälä frequency $N^2 = (g/\theta)(\partial \theta/\partial z)$
$l$ Scorer parameter $l^2 = N^2/U^2 - (1/U)(\partial^2 U/\partial z^2)$
$L_e$ value at upper boundary
$L$ run length
$k_n$ lee wave eigenvalue
$w^*(k_n, z)$ lee wave eigenfunction
$F$ vertical flux of alongwind horizontal momentum
$G(x^*) = \int u'w' \, dx$, integrated from $x_1$ to $x^*$
$\sigma_w^2$ measured variance of $w$
$\bar{w}$ run-mean wave amplitude $= (2\sigma_w^2)^{1/2}$

Other symbols have their usual meaning.
REFERENCES


