Planetary wave modelling of the middle atmosphere: 
the importance of transients

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SUMMARY

A linear planetary wave model of the middle atmosphere is used to diagnose the wavenumber-1 structure for a day during December 1979 given the wave forcing and the time-derivative terms. The time derivatives are calculated using data from the stratospheric sounding unit (SSU) on NOAA 6 for days on either side of the principal day of interest. The effect of the time-derivative terms is significant and the model is found to be in fair agreement with the data in high latitudes although low latitude features are poorly reproduced. The influence of the transient wave components is discussed and in particular some dissipation-like effects are demonstrated which might clarify the role of Rayleigh friction in planetary wave models.

1. INTRODUCTION

Following the work of Matsuno (1970), stationary wave modelling has been increasingly used to study, for example, the influence of the stratosphere on the troposphere (Bates 1977); the effect of troposphere-stratosphere coupling as a sun–weather mechanism (Geller and Alpert 1980); and has also been used to calculate chemical transports (Pyle and Rogers 1980). However, in a companion paper Austin (1982), referred to as JA, has shown that stationary wave modelling may lead to serious discrepancies in comparison with observations and thus the implications of stationary wave modelling in general and of the above studies in particular need to be treated with caution.

JA considered the impact of a travelling wave on the model results while neglecting the influence of changes in wave amplitude. The main conclusions of this work were that travelling waves with periods even as long as 30 days can have a large effect on the solutions and that the stationary solution may have smaller wave amplitude than both retrogressive and progressive waves. This picture is extended in the current work by considering the influence of changing wave amplitudes on planetary wave model results. Whereas stationary wave model results are often considered in relation to a monthly timescale, this might obscure the effects of changes in wave amplitude and the current work therefore considers a shorter period for which the rates of change of the amplitude and phase of the wave may be regarded as fixed.

In practice, to simplify the mathematics we assume that the phase of the wave varies linearly with time and that the amplitude is exponentially growing or decaying within the period chosen. Thus in principle we can consider the propagation of quasi-linear waves while studying the impact of the observed wave transients, whether generated by linear or nonlinear processes, as a perturbation. The period 4–10 December 1979 was chosen, as satellite data demonstrated the presence of a high latitude travelling wave (wavenumber 1) with amplitude decreasing monotonically with time (assumed exponential).

First, stationary wave and travelling wave calculations were performed, based on SSU data from NOAA 6 and NMC data for 7 December 1979, in order to compare the characteristics of the solutions with those of the monthly mean for December 1979 given in JA. Second, by diagnosing time-derivative terms from the satellite data and using these values in a planetary wave model calculation for 7 December 1979, the influence of transients on the results was considered. Further, the importance of these terms for a monthly mean is discussed and in particular the use of Rayleigh friction in stratospheric stationary wave models is questioned.

Crane et al. (1980) have shown with the use of satellite data that there is an apparent momentum imbalance in the mesosphere. It can be assumed that the implied zonal flow deceleration is due to the effects of gravity waves and tides since Holton (1982), for

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example, has calculated a similar zonal flow deceleration generated in an internal gravity-wave-breaking model. These effects are usually crudely parametrized, in numerical models, by use of Rayleigh friction, i.e. the friction term is proportional to the eddy or total velocity. No such imbalance exists in the stratosphere but nonetheless substantial Rayleigh friction is usually employed there (e.g. Matsuno 1970). In this paper it is shown that the stratospheric Rayleigh friction used in JA has a similar effect on solutions as the diagnosed transient terms. Thus it is suggested that the stratospheric Rayleigh friction used in stationary wave models is necessary because of the neglect of wave transience.

2. SATELLITE DATA ANALYSIS

The observational data used both in the model and for the verification of results were obtained from radiances measured by the stratospheric sounding unit of NOAA 6. Thickness data came from the daily analyses carried out by the Meteorological Office (Pick and Bonsombe 1980), which were vertically interpolated to model levels by the fitting of bicubic splines. NMC 100mb analyses were then used to produce geopotential height data. In JA the daily analyses, obtained in this way, were subsequently averaged to provide a monthly mean data set.

The mean zonal winds required for the wave calculations were obtained from the appropriate geopotential height data set using an approximation to the gradient wind. During the periods studied the waves do not have large amplitudes which implies that zonal symmetry can be used as a first approximation in calculating the radius of curvature of the isobars. The gradient wind in the zonal direction, \( u_{gr} \), therefore, is given approximately by

\[
-(1/\alpha)(\partial \psi / \partial \theta) = u_{gr} + u_{gr}^2 / f r .
\]

(1)

\( r \) being the radius of curvature of the height contours in the horizontal plane, which is approximated by \( a/\tan \theta \). Equation (1) was solved at each model point for \( u_{gr} \) given \( \psi \), the geopotential. All other variables are as defined in appendix A. The local gradient winds were averaged to give the zonal means used in the model. To ensure that the upper boundary condition for the model had little influence on the wave calculations in the region up to 1mb (where verification data exist) the model upper boundary was set at 0.01mb. This requires a specification of the zonal winds above the data region and these were obtained from climatological temperatures using gradient wind and hydrostatic assumptions. The mean zonal wind for 7 December 1979 is shown in Fig. 1.

3. DESCRIPTION OF THE DIAGNOSTIC MODEL WITH SPECIFIED TIME DERIVATIVES

We start by considering a general time-dependent wave field and go on from the primitive equations to investigate how the time-derivative terms can be incorporated into the equation describing the conservation of potential vorticity used by JA. Section 4 discusses how the satellite data may then be used to solve the equation derived in this section.

A linear time-dependent geopotential wave field \( \psi' \) can be written without loss of generality

\[
\psi'(\theta, \lambda, p, t) = \text{Re} \left[ \psi_k(\theta, p, t) \exp(ik\lambda - \sigma(\theta, p, t)t) \right]
\]

(2)

where the wave perturbation \( \psi_k = A(\theta, p, t) \exp(i\phi(\theta, p)) \) for a phase angle \( \phi \) and amplitude \( A \). \( k \) is the wavenumber and \( \sigma(\theta, p, t) \) is the phase speed which, in general, is a function of latitude, pressure and time. For the steady, stationary wave case \( \sigma = 0 \) and \( A(\theta, p, t) = A(\theta, p) \); for the travelling wave case \( \sigma \) is constant and \( A(\theta, p, t) = A(\theta, p) \).

The linearized primitive equations are (Schoeberl et al. 1977, 1979)
\[ \partial v'/\partial t + Z* u' = -(1/a) \partial \psi'/\partial \theta - \beta_R v' - \bar{\omega} \partial v'/\partial \lambda \]  
(3)

\[ \partial u'/\partial t + \bar{\omega} \partial u'/\partial \lambda + w' \partial \bar{u}/\partial z - Zv' = -(1/a \cos \theta) \partial \psi'/\partial \lambda - \beta_R u' \]
(4)

\[ \partial T'/\partial t + H'/c_p = \bar{\omega} \partial T'/\partial \lambda + (u'/a) \partial T/\partial \theta + w's/R \]
(5)

\[ (1/a \cos \theta) \partial (v' \cos \theta)/\partial \theta + \partial u'/\partial \lambda \]

\[ = w' - \partial w'/\partial z \]
(6)

where

\[ Z = 2\Omega \sin \theta - (1/a)(\partial \bar{u}/\partial \theta) + (\bar{u}/a) \tan \theta \]

\[ Z* = 2\Omega \sin \theta + (2\bar{u}/a) \tan \theta \]

and all other variables are defined in appendix A.

We wish to specify the time-derivative terms which cannot be calculated with a steady state model, in order to calculate the planetary wave amplitude at a single time. We do this by diagnosing appropriate terms from the satellite data. It turns out to be simple, mathematically, to represent these terms in the form of 'friction'- and 'travelling wave frequency'-like terms. Further, such a representation provides useful physical insight, as we shall see later. Consequently, we estimate the terms \((1/u')(\partial u'/\partial t), (1/v')(\partial v'/\partial t), (1/T')(\partial T'/\partial t)\) from the satellite data. Note that we are here assuming that the amplitudes of \(u', v'\) and \(T'\) vary exponentially and that the phases vary linearly with time within the period chosen. Also, in order to expose the analogy with 'friction' and travelling wave frequency' terms we write the diagnosed terms in the form

\[ (1/u')(\partial u'/\partial t) \equiv \alpha_u - i\sigma_u \]  
(7a)

\[ (1/v')(\partial v'/\partial t) \equiv \alpha_v - i\sigma_v \]  
(7b)

\[ (1/T')(\partial T'/\partial t) \equiv \alpha_T - i\sigma_T \]  
(7c)

where complex notation is used throughout. As will emerge later in this section and in section 4 the real parts of the above representations are analogous to friction or Newtonian cooling while the imaginary parts are analogous to travelling wave terms. Substi-
tuting Eqs. (7a), (7b) and (7c) into the primitive equations and solving Eqs. (3) and (4) for \( v' \) and \( u' \) gives

\[
\xi_0 v' = \frac{iv_u}{a} \frac{\partial \psi'}{\partial \theta} - \hat{Z}^* \left( \frac{ik \psi'}{a \cos \theta} + w' \frac{\partial \hat{u}}{\partial \zeta} \right) \tag{8}
\]

\[
\hat{Z}^* u' = - \frac{1}{a} \frac{\partial \hat{u}}{\partial \theta} - \frac{i v_u}{\xi_0} \left( \frac{iv_u}{\xi_0} \frac{\partial \psi'}{\partial \theta} - \hat{Z}^* \left( \frac{ik \psi'}{a \cos \theta} + w' \frac{\partial \hat{u}}{\partial \zeta} \right) \right) \tag{9}
\]

The above equations have been written for zonal wavenumber \( k \) so that \( \partial / \partial \lambda = ik \).

Using a Newtonian cooling approximation, \( H' = -a_0 c_p T' \), the thermodynamic equation can be written in terms of \( \psi' \)

\[
w' \left( s - \frac{\hat{Z}^*}{a \xi_0} \frac{\partial \hat{u}}{\partial z} \frac{\partial^2 \psi'}{\partial \theta \partial z} \right) = -iv_T \frac{\partial \psi'}{\partial z} - \frac{i}{\xi_0} \left( -\frac{v_u}{a^2} \frac{\partial \psi'}{\partial \theta} + \frac{\hat{Z}^* k \psi'}{a \cos \theta} \frac{\partial^2 \psi'}{\partial \theta \partial z} \right) \tag{10}
\]

where

\[
\xi_0 = v_u v_v - \hat{Z} \hat{Z}^*
\]

\[
v_u = k \omega - \sigma_u - i \alpha_u - i \beta_R
\]

\[
v_v = k \omega - \sigma_v - i \alpha_v - i \beta_R
\]

\[
v_T = k \omega - \sigma_T - i \alpha_T - i \alpha_0
\]

Other variables are defined in appendix A.

Equations (8) and (9) are substituted into the continuity equation (6). The resulting equation and Eq. (10) are scaled to the planetary wave regime as in Schoeberl et al. (1977, 1979), except that greater accuracy is required in low latitudes. Thus, the full accuracy of \( \hat{Z} \), \( \hat{Z}^* \) and \( \xi_0 \) is retained as in the above expressions. \( w' \) is then eliminated between the scaled equations giving the planetary wave equation

\[
\frac{\xi_0}{\cos \theta} \frac{\partial}{\partial \theta} \left( \frac{\cos \theta}{\xi_0} \frac{\partial \psi'}{\partial \theta} \right) + Q_k \psi' - \left( \frac{a^2 \xi_0 v_T}{s v_u} \right) \frac{\partial^2 \psi'}{\partial z^2} + \left( \frac{a^2 \xi_0 v_T}{s v_u} \right) \frac{\partial \psi'}{\partial z} \left\{ 1 - \frac{s}{v_T} \frac{\partial}{\partial z} \left( \frac{v_T}{s} \right) - \frac{k \hat{Z}^* z^2}{s v_T \xi_0} \frac{\partial \hat{u}}{\partial z} \right\} = 0 \tag{11}
\]

where

\[Q_k = \frac{k^2 v_u}{\cos^2 \theta v_u} - \frac{k \xi_0}{v_u \cos \theta} \frac{\partial}{\partial \theta} \left( \frac{\hat{Z}^*}{\xi_0} \right) + \frac{k a^2 \xi_0}{v_u} \left\{ \frac{\hat{Z}^* z^2}{s v_T \xi_0} \frac{\partial \hat{u}}{\partial z} - \frac{\partial}{\partial z} \left( \frac{1}{s} \frac{\partial \hat{u}}{\partial \zeta} \frac{\hat{Z}^* z^2}{\xi_0} \right) \right\} \]

Equation (11) is similar in form to the Matsuno (1970) stationary wave equation which can be recovered by: (i) casting the equation into 'normal form' with the substitution \( \Psi' = \psi' \exp(-z/2) \); (ii) using the approximations \( \hat{Z}^* = 2 \Omega \sin \theta, \xi_0 = -4 \Omega^2 \sin^2 \theta \), \( v_u = v_v = v_T = k \omega - i \alpha \), where \( \alpha = \beta = \beta_R \) is dissipation; (iii) setting \( s \) to a constant; and (iv) neglecting the term 'A' of Eq. (11).

\( \alpha_u \) and \( \alpha_v \), when positive formally have the same effect as friction of a more general nature than Rayleigh friction. \( x_T \), when positive, has a similar effect to Newtonian cooling as can be seen from the expression for \( v_T \). The terms \( \sigma_u, \sigma_v \) and \( \sigma_T \) may be regarded as describing travelling wave disturbances, as more fully explained in the next section. It is worth remarking here that for a uniformly rotating atmosphere we find that \( \sigma_u = \sigma_v = \sigma_T \) = the travelling wave frequency. In general \( \alpha_u \) and \( \alpha_v \) would be expected to have the largest effect on the wave solutions since they both appear in the term \( Q_k \), the 'refractive
index squared', which is known to have a large effect on the wave solutions (Matsuno 1970). Conceptually, then, Eq. (11) can be used to describe all time-dependent processes which can be represented by the exponential assumption, from the simple steady, stationary wave model ($x_u = x_v = x_T = 0$, $\sigma_u = \sigma_v = \sigma_T = 0$, as in JA), through the travelling wave model ($x_u = x_v = x_T = 0$, $\sigma_u = \sigma_v = \sigma_T = \text{constant}$, also in JA), to the fully time-dependent case in which the $\alpha$s and $\sigma$s are specified.

Further, we could, in principle, consider the problem in which the instantaneous values of $\partial u'/\partial t$, $\partial v'/\partial t$ and $\partial T'/\partial t$ are specified and use a 'first guess' solution to estimate $u'$, $v'$ and $T'$ and hence the terms $\alpha_u$, $\alpha_v$, $\alpha_T$ etc. We might then use an iterative technique to determine the wave solution although there would be no guarantee that the iterations would converge. Note, however, that we cannot consider a time-stepping problem but we are constrained to evaluate the wave field at a given instant. Nonetheless the nature of the $\alpha$s and $\sigma$s allows us to make some general remarks concerning the properties of the transient terms that they represent.

In this work the terms $\alpha_u$, $\alpha_v$, $\alpha_T$, $\sigma_u$, $\sigma_v$ and $\sigma_T$ were diagnosed from satellite data (section 4) and Eq. (11) was solved using the Lindzen and Kuo (1969) method, subject to the following boundary conditions:

(i) geopotential perturbation $\psi' = 0$ at equator and pole;
(ii) $\psi'$ specified on the lower boundary from atmospheric data for 7 December 1979;
(iii) the upper boundary condition a radiating one, implying upward propagation of wave energy.

These boundary conditions are similar to those used by Matsuno (1970) where further discussion may be found.

Calculations were performed with a hemispheric grid-point model with 19 equally spaced horizontal points at 5°-latitude intervals. The vertical resolution was approximately 2 km with 33 levels equally spaced in log(pressure) between 100 and 0-01 mb.

4. Method of Estimating the Time-Derivative Terms

Equation (11), derived in the previous section, can be solved for the wave amplitude and phase providing we can obtain a good approximation to the terms $(1/\nu')(\partial u'/\partial t)$, $(1/\nu')(\partial v'/\partial t)$ and $(1/T')(\partial T'/\partial t)$ at a given instant. These terms are evaluated using centred differencing in time so that the centre of the period must correspond to a time at which data are also available. Since the geopotential data are only available at intervals of one day the minimum period has a span of two days. Because of random errors in the data, however, it is important to select an appropriate length of time over which the time-derivative terms are estimated. The period must not be too short otherwise noise in the data may dominate the real changes taking place. In addition the period must not be too long that the rates of change vary significantly during the period. Because of the related calculations of JA, December 1979 was considered and during this month the period 4–10 December 1979 was suitable as the changes were monotonic throughout.

Model calculations are therefore pertinent to 7 December and furthermore are restricted to wavenumber 1.

The time-derivative terms are calculated using the geostrophic wind approximation and hydrostatic assumption. The terms can thus be written

$$\frac{1}{u'} \frac{\partial u'}{\partial t} = \frac{\partial}{\partial t} \left( \ln u' \right) = \frac{\partial}{\partial t} \left( \ln \frac{\partial \psi'}{\partial \theta} \right) \quad (12a)$$

$$\frac{1}{v'} \frac{\partial v'}{\partial t} = \frac{\partial}{\partial t} \left( \ln v' \right) = \frac{\partial}{\partial t} \left( \ln \psi' \right) \quad (12b)$$

$$\frac{1}{T'} \frac{\partial T'}{\partial t} = \frac{\partial}{\partial t} \left( \ln T' \right) = \frac{\partial}{\partial t} \left( \ln \frac{\partial \psi}{\partial z} \right). \quad (12c)$$
\[ \psi' \] is the complex representation of the wave defined in Eq. (2). Thus Eq. (12b), in particular, simplifies to

\[ \alpha_v \equiv i\sigma_v \equiv (1/v')(\partial \psi' / \partial t) = (1/A)(\partial A / \partial t) - i\sigma. \]

Accordingly we obtain the useful results \( \alpha_v = (1/A)(\partial A / \partial t) \) and \( \sigma_v = \sigma \), the travelling wave frequency.

Expressions for \( \alpha_u, \alpha_T, \sigma_u, \sigma_T \), defined in Eq. (7), can be found in a similar way although they are more complicated by the presence of space derivatives of \( \psi' \). For the simple case \( \sigma(\theta, p) = \text{constant} \), we obtain \( \sigma_u = \sigma_v = \sigma_T = \text{constant} \).

Thus non-zero \( \sigma_u, \sigma_v \) and \( \sigma_T \) are associated with travelling wave disturbances and non-zero \( \alpha_u, \alpha_v \) and \( \alpha_T \) with transient wave disturbances (i.e., changes in wave amplitude). Equation (11) shows that, mathematically, a transient wave disturbance results in terms similar to Rayleigh friction and Newtonian cooling, and a travelling wave disturbance acts to change the effective mean wind velocity. These are quite different effects as JA and the current work demonstrate.

We note that the terms of Eqs. (12a) and (12c) are 2nd derivatives of \( \psi' \) (one derivative in time and one in space) which are therefore prone to error when estimated from observational data. Furthermore, we note that large errors are likely to occur where \( \mid \partial \psi' / \partial \theta \mid, \mid \psi' \mid \) and \( \mid \partial \psi' / \partial z \mid \) are small. \( \mid \partial \psi' / \partial \theta \mid \) is small where the wave amplitude reaches a maximum or minimum with respect to latitude. On 7 December 1979 this occurred near 70°N and 25°N (maxima) and 40°N (minimum) at all levels (see Fig. 2). Similarly \( \mid \psi' \mid \) is small near 40°N and finally \( \mid \partial \psi' / \partial z \mid \) is small where the amplitude reaches a maximum with respect to log(pressure) i.e., near 5 mb at all latitudes. Further, in low latitudes the wave amplitudes are not much greater than the noise level in the data (\( \sim 50-100 \text{m} \)) and therefore model results south of 45°N may be particularly poor.

The terms \( \alpha_u, \alpha_v, \alpha_T, \sigma_u, \sigma_v \) and \( \sigma_T \), defined in Eq. (7), were calculated from the SSU data using the expressions in Eq. (12) with the time derivatives calculated as the difference in the appropriate term between 10th and 4th December 1979 divided by the time interval.
(6 days). The small values of $|\partial \psi / \partial \theta|$, $|\psi|$ or $|\partial \psi / \partial z|$ resulted in large gradients in the calculated values of the time-derivative terms at certain latitudes or heights. These large gradients were removed by linear interpolation across the regions concerned.

5. Satellite-derived data for use in the model

(a) Description of the geopotential data

For 7 December 1979 the wave-1 amplitude (Fig. 2) has a similar structure to that obtained in JA for the monthly mean, with low and high latitude peaks in approximately the same positions. However, for 7 December the high latitude peak is much greater than in the monthly mean. There is a monotonic decrease in peak amplitude from 4 December (1390 m) to 10 December (913 m) although the low latitude peak is approximately constant and averages about 175 m.

The phase of the wave changes steadily throughout the period in high latitudes but is less steady at low and middle latitudes where the amplitudes are small and the phase is not well determined. Both vertical and horizontal phase gradients for 7 December (Fig. 2) are small, especially north of 55° where the minimum in geopotential height is observed to be at or near 135°E right up to the pole.

The gradient wind profile of 7 December (Fig. 1) has a peak value in the polar night jet of 91 m s$^{-1}$ and is similar to recent climatology (Quirroz 1981). The day to day variation is quite small in low and middle latitudes with the peak wind changing from 90 m s$^{-1}$ on the 4th to 98 m s$^{-1}$ on the 10th. In high latitudes the daily variation is more marked with weak easterlies north of 80° on 4 December which becomes westerly by 7 December. The winds on the 7th and 10th are more similar qualitatively.

The static stability values used in the model were calculated as a function of height and latitude from the zonally averaged geopotential data for 7 December. Above 1 mb climatological temperatures were used to compute the stability. The stability profiles are given as a cross-section in Fig. 3.

(b) Description of the time-derivative terms

Figure 4 gives the values of the travelling wavenumber-1 frequency, $\sigma$, as a function

![Image of Figure 3](image-url)

Figure 3. Zonal mean static stability 7 December 1979.
of latitude and pressure. North of 60° the frequency is approximately constant and is equivalent to a travelling wave period of about 30 days. The travelling wave frequency reaches a minimum near 50°, where the waves are stationary at 100 mb, and has a maximum at 35°.

The values of $\sigma_u$ are similar in some respects to the values of $\sigma$. In high latitudes $\sigma_u$ is approximately constant although it is more variable than $\sigma$. The magnitude of $\sigma_u$ also reaches a minimum in middle latitudes although slightly further south than for $\sigma$. However, $\sigma_u$ continues to decrease with decreasing latitude reaching a minimum at 30° where $\sigma$ is a maximum. Mathematically, $\sigma_u$ is more important than $\sigma$ since $\sigma_u$ appears in the term $\nu_u$ of Eq. (11), which has a larger effect on the solutions than the corresponding term $\nu_c$, which depends on $\sigma$. Note that the maxima in $|\sigma_u|$ corresponds to periods conventionally ignored in stationary wave calculations but as shown in JA such periods can have a marked effect on the results. Indeed it may be significant in this respect that the maxima in $|\sigma_u|$ correspond closely to maxima in the wave amplitude (Fig. 2(a)), although the values of $\sigma_u$ are not known accurately in low latitudes because of the small wave amplitudes.
The values of $\sigma_T$ have much less influence on the results and tend to be quite different from both $\sigma$ and $\sigma_u$. $\sigma_T$ has a maximum at $40^\circ$ and a minimum near $20^\circ$. However, these features occur south of $45^\circ$ where the calculated values must be considered doubtful because of the small wave amplitudes there.

Figure 5 shows a cross-section of the values of $\alpha_u$ which was shown, in section 4, to be equal to $\left(1/A\right)(\partial A/\partial t)$. North of $40^\circ$ $\alpha_u$ is negative since the wave amplitudes were decreasing with time. $\alpha_u$ reaches a minimum at about $45^\circ$N and increases steadily towards the pole. South of $40^\circ$N the amplitude features generally remained constant throughout the period resulting in small values of $\alpha_u$.

Values of $\alpha_u$ are of the same order of magnitude but are quite different qualitatively. The values are, to a first approximation, independent of height and can be grouped into positive and negative regions. $\alpha_u$ is generally positive in the bands $20^\circ$–$35^\circ$ and $55^\circ$–$70^\circ$ and negative in the bands $35^\circ$–$55^\circ$ and $70^\circ$–$90^\circ$. These bands correspond roughly to regions in which wave amplitudes increase and decrease, respectively, with latitude. $\alpha_u$ tends to have a large effect on the model calculations because of its occurrence in the $v_u$ term whereas $\alpha_u$ tends to be less important since it appears only in the $v_v$ term. The

Figure 5. $\alpha_v$, $\alpha_u$ and $(\alpha_T + \alpha_0)$ on 7 December 1979.
magnitude of $\alpha_u$ is typically $1 \times 10^{-6} \text{s}^{-1}$, which may be compared with the Rayleigh friction used in JA (profile A, Fig. 6) which increases from $2 \times 10^{-7} \text{s}^{-1}$ at 100 mb to $1 \times 10^{-6} \text{s}^{-1}$ at about 3 mb. Above 3 mb Rayleigh friction is generally greater than $\alpha_u$.

$\alpha_T$ has less effect on the model solutions than either $\alpha_u$ or $\alpha_v$. $\alpha_T$ appears in Eq. (11) only as the sum $x_T + a_0$, which is shown in Fig. 5 for the Newtonian cooling coefficients, $a_0$, of Fig. 6. The values of $x_T + a_0$ tend to be about the same size as $\alpha_u$ and reach a minimum at 50°N. The values are negative north of 40° up to 10 mb and become small in high latitudes. South of 40° the values are positive and of order $1 \times 10^{-6}$. The magnitude of $\alpha_T$ is comparable with or greater than the Newtonian cooling, $a_0$, in the lower stratosphere whereas $a_0$ dominates in the upper stratosphere. $\alpha_T$ is inversely proportional to the vertical gradient of the wave amplitude, which is small near 5 mb (see Fig. 2) in high latitudes. The values of $\alpha_T$ in high latitudes must therefore be considered doubtful.

6. DISCUSSION OF MODEL RESULTS

First, a stationary wavenumber-1 calculation was performed ($x_u = \alpha_u = \alpha_T = \sigma_u = \sigma_v = \sigma_T = 0$) using the forcing and mean zonal wind data for 7 December 1979. The Rayleigh friction profile, A (Fig. 6), was used, which was found in JA to give the best agreement with data in the monthly mean calculation. In this calculation and in all subsequent calculations the Newtonian cooling profile of Dickinson (1973) was used. The results (Fig. 7) show similar discrepancies as were noted for the monthly mean calculation with a maximum amplitude (540 m) considerably less than observed (1200 m, Fig. 2). Additionally, the high latitude maximum was predicted to be at a lower level than observed and the low latitude maximum was absent from the model results. Although one would not expect good agreement with a stationary wave model applied to a single day it is perhaps significant that the model gives similar discrepancies from the appropriate observational data as in the monthly mean calculation. In JA it was suggested that much of this discrepancy might arise through neglect of the travelling wave frequency.

During the period chosen for study wavenumber 1 was travelling with a period of approximately 30 days in high latitudes, as shown in Fig. 4. Before considering the effect of the observed travelling wave frequency on the results it is of interest to investigate the
influence of a constant travelling wave frequency on the wave characteristics. Figure 8 shows the wavenumber-1 amplitude results obtained with the forcing and zonal wind of 7 December 1979 and the friction profile, A, with a uniform travelling wave period of 30 days ($\sigma_u = \sigma_v = \sigma_T = 2\pi/30\text{d}^{-1}$; $N_u = N_v = N_T = 0$). Both the peak amplitude (1070 m) and the height of the maximum are considerably greater than in the stationary wave calculation and are encouragingly close to the behavior observed (Fig. 2). However, the low latitude maximum is still absent from the results.

We next consider the problem in which the exact travelling wave frequency of wavenumber 1 as a function of latitude is incorporated into the model and the 'transient wave' terms are ignored, i.e. $\sigma_u$, $\sigma_v$ and $\sigma_T$ are given by Fig. 4 and $\alpha_u = \alpha_v = \alpha_T = 0$. As in the previous calculations, profile A of Fig. 6 was used for the Rayleigh friction. The amplitude results (Fig. 9) are in good agreement with the observed values (Fig. 2) with a high latitude maximum of 1170 m compared with the observed value of 1200 m. In addition the low latitude maximum is present in the results confirming the suggestion in section 5 that the maximum in $|\sigma_u|$ near 30° is related to the maximum in wave amplitude.

The phase results (Fig. 9) are in reasonable agreement with observations (Fig. 2) although the vertical gradients are too large.

When the calculation is repeated with the observed values of $\alpha_u$, $\alpha_v$ and $\alpha_T$ and the same Rayleigh friction profile, considerably worse results are obtained. Figure 10 shows the amplitude values which reach a peak of 2020 m in high latitudes and are in excess of 4000 m in middle latitudes. However, it was noted in section 5 that in large regions of the domain $N_u$ is negative. Since $N_u$ appears in Eq. (11) only in the form $N_u + \beta_R$ substantial cancellation may occur in regions of negative $N_u$, resulting in small values of the 'effective' friction $N_u + \beta_R$. It is likely that this caused the very large wave amplitudes in the above results.
These results were obtained with a friction profile which was adjusted, within limits, to give the best results for the monthly mean calculation of JA. Such a profile is not justifiable \textit{a priori} for the calculation of 7 December. The calculation was therefore repeated with the 'Oxford' profile of Fig. 6 (Crane et al. 1980), which was calculated by considering the momentum budget derived from satellite data. For the Oxford profile $\beta_K$ is zero in the stratosphere.

The amplitude results (Fig. 11) are much less than was obtained with profile A (Fig. 10), with a high latitude maximum of 1320 m, in good agreement with the observed value (1200 m). In low latitudes above about 25 mb the amplitudes increase rapidly with height reaching a local maximum of 880 m before increasing again. This is in contrast with the data (Fig. 2) and moreover is worse than the results obtained with friction profile, A, and $\alpha_u = \alpha_v = \alpha_T = 0$ (Fig. 9). However, in low levels the amplitude gradients are more nearly in agreement with the data for the full time-dependent case (Fig. 11).
The phase results (Fig. 11) for this calculation are in poor agreement with the data. The phase gradients in middle and low latitudes are much larger than observed (Fig. 2) and appear to be associated with the boundary in middle latitudes separating large from small vertical amplitude gradients. In high latitudes the calculated phase gradients are small in magnitude, as observed, but do not have the correct sign in the lower stratosphere.

As noted in sections 4 and 5, model results are likely to be poor in low latitudes because of difficulties in estimating the time-derivative terms where the wave amplitudes are small. Near the lower boundary, however, data are dominated by the 100 mb NMC data, which has a higher resolution (in both space and time) than the satellite data, enabling reasonably accurate time-derivative terms to be calculated even where the wave amplitude is small.

In high latitudes, where the time-dependent calculation produced good results, the wave amplitudes were very similar to the values obtained with friction profile A and \( \alpha_u = \alpha_v = \alpha_T = 0 \). This suggests that the Rayleigh friction is, in effect, used to parametrize the transient effects described by \( \alpha_u \), \( \alpha_v \), and \( \alpha_T \). In section 5 it was shown that \( \alpha_u \) was negative in a substantial part of the domain. Thus we have the important result that negative \( \alpha_u \) as well as positive \( \alpha_u \) can have the same effect as Rayleigh friction, which is always positive.

This suggests that negative values of the dissipation parameters \( \beta_R \) and \( a_0 \) should have a similar effect on the wave amplitudes as values of opposite sign. The calculation relating to Fig. 9 was therefore repeated with the sign of \( \beta_R \) and \( a_0 \) changed. The amplitude results obtained (Fig. 12(a)) are in very good agreement, with typical differences of about 20%. These results should be compared with Fig. 12(b), which shows the results obtained with \( \beta_R = 0 \) and \( a_0 = 0 \), demonstrating the dissipative nature of \( \beta_R \) and \( a_0 \) even when the sign of the terms is changed. This is discussed more fully in the next section and conclusions are drawn in section 8.
7. THE EFFECT OF NEGATIVE DISSIPATION PARAMETERS

The results described in the previous section demonstrate that when the dissipation parameters $\beta_R$ and $a_0$ have negative signs the model predicts very similar wave amplitudes to that obtained when the dissipation parameters have their normal (positive) sign. This result is essential to our understanding of the impact of transients on model results since if both increasing and decreasing wave amplitudes act as dissipation then we will need non-zero Rayleigh friction in monthly mean calculations.

In this section we attempt to explain the results obtained with negative dissipation parameters by considering, to simplify the mathematics, the Matsuno (1970) model. In this model the planetary wave equation for wavenumber $k$ is

$$L(\Psi) + Q_k(\alpha)\Psi' = 0$$  \hspace{1cm} (13)

where $L$ is a 2nd-order, real linear differential operator (given in appendix B) and

$$Q_k = \frac{\partial \tilde{q}}{\partial \theta} \{ \tilde{u}/a - i\alpha \cos \theta/k \}^{-1} - k^2/\cos^2 \theta - \Omega^2 a^2 \sin^2 \theta/s.$$

$\tilde{q}$ is the potential vorticity of the zonally meaned basic state and $\alpha$ is the Rayleigh friction and Newtonian cooling coefficient, which in this model are assumed to be equal and independent of latitude and height. Taking the complex conjugate (denoted by $^*$) of Eq. (13)

$$L(\Psi^*) + Q_k^*(\alpha)\Psi'^* = 0,$$

i.e. if $\Psi'$ is the wave solution for positive dissipation $\Psi'^*$ is the solution for negative dissipation providing the boundary conditions are satisfied.

The upper boundary condition could not be satisfied if $\Psi'$ had an upward propagating boundary condition since $\Psi'^*$ would then be downward propagating. However, Beaudoin and Derome (1976) have shown that providing the upper boundary is sufficiently
high, a rigid lid upper boundary condition ($w' = 0$) produces similar wave amplitudes as the radiative condition. In fact further calculations (not shown here) by the current author demonstrate that wave amplitudes are not significantly affected by a $\Psi'' = 0$ upper boundary when the boundary is some 30 km from the region of interest. Consequently, if we are only interested in the stratosphere we can set the upper boundary at 0.01 mb where we can set $\Psi'' = 0$. Both $\Psi'$ and $\Psi'^*\Psi''$ will then satisfy the boundary condition.

Suppose the lower boundary condition is $\Psi' = A_0 \exp(i\phi_0)$ for given values of $A_0$ and $\phi_0$, which are functions of latitude only. Then $\Psi'^*\Psi''$, where $E = \exp(2i\phi_0)$, satisfies both boundary conditions when $\Psi'$ satisfies the boundary conditions. We therefore have the final result that given $\Psi'$, a solution for positive dissipation, the approximate solution for negative dissipation is $\Psi'^*\Psi''$ providing

$$\left| L(E, \Psi'^\ast) \right| \ll \left| Q_{\ast}(\alpha) \right|$$

(14)

within the interior of the domain. $L$ is a second-order differential operator (related to $L$) and is given in appendix B. $L(E, \Psi'^\ast)$ is small if the horizontal phase gradient at the lower boundary can be neglected, which tends to be the case in middle to high latitudes, where the wave amplitude is a maximum. Physically this implies that when the waves have very small momentum fluxes at the lower boundary the amplitudes of the positive and negative solutions are identical.

Although the above theory is strictly valid only for waves which satisfy Eq. (13), it can, however, help us to understand the results obtained with a more realistic model. Using Eq. (11) with the transient terms set to zero, in which many of the Matsuno model assumptions have been relaxed, very good agreement in the amplitudes was obtained (see Figs. 9(a), 12(a)) after changing the sign of the dissipation terms. In addition, the vertical phase gradients away from the upper boundary were reversed, as implied by the theory, but they were stronger than predicted in some parts of the domain and weaker than predicted in other parts. This reversal of the phase gradients suggests that in the full time-dependent case the phase is likely to be sensitive to errors in the transient terms particularly where $z_\alpha$ changes sign. This may explain the poor phase results given by the model in comparison with observed data.

8. Summary and Conclusions

Wavenumber-1 calculations for the stratosphere were performed for 7 December 1979 using an extended version of the stationary wave model of Schoeberl et al. (1977, 1979), with the geopotential height specified at 100 mb. Wavenumber-1 transient terms were empirically determined from satellite data from the stratospheric sounding unit (SSU) onboard NOAA 6. The impact of these transients on the model results was investigated by inserting the empirically determined terms into the wave equation. The terms were calculated over the period 4–10 December 1979 which contained an appreciable travelling wave component in the data as well as transients of fairly fixed behaviour throughout the period.

In the first calculation wavenumber 1 was assumed to be steady and stationary and the model predicted much lower amplitudes than observed, as was also noted by Austin (1982) for the monthly mean calculation. Although one would not expect very good agreement with observations using a stationary wave model applied to a single day, introducing the travelling wave frequency had a very marked effect on the results, as was also found by Austin (1982) for the monthly mean. Further, in the calculation of 7 December 1979 with the empirically determined wave frequency (as a function of latitude and height) and using friction profile A of Austin (1982), good qualitative agreement with the SSU data was achieved. Considering the effects of changes in wave amplitude introduces terms which are analogous to Rayleigh friction except that the terms can have a positive or negative sign. When these effects were included in the model, again using
friction profile A, very large amplitudes were calculated. This was because large parts of the model domain contained regions of decreasing wave amplitude the effect of which was a term of opposite sign to Rayleigh friction. Consequently the effective dissipation was much reduced resulting in large wave amplitudes.

Friction profile A was adjusted to improve the results of the stationary wave calculation for the December 1979 monthly mean and is not justifiable a priori. In order to obtain a more realistic profile the results of Crane et al. (1980) were considered. They found that in a 2D circulation model the mesospheric zonal flow acceleration induced by mean meridional motion could not be balanced by momentum fluxes calculated from satellite observations. This momentum imbalance was represented by a Rayleigh friction profile (the ‘Oxford’ profile) which has zero values in the stratosphere increasing rapidly in the mesosphere. Much of this momentum imbalance is thought to be caused by the breaking of internal gravity waves in the mesosphere as calculated for example by Holton (1982).

When the wavenumber-1 calculation was repeated with all the transient terms and wave frequency terms included and using the Oxford friction profile, the results in high latitudes were again in good agreement with observations. This suggests that the transient terms act like dissipation in their effect on wave amplitudes. Moreover, both increasing and decreasing wave amplitudes can act as dissipation.

It should be stressed that the results obtained with the empirically determined transients can only be interpreted in qualitative terms since broad assumptions have been made in their calculation. Firstly, the wave amplitudes from the satellite data have a typical ‘noise’ level of 50–100 m, which is a significant portion of the observed decrease during the 6-day period (approximately 500 m at the peak). Secondly, the required empirical terms are second-derivative quantities which are particularly prone to error when estimated from observational data. Also, although the transient terms were assumed to be constant for the period, thus implying exponential time variation in the wave amplitude for 6 days, we could in principle have made the empirical terms a function of time and used only those values appropriate to 7 December 1979. Only lack of data prevented us from doing so. Consequently, although we have artificially considered a specific type of time dependence (which is apparently reasonably well satisfied during the period chosen), this should not be a limitation on the conclusions reached.

One might conclude from this that, because of the analogy between transients and friction, both positive and negative values of Rayleigh friction can act as a dissipative effect on the wave amplitudes calculated by stationary wave models. Indeed, mathematical analysis (section 7) shows that when the momentum fluxes at the lower boundary are small, stationary wave models would predict the same amplitudes for the positive dissipation case as for the negative dissipation case. Further, although these momentum fluxes were not quite negligible for 7 December 1979, good agreement between the two cases was nonetheless obtained when the appropriate calculations were performed.

These results have important implications for the use of stationary wave models, such as Matsuno (1970), to predict monthly mean amplitudes. If both increasing and decreasing wave amplitudes act like dissipation the mean effect over a long time will not be zero. This suggests that non-zero friction will improve the results of stratospheric stationary wave models even though satellite data (Crane et al. 1980) show that there is no physical basis for the amount of dissipation typically assumed.

Finally, although the wave equation has been assumed to be linear, Austin and Palmer (in preparation) have shown that this cannot necessarily be assumed a priori. In addition nonlinear wave interactions may have been responsible for at least part of the empirically determined transient effects and indeed using a fully time-dependent primitive equation model Austin and Palmer also found some dissipation-like effects due to nonlinearities.
APPENDIX A [SYMBOLS]

\[ \begin{align*}
\psi & \quad \text{geopotential} = \tilde{\psi} + \psi' \\
\Psi' & \quad \text{normalized geopotential perturbation} = \psi' \exp(-z/2) \\
a & \quad \text{radius of earth} \\
\theta & \quad \text{latitude} \\
\lambda & \quad \text{longitude} \\
\Omega & \quad \text{angular velocity of earth} \\
p_0 & \quad 100 \text{ mb} \\
z & \quad -\ln(p/p_0) \\
u & \quad \text{zonal wind} \\
v & \quad \text{meridional wind} \\
w & \quad \text{vertical wind} = \frac{dz}{dt} \\
\bar{\omega} & \quad \frac{\bar{u}}{a} \cos \theta \\
H & \quad \text{atmospheric heating rate} \\
R & \quad \text{gas constant for dry air} \\
s & \quad \text{static stability} = \frac{R(T/c_p + \partial T/\partial z)}{s} \\
k & \quad \text{wavenumber} \\
\alpha_u, \alpha_v, \alpha_T & \quad \text{transient wave terms defined by Eqs. (7a), (7b) and (7c)} \\
\sigma_u, \sigma_T, \sigma_v & \quad \text{travelling wave terms defined by Eqs. (7a), (7b) and (7c)} \\
\sigma & \quad \text{phase speed of wave} \\
\beta_R & \quad \text{Rayleigh friction coefficient} \\
a_0 & \quad \text{Newtonian cooling coefficient}
\end{align*} \]

In addition, overbars are used to denote zonal means; primes, perturbations from the zonal mean. Other symbols have their usual meaning.

APPENDIX B

The operator \( L \) for the Matsuno (1970) model is given by

\[
L(\Psi') = \frac{\sin^2 \theta}{\cos \theta} \frac{\partial}{\partial \theta} \left( \frac{\cos \theta \partial \Psi'}{\sin^2 \theta \partial \theta} \right) + \frac{4\Omega^2 a^2 \sin^2 \theta}{s} \frac{\partial^2 \Psi'}{\partial z^2}.
\]

For \( \Psi^*E \), where \( E = \exp(2i\phi_0) \), to be a solution of the Matsuno equation with negative dissipation we need to consider \( L(E, \Psi^*) \).

\[
L(E\Psi^*) = EL(\Psi^*) + \Psi^*L(E) + 2(\partial \Psi^*/\partial \theta) \frac{dE}{d\theta}
\]

since \( E \) is a function of latitude only.

Define

\[
L'(E\Psi^*) = L(E) + (2/E\Psi^*)(\partial \Psi^*/\partial \theta) \frac{dE}{d\theta}
\]

then

\[
L(E\Psi^*) + Q^*(x)E\Psi^* = E\{L(\Psi^*) + Q^*(x)\Psi^*\} + \Psi^*L(E, \Psi^*).
\]

Hence \( E\Psi^* \) is an approximate solution of the Matsuno equation for negative dissipation providing

\[
|L(E, \Psi^*)| \leq |Q^*_x(x)|.
\]

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