A radiation boundary condition for multi-dimensional flows

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(Received 17 November 1982; revised 23 June 1983. Communicated by Professor Herman)

**SUMMARY**

A radiation boundary condition designed for use at open or computational boundaries with multi-dimensional flows is formulated and tested on three two-dimensional problems. Two of the problems simulate a simple wave propagation and possess analytic solutions so that the effectiveness of the boundary condition can be measured in terms of a r.m.s. error. A more subjective analysis must be used in the final problem, which is the simulation of an atmospheric cold front. The proposed radiation boundary condition utilizes the phase velocity projection in each coordinate direction or equivalently a weighted configuration of the phase velocities. This approach greatly reduces the sensitivity on the spatial derivative normally displayed in the phase velocity calculations. Comparisons are made against the traditionally used Sommerfeld radiation condition and various numerical schemes are tested. For multi-dimensional flows the proposed radiation boundary condition is found to give significant improvement over the traditional Sommerfeld radiation condition.

1. **INTRODUCTION**

One of the problems facing modellers of meso- and other small-scale atmospheric phenomena is that in these finite area simulations there is a difficulty in prescribing lateral boundary conditions since no true physical boundary exists. The nature of the environment outside the region under investigation is also strictly unknown and usually changing. When the flow is directed outward at lateral boundaries or where the pattern is exiting a variety of conditions have been utilized, e.g. (a) a Sommerfeld radiation condition; (b) an absorbing boundary; (c) one-sided differencing of the equations; (d) various other types of extrapolation.

Commonly, these procedures are utilized on some very complicated problems where analytical solutions do not exist hence the impact of any one of these boundary or extrapolation techniques is not known fully. For example, Clark (1979) using different expressions for the phase velocity associated with method (a) has found, for flow over a bell-shaped mountain, significant variations in the interior calculations.

What is needed in problems where advection or wave motions dominate, as pointed out by Orlanski (1976), is an ‘open’ boundary condition. Such a condition entails determining whether the ‘flow pattern’ is entering or exiting across a boundary. In the latter case the disturbance should be allowed to propagate out without reflection. It is in this spirit that Orlanski (1976) and Pearson (1974) proposed to use the following form of the Sommerfeld radiation condition at the boundary:

\[
\frac{\partial \phi}{\partial t} + C \frac{\partial \phi}{\partial n} = 0,
\]

where \(\phi\) is the variable, \(C\) the phase velocity, \(t\) the time and \(n\), the coordinate perpendicular to the boundary in question. Pearson proposed estimating \(C\) from a linearized dispersion relation while Orlanski proposed to determine \(C\) locally and hence to predict the boundary value of \(\phi\) without finding the dispersion relation which as a rule is unknown. Unfortunately this determination of \(C\) via Eq. (1), i.e. \(C = - (\partial \phi / \partial t)(\partial \phi / \partial n)^{-1}\), predicts unrealistic phase speeds in the \(n\) direction that range between plus and minus infinity over the cycle of a simple two-dimensional wave (see section 3). This large variation in \(C\) is a direct consequence of substituting \(\phi\) that is multi-dimensional in a formula that only allows for advection in one direction. In this study a radiation boundary condition

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is proposed that contains advection terms for each coordinate direction. The proposed radiation condition is derived by adding together weighted versions of the Sommerfeld radiation condition each valid for a separate coordinate direction, i.e. $n$, may be $x$, $y$ or $z$. This boundary condition is then tested on three two-dimensional problems two of which possess analytic solutions. For multi-dimensional flow the proposed technique is shown to be superior to the traditionally used Sommerfeld radiation boundary condition. We do not make extensive comparisons with other procedures since at outflow boundaries the traditional radiation procedure has already been shown to compare favourably against other extrapolation techniques (Miller and Thorpe 1981).

It should be pointed out that the current concept of a ‘radiation condition’ is not that traditionally used in mathematical physics, but is a numerical procedure to determine inflow or outflow and represents an extrapolation procedure for outflow conditions. Here inflow and outflow refer to the movement of a pattern and are not restricted to the direction of motion across a boundary of a specified velocity component. The terminology ‘radiation condition’ has been retained because these procedures predict the direction a pattern radiates.

Typically in limited area or mesoscale atmosphere studies the boundaries are placed as far as possible from the centre of activity. To test and evaluate the proposed boundary condition a somewhat different philosophy is taken, in that we study the distortion as a phenomenon nears and passes through a boundary.

2. The Radiation Boundary Condition

The procedure proposed by Orlanski (1976) to determine $\mathcal{C}$ locally involves evaluating $\mathcal{C} = (\partial \phi / \partial t)(\partial \phi / \partial n_r)^{-1}$ just one point interior to a specified boundary point using values of $\phi$ previously calculated. These $\phi$ may represent multi-dimensional flow and even though Eq. (1) is valid for multi-dimensional flow note that the determination of the apparent phase velocity in the $n_r$ direction by this equation depends on the spatial variation of $\phi$ in only one direction, i.e. normal to the boundary. Consequently $\mathcal{C}$ can range between plus and minus infinity as the spatial derivative varies over the cycle of a simple two-dimensional wave, as will be shown in section 3. Our goal then is to introduce a radiation boundary condition suitable for multi-dimensional flow which contains modified or weighted phase velocities which are less sensitive to variations in any one spatial derivative.

In the individual radiation conditions

\begin{align*}
\frac{\partial \phi}{\partial t} &= -\hat{C}_x \frac{\partial \phi}{\partial x} \\
\frac{\partial \phi}{\partial t} &= -\hat{C}_y \frac{\partial \phi}{\partial y} \\
\frac{\partial \phi}{\partial t} &= -\hat{C}_z \frac{\partial \phi}{\partial z}
\end{align*}

(2a) (2b) (2c)

the coefficients $\hat{C}_x$, $\hat{C}_y$, $\hat{C}_z$ represent the apparent phase velocities in the $x$, $y$, $z$ coordinate directions measured relative to the mean currents in these directions, and they are given respectively by $\hat{C}_x = \omega / k_x$, where $\omega$ is the frequency of the disturbance and $k_x$ is the wavenumber in the $x$ direction. These apparent phase velocities are not the projections or components of the vector phase velocity $\mathcal{C}$ since the phase speed does not satisfy the rules of vector decomposition. To find the relations between these apparent phase velocities, $\mathcal{C}$, their projections, and $\phi$, we multiply the above equations by the weighting factors $\alpha_x$, $\alpha_y$, $\alpha_z$ under the restriction that we have $\alpha_x + \alpha_y + \alpha_z = 1$. Thus, on taking
the sum of the three resulting equations, we find
\[ \frac{\partial \phi}{\partial t} = -(\alpha_x \hat{C}_x \frac{\partial \phi}{\partial x} + \alpha_y \hat{C}_y \frac{\partial \phi}{\partial y} + \alpha_z \hat{C}_z \frac{\partial \phi}{\partial z}). \tag{3} \]

We can also write
\[ \frac{\partial \phi}{\partial t} = -C \cdot \nabla \phi = -(C_x \frac{\partial \phi}{\partial x} + C_y \frac{\partial \phi}{\partial y} + C_z \frac{\partial \phi}{\partial z}), \tag{4} \]
where $C$ is the vector phase velocity and $C_x$, $C_y$, $C_z$ are the projections of $C$ in the $x$, $y$, $z$ directions. According to the expressions given in the appendix, we find when relating Eqs. (3) and (4) that
\[ \alpha_x = k_x^2 / k^2 = \left(\frac{\partial \phi}{\partial x}\right)^2 / G, \quad \alpha_y = k_y^2 / k^2 = \left(\frac{\partial \phi}{\partial y}\right)^2 / G, \]
\[ \alpha_z = k_z^2 / k^2 = \left(\frac{\partial \phi}{\partial z}\right)^2 / G. \tag{5a, b, c} \]

where
\[ k^2 = k_x^2 + k_y^2 + k_z^2, \quad G = \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2. \tag{5d, e} \]

Thus utilizing Eq. (3) with weights as defined in Eqs. (5a, b, c) is equivalent to using Eq. (4). An expression for the coefficients $C_x$, $C_y$ and $C_z$ in terms of the actual phase velocities is found in the appendix.

Let $n$ be the number of advection terms in the radiation condition then for Eq. (4) $n = 3$, i.e. one advection term for each coordinate direction, while for the Sommerfeld radiation condition $n = 1$. In this study three two-dimensional problems are considered so we rewrite Eq. (4) or equivalently Eqs. (3) and (5) for the $n = 2$ case as
\[ \frac{\partial \phi}{\partial t} = -(C_x \frac{\partial \phi}{\partial x} + C_y \frac{\partial \phi}{\partial y}) \tag{6a} \]
where
\[ C_x = F \frac{1}{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} \tag{6b} \]
and
\[ C_y = F \frac{1}{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} \tag{6c} \]

Note that the relationship between $C_x$ and $C_y$ is given by
\[ C_x = C_y \left(\frac{\partial \phi}{\partial x}\right) \left(\frac{\partial \phi}{\partial y}\right)^{-1}. \tag{7} \]

In (6b) and (6c), $-\frac{\partial \phi}{\partial t}$ has been replaced by $F$ since the general equation describing $\phi$ can be written formally as
\[ \frac{\partial \phi}{\partial t} = -F(x, y, z, t, \phi, \frac{\partial \phi}{\partial x}, \ldots, \frac{\partial^2 \phi}{\partial x^2}, \ldots, \psi, \chi, \ldots). \tag{8} \]

Here $x$, $y$ and $z$ represent the coordinate directions while $\psi, \chi, \ldots$ represent other meteorological variables on which $\phi$ depends. Two methods of evaluating Eqs. (6b) and (6c) are apparent and are now presented.

*(a) An extrapolation approach*

The values of $\phi$ at any lateral boundary, say $\phi_{N,j} = \phi(x_N, y_j)$, $j = 2, \ldots, M - 1$, as predicted by Eq. (6a) must be determined numerically. If Eq. (6a) is evaluated by a leap-frog approach for a time step centred at $t = \tau$ then the correct formulation for $C_x$ and $C_y$ should correspond to these coefficients evaluated at time $\tau$ and centred correctly.
in space. Provided these $C_x$ and $C_y$ are not rapidly varying in time and space an approximation based on previous interior values of $\phi$ can be made, e.g. Orlanski (1976) used $C^*_k = C^{-1}_{k-1}$ in the Sommerfeld radiation boundary condition ($n = 1$), where $C^*_k \approx C(x_N, t = \tau)$. Other schemes are possible but this approach has the advantage that it is independent of the numerical procedure used in the interior, i.e. to solve Eq. (8). However, various improvements can be made and these will be indicated later.

Assuming, like Orlanski, that $F$ is given by $-\partial \phi / \partial t$ centred at $(\tau - 1)$ and $(N - 1)$ we find that at the right-most boundary $(x_N, y_j)$, excluding corners, of the rectangle region bounded by $0 < x < x_N$ and $0 < y < y_M$, Eqs. (6b) and (7) can be approximated by

\[
C_x = -(1/2\Delta t) \left[ \phi^*_{N-1,j} - \phi^*_{N-1,j-1} \right] \left( \partial \phi / \partial x \right) \left( \partial \phi / \partial x \right) + \left( \partial \phi / \partial y \right)^2 \right) \right]^{-1} \tag{9a}
\]

\[
C_x = C_x \left( \partial \phi / \partial y \right) \left( \partial \phi / \partial x \right) \right)^{-1} \tag{9b}
\]

where

\[
\partial \phi / \partial x = \left\{ \left[ \left( \phi^*_{N-1,j} + \phi^*_{N-1,j} - \phi^*_{N-2,j} \right) / \Delta x \right] \right\} \tag{9c}
\]

and

\[
\partial \phi / \partial y = \left\{ \left( \phi^*_{N-1,j+1} - \phi^*_{N-1,j} \right) / \Delta y \right\} \tag{9d}
\]

Thus each of the $C_x$ and $C_y$ are located spatially, in the upstream sense, at $(x_{N-1}, y_j)$ and about the $(\tau - 1)$ time step. These coefficients can be used to predict $\phi^*_{N,j}$ via, e.g., an implicit formulation of Eq. (6a), yielding

\[
\phi^*_{N,j} = \left( 1 - C_x \Delta t / \Delta x \right) \phi^*_{N,j-1} + \frac{2C_x \Delta t / \Delta x \Delta t C_y}{\Delta y} \left[ \phi^*_{N,j-1} - \phi^*_{N,j-1} \right] \right] \tag{10}
\]

where $D = 1 + C_x \Delta t / \Delta x$ and with the restriction given in Orlanski expanded so that

\[
C_x = \left\{ \begin{array}{ll}
0, & \text{if } C_x < 0 \\
\Delta x / \Delta t, & \text{if } 0 \leq C_x \leq \Delta x / \Delta t \\
\Delta y / \Delta t, & \text{if } C_x > \Delta x / \Delta t \end{array} \right. \tag{11a}
\]

\[
C_y = \left\{ \begin{array}{ll}
\Delta x / \Delta t, & \text{if } -\Delta y / \Delta t \leq C_y \leq \Delta y / \Delta t \\
C_y, & \text{if } C_y < -\Delta y / \Delta t \end{array} \right. \tag{11b}
\]

Similar formulae can be written for the lateral boundaries at $(x_i, y_M)$, etc. If corners are to be computed then both $C_x$ and $C_y$ must be computed using one-sided differencing. Note that the Orlanski formulation for the Sommerfeld radiation condition is recovered above if $C_x = 0$. Also if the properties at an inflow boundary are known then Eq. (10) need not be used. In that situation, a scheme given by Carpenter (1982) allowing for both inflow and outflow may be useful, especially on nested grids. Other finite differencing schemes may also be used to express Eqs. (9) and (10).

Bannon (1979) and Miller and Thorpe (1981) have both proposed an upstream time differencing scheme to evaluate the Sommerfeld radiation boundary condition. The latter authors suggest several possibilities one of which is

\[
\phi^*_{N,j} = \phi^*_{N,j} - r(\phi^*_{N,j} - \phi^*_{N,j-1}) \tag{12}
\]
where $r = C \Delta t / \Delta x$, $0 \leq r \leq 1$, and $r$ is determined from

$$r = (\phi_{N-1}^{k+1} - \phi_{N-1}^{k-1}) / (\phi_{N-2}^{k} - \phi_{N-1}^{k-1}).$$ \hspace{1cm} (13)

In addition, Miller and Thorpe performed a truncation error analysis and found that improved accuracy is obtained when

$$r = (\phi_{N-1}^{k+1} - \phi_{N-1}^{k-1}) / (\phi_{N-2}^{k} - \phi_{N-1}^{k-1}) + (\phi_{N}^{k} - \phi_{N-1}^{k-1}) / (\phi_{N-1}^{k} - \phi_{N-1}^{k-1}) - (\phi_{N-1}^{k} - \phi_{N-1}^{k-1}) / (\phi_{N-2}^{k} - \phi_{N-1}^{k-1}).$$ \hspace{1cm} (14)

We will make comparisons between both the leap-frog and upstream time differencing schemes. Miller and Thorpe have also suggested other amendments that are valid for the leap-frog approach including higher-order-accurate numerical approximations.

Note that Eqs. (13) and (14) require an interior value evaluated for time step $(r+1)$. If the interior calculations are made using a leap-frog or some explicit scheme then the computed solutions at time $(r+1)$ can be utilized in the lateral boundary calculations. Under these conditions $r$ may be replaced with $(r+1)$ in Eqs. (9a) to (9d). The coefficients $C_x$ and $C_y$ are then centred correctly in time for the leap-frog integration scheme given above. For some problems it is possible to compute the phase velocity correctly in both time and space.

(b) Equating to the equation technique

Again assume a leap-frog time integration scheme for Eq. (6a). If in Eqs. (6b) and (6c) it is possible to evaluate $F$, described by Eq. (8), at the boundary at time $\tau$ using one-sided differencing then $C_x$ and $C_y$ are obtained centred spatially at $(x_{N-1}, y_{j})$ and at the $\tau$ time step. If $F$ is a complicated function then this one-sided differencing approach will be more involved than the extrapolation procedure. In addition, one-sided differencing may itself introduce large errors especially if the equation contains terms that are in a state of near quasi-balance, e.g. the geostrophic balance condition in the primitive equations of motion and the thermal wind balance condition in the vorticity equations. This error can be removed to a certain extent provided higher-order finite differencing approximations are used at the boundaries. It is best to avoid this type of error if possible. Nevertheless in our model problems A and B this technique can be used to help measure the error introduced by using the extrapolation procedure described above. Thus given the values of $C_x$ and $C_y$ for Eqs. (6b) and (6c) the value of $\phi_{N,j}^{*+1}$ is obtained as before via Eq. (10).

3. Three problems

(a) Problem A: constant advection

A two-dimensional constant advection problem in non-dimensional form given by

$$\frac{\partial u}{\partial t} + \bar{u} \frac{\partial u}{\partial x} + \bar{v} \frac{\partial u}{\partial y} = 0, \hspace{1cm} 0 \leq x \leq 1, \hspace{0.5cm} 0 \leq y \leq 1, \hspace{0.5cm} t > 0$$ \hspace{1cm} (15)

is solved for $\bar{u} = \bar{v} = 1$ with initial conditions

$$u(x, y, 0) = A (\sin 2\pi x)(\sin 2\pi y).$$ \hspace{1cm} (16)

The boundary conditions along the inflow boundaries at $y = 0$ and $x = 0$ are made to satisfy

$$u(x, y, t) = A(\sin 2\pi(x - t))(\sin 2\pi(y - t)).$$ \hspace{1cm} (17)
This solution projects a pattern which moves at a 45° angle toward the upper right hand corner.

Equation (15) is solved numerically using a leap-frog scheme, with Robert’s filter, for a time step of 0.01 on a 21 by 21 equally spaced grid having \( \Delta x = \Delta y = 0.05 \) and applying the open boundary conditions of the form discussed in section 2 at the outflow boundaries along \( x = 1 \) and \( y = 1 \). The constant \( A \) is assigned the value of 100. The r.m.s. error is computed in the traditional manner.

\[(b) \quad \text{Discussion of results of problem A}\]

We show in Table 1 the r.m.s. error after one hundred time steps. The results using both the Sommerfeld \( (n = 1) \) and the newly proposed \( (n = 2) \) radiation conditions are shown for two methods of evaluating the coefficients \( C_T, C_r \) or \( C \). The extrapolation approach, described in Eqs. (9a-d), as indicated goes back one time step from the last known value at the boundary and interior to the boundary one space step to determine the \( C_r \) and \( C_T \) while the equating to the equation technique evaluates these coefficients correctly in time and space using one-sided finite differencing of terms in the governing equations. Thus \( F = -\partial u/\partial t \) in Eqs. (6b, c) for the extrapolation approach while in the equating to the equation technique \( F = \partial u/\partial x + \partial u/\partial y \). Orlanski first proposed the extrapolation procedure with the Sommerfeld radiation boundary condition. Under the extrapolation approach note that there is a five-fold reduction in the r.m.s. error when the number of advection terms in the radiation boundary condition is increased from one \( (n = 1) \) to two \( (n = 2) \). Next note that the r.m.s. error is further reduced for both radiation conditions by using the equating to the equation technique. In this problem the radiation boundary condition for \( n = 2 \) has exactly the same form as the equation being solved so consequently the r.m.s. error from the equating to the equation technique is essentially identical to that given by using one-sided finite differencing. Note that one-sided differencing is equivalent to taking \( C_T = C_r = 1 \). From Table 1 we conclude that limiting the radiation condition to the Sommerfeld \( (n = 1) \) boundary technique increases the error substantially, while correctly centring the \( C_s, C_r \) or \( C \) coefficients in time and space reduces the error. Also using the known analytical solution on all boundaries does not reduce the error due to inaccuracies in the interior numerical calculations.

In Fig. 1 values of \( C_s \) for the first fifty time steps are shown at location \( (x, y) \). Notice how the phase velocity predicted by the Sommerfeld equation (solid line) fluctuates and must be confined below \( \Delta x/\Delta t = 5 \). In contrast the dotted curve \( (n = 2, \text{equated to} \)
the equation technique) compares favourably with \( C_x = 1 \) which together with \( C_y = 1 \) give results equivalent to one-sided differencing of the governing equation, i.e. Eq. (15). The dotted curve is obtained from

\[
C_x = \frac{(\partial u/\partial x + \partial u/\partial y)(\partial u/\partial x)((\partial u/\partial x)^2 + (\partial u/\partial y)^2)}{(\partial u/\partial y)^2}^{-1}
\]

(18a)

using one-sided differencing for \( \partial u/\partial x \) at the boundary but otherwise centred correctly in time and space. The dashed curve \( (n = 2) \), extrapolation technique indicates that the extrapolation technique is not completely satisfactory in this problem during the first twenty-five time steps. A higher-order numerical approximation similar to those suggested by Miller and Thorpe would be beneficial. As a consequence of \( \partial u/\partial x \) approaching zero note that the curves in Fig. 1 are nearly zero at time step twenty-seven. In the exact solution this occurred at time step twenty-five.

Substituting the analytical solution into the Sommerfeld boundary condition, Eq. (1), yields an analytically predicted time-dependent solution for the phase velocity, i.e.

\[
C = 1 + \frac{\tan 2\pi(x - t)/\{\tan 2\pi(y - t)\}}{2\pi(y - t)}
\]

(18b)

which becomes infinitely large in magnitude when

\[
2\pi(y - t) = \pm j\pi, \quad j = 0, 1, \ldots \quad \text{or} \quad 2\pi(x - t) = \pm \frac{1}{2}h\pi, \quad h = 1, 3, 5, \ldots
\]

This explains why in the numerical procedure the value of \( C \) (solid line, Fig. 1) fluctuates rapidly and is quasi-periodic in time. Nevertheless the value of \( C \) must be restricted (Orlanski 1976), i.e. \( 0 \leq C \leq \Delta x/\Delta t \), otherwise substitution of \( C \) back into the formula (Eq. (10), \( n = 1 \)) to predict the new boundary value of the dependent variable \( u \) would make no sense.

(c) **Problem B: barotropic vorticity equation**

Rossby waves, a commonly known large-scale atmospheric motion feature, are represented by the solution of the barotropic non-divergent vorticity equation whose linearized version is given by

\[
\frac{\partial \zeta'}{\partial t} + \bar{u} \frac{\partial \zeta'}{\partial x} + \bar{v} \frac{\partial \zeta'}{\partial y} + \nu' \frac{\partial f}{\partial y} = 0
\]

(19)
where $\bar{u}$ and $\bar{v}$ are the basic currents in $x$ and $y$ directions, $f$ is the Coriolis parameter and $u'$, $v'$ and $\xi'$ are the perturbation velocities and perturbation vorticity which are expressed in terms of the perturbation streamfunction, $\psi$, by

$$
\begin{align*}
\psi' &= -\frac{\partial \psi}{\partial y}, \\
v' &= \frac{\partial \psi}{\partial x}, \\
\xi' &= \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2},
\end{align*}
$$

$$
0 \leq x \leq x_N, 0 \leq y \leq y_M. \tag{20}
$$

It is well understood that Rossby waves owe their existence to the variation of the Coriolis parameter with latitude.

For our testing purposes, i.e. calculations limited to a finite region, we will assume a solution of the form

$$
\psi = A \cos(k(x - \bar{C}_x t)) \cos(m(y - \bar{C}_y t))
$$

$$
\xi' = -A(k^2 + m^2) \cos(k(x - \bar{C}_x t)) \cos(m(y - \bar{C}_y t)) \tag{22}
$$

where $k = 2\pi / L_x$ is the zonal wavenumber, $m = 2\pi / L_y$ is the meridional wavenumber while $L_x$ and $L_y$ are the wavelengths in the $x$ and $y$ directions, respectively. The $x$ component of the phase velocity is

$$
\bar{C}_x = \bar{u} - \beta/(k^2 + m^2), \tag{23}
$$

here $\beta = \partial f / \partial y$, while the $y$ component is given by

$$
\bar{C}_y = \bar{v}. \tag{24}
$$

The initial conditions are given by Eqs. (21) and (22) for $t = 0$ while the conditions along $x = 0$ and $y = 0$ are also obtained from the analytical solution. The known outflow boundaries along $x_N = 3000$ km and $y_M = 2250$ km, for both the vorticity and streamfunction calculations, are used for testing. The unknown streamfunction boundary values are computed using the radiation formulation with the coefficients $C_x$ and $C_y$ obtained from the vorticity equation. A leap-frog scheme is used to solve Eq. (19) and SOR technique is used for Eq. (20). A 21 by 16 evenly spaced grid is used in the $x$ and $y$ directions respectively for a grid size of 150 km along with a time step of 2000 s. In addition $\beta = 1.6 \times 10^{-11}$ s$^{-1}$ m$^{-1}$, $A = 10^6$ m$^2$ s$^{-1}$, $\bar{u} = 15$ m s$^{-1}$, $\bar{v} = 0$ or 5 m s$^{-1}$, $L_x = 6000$ km and $L_y = 3000$ km.

(d) Discussion of results of problem B

In Table 2 some r.m.s. errors are given after a total of one hundred time steps or 55-55 h. The results are presented in a format similar to that given in Table 1 except two cases are given. The first two columns of the r.m.s. errors for the vorticity and streamfunction, respectively, are for when the true phase velocity is entirely in the $x$ direction, i.e. $\bar{C}_y = 0$, while columns three and four are for when the phase velocity has components in both coordinates. The latter occurs when $\bar{v}$ is non-zero.

In Table 2 note that when $\bar{v} = 0$ both radiation conditions give approximately the same r.m.s. errors for each category. The extrapolation approach gives the largest errors, as compared to the equating to the equation technique, and predicts changes along the $y = y_M$ boundary that are not very desirable. When the mean flow contains components in both the $x$ and $y$ directions the radiation condition with $n = 2$ again becomes superior, as seen in columns three and four. We also solve Eq. (19) by utilizing standard first-
TABLE 2. R.M.S. ERRORS AFTER 100 TIME STEPS FOR PROBLEM B.

\[ \bar{u} = 15 \text{ m s}^{-1}; \ \bar{v} = 0 \quad \bar{u} = 15 \text{ m s}^{-1}; \ \bar{v} = 5 \text{ m s}^{-1} \]

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Vorticity (10^{-2} s^{-1})</th>
<th>Streamfunction (10^{8} m^2 s^{-1})</th>
<th>Vorticity (10^{-2} s^{-1})</th>
<th>Streamfunction (10^{8} m^2 s^{-1})</th>
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</thead>
<tbody>
<tr>
<td>(I) Extrapolated</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \partial \phi / \partial t + C, \ \partial \phi / \partial n  = 0 )</td>
<td>0.1336</td>
<td>0.8650</td>
<td>0.5971</td>
<td>1.5072</td>
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<td>( \partial \phi / \partial t + C, \ \partial \phi / \partial x + C_x, \ \partial \phi / \partial y = 0 )</td>
<td>0.1308</td>
<td>0.7180</td>
<td>0.1521</td>
<td>0.2878</td>
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<td>(II) Equated to the equation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \partial \phi / \partial t + C, \ \partial \phi / \partial n  = 0 )</td>
<td>0.0112</td>
<td>0.0296</td>
<td>0.0674</td>
<td>0.6995</td>
</tr>
<tr>
<td>( \partial \phi / \partial t + C, \ \partial \phi / \partial x + C_x, \ \partial \phi / \partial y = 0 )</td>
<td>0.0112</td>
<td>0.0270</td>
<td>0.0475</td>
<td>0.1929</td>
</tr>
<tr>
<td>(III) One-sided finite differences</td>
<td>0.0113</td>
<td>0.0270</td>
<td>0.0448</td>
<td>0.1977</td>
</tr>
</tbody>
</table>

order finite difference forms at the known outflow boundaries using time averaging to keep the scheme stable. Comparing the equating to the equation technique \((n = 2)\) and one-sided finite differencing we find that they give nearly identical solutions. This implies at the known outflow boundaries that the coefficients \(C_x\) and \(C_y\) are predicting outflow and thus satisfy the restrictions imposed by Eqs. (11a, b).

In Fig. 2 values of \(C_x\) along the boundary \(x = x_N\) are displayed at the fiftieth time step. The solid and dashed curves are \(C\) for \(n = 1\) and \(C_x\) for \(n = 2\) cases, respectively, obtained by the extrapolation approach. The dotted curve is found from the two-

![Figure 2](image-url)  

**Figure 2.** Same as Fig. 1 except for problem B and computed for the fiftieth time step along the right lateral boundary \((x = x_N)\). The known analytical solution is shown by the \(C_x = \bar{C}_x\) curve.
dimensional equated to the equation technique, i.e. in Eqs. (6) and (7) we let
\( F = \frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial y} + v' \frac{\partial f}{\partial y} \) and evaluate \( \frac{\partial \xi}{\partial x} \) using one-sided differencing but otherwise \( F \) is centred correctly in time and space. The analytically predicted value for the coefficients \( C_x \) and \( C_y \) are \( C_x = \tilde{C}_x \) and \( C_y = \tilde{C}_y \). In contrast to problem A the extrapolated and the equating to the equation approaches, dashed and dotted lines respectively, both predict values for \( C_x \) that agree with \( \tilde{C}_x \) except near location 8 where \( C_x \) is nearly zero since \( \frac{\partial \zeta}{\partial x} \) approaches zero when \( \cos(k(y - \tilde{C}_x t)) \) changes sign. On the other hand just past location 8 the value of \( C \) in the Sommerfeld equation has a large spike since here \( \frac{\partial \xi}{\partial x} \) is small in comparison with \( \frac{\partial \zeta}{\partial t} \). This behaviour is a direct consequence of \( \xi \) being multi-dimensional as revealed in the analytically predicted value of \( C \) found below.

Substituting the analytical solution for the vorticity, Eq. (22), into the Sommerfeld radiation condition gives an analytically predicted value for the phase velocity in the \( x \) direction, i.e.
\[
C = \tilde{C}_x + \tilde{C}_y (m/k) \frac{\tan(m(y - \tilde{C}_x t))}{\tan(k(x - \tilde{C}_x t))}. \tag{25}
\]
Here our previous finding is again repeated since this expression for \( C \) predicts values much in excess of the acceptable upper limit, i.e. \( C = \Delta x/\Delta t \), and values much less than the lower limit \( C = 0 \). The latter is obtained in spite of the fact that the flow is continuously outward. The sharp spike in the solid curve, Fig. 2, reflects this large variability and is commonly found at least at one grid point at almost every time step and often exceeds the maximum magnitude allowable. Note that \( C \) is well behaved if \( \tilde{C}_y = 0 \), i.e. if the mean flow is one-dimensional. It is clearly seen in Eq. (25) and in Tables 1 and 2 that the use of the traditional Sommerfeld radiation boundary condition incurs greater errors as the flow becomes increasingly multi-dimensional.

In Table 3 the r.m.s. errors associated with the upstream time differencing schemes of Miller and Thorpe (1981) are given. The simpler of the two schemes, computed using Eq. (13) for \( n = 1 \), gives nearly identical results with the \( (n = 1) \) case evaluated using Eqs. (9a-d) with \( \tau \) replaced by \( (\tau + 1) \). Utilizing the interior solutions evaluated at time \( (\tau + 1) \) in the lateral boundary calculations does not change the r.m.s. error appreciably either positively or negatively, as seen by comparing Tables 2 and 3.

<table>
<thead>
<tr>
<th>Boundary condition schemes</th>
<th>Vorticity ((10^{-5} \text{s}^{-1}))</th>
<th>Streamfunction ((10^6 \text{m}^2\text{s}^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upstream Eq. (13)</td>
<td>0.128</td>
<td>0.8166</td>
</tr>
<tr>
<td>Eq. (14)</td>
<td>0.0298</td>
<td>0.1245</td>
</tr>
<tr>
<td>Leap-frog ( n = 1 )</td>
<td>0.1285</td>
<td>0.8171</td>
</tr>
<tr>
<td>( n = 2 )</td>
<td>0.1287</td>
<td>0.6861</td>
</tr>
</tbody>
</table>

of \( \tau \) and hence \( C \) by Eq. (14) does show definite improvement except for the streamfunction calculations when there is two-dimensional mean flow. With further integrations the large error in the latter category will eventually deteriorate the vorticity solutions.

Some severe storm modellers have had success using a constant phase velocity in the Sommerfeld radiation condition, e.g. see Klemp and Wilhelmson (1978). Table 4 shows the r.m.s. error when \( C_x \) and \( C_y \) are each held fixed. Using the analytically
TABLE 4. R.M.S. ERRORS AFTER 100 TIME STEPS FOR CONSTANT COEFFICIENTS IN THE RADIATION CONDITION USED IN PROBLEM B.

\[ \begin{array}{c|c|c|c|c|c}
\text{Boundary condition} & a = 15 \text{ m s}^{-1}; \beta = 0 & a = 15 \text{ m s}^{-1}; \beta = 5 \text{ m s}^{-1} \\
C_x & C_y & \text{Vorticity} & \text{Streamfunction} & \text{Vorticity} & \text{Streamfunction} \\
\hline
12.082 & 0.0 & 0.0114 & 0.0298 & 0.6846 & 1.2197 \\
12.082 & 5.0 & & & 0.0434 & 0.1974 \\
5.0 & 0.0 & 0.1521 & 0.6614 & 0.8748 & 1.4602 \\
25.0 & 0.0 & 0.0640 & 0.3658 & 0.6116 & 1.6686 \\
\end{array} \]

predicted values of \( C_x = 12.082 \) and \( C_y = 0 \) or 5 in Eq. (10) gives r.m.s. errors essentially identical to the one-sided finite differencing. From Table 4 we also see that for one-dimensional flow the error incurred by overestimating \( C_x \) is less than when \( C_x \) is underestimated. However, this pattern is not clearly reproduced when the flow is two-dimensional. With the proposed radiation condition it may also be acceptable, under some circumstances, to use a fixed or constant value for each of the coefficients \( C_x \), \( C_y \) and \( C_z \) provided enough information is available to determine the nearly correct magnitudes and directions.

(e) Problem C: cold front model

For the final problem we choose a purely two-dimensional anelastic moist cold front model to simulate the circulation associated with an atmospheric cold front (Ross and Orlanski 1978). A cold front represents a propagating disturbance that cannot be described completely as wave motion, thus problem C differs from problems A and B. The governing equations for the \((x, z)\) plane are of the form

\[
\frac{\partial \eta}{\partial t} = -J(\psi, \alpha_0 \eta) + \frac{g}{\theta_0 \partial x} \frac{\partial \theta}{\partial x} - f \frac{\partial v}{\partial z} + \frac{\partial}{\partial x} \left( \nu K \frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial z} \left( \nu \frac{\partial \eta}{\partial z} \right) - \frac{\partial}{\partial x} \\
\frac{\partial v}{\partial t} = -J(\psi, \alpha_0 \nu) + \sigma \nu w + f(u - U_e) + \frac{\partial}{\partial x} \left( \nu K \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial z} \left( \nu \frac{\partial v}{\partial z} \right) \\
\frac{\partial \theta}{\partial t} = -J(\psi, \alpha_0 \theta) + \sigma \theta w + f \frac{\theta}{g} \frac{\partial U_e}{\partial z} + \frac{\partial}{\partial x} \left( \kappa \frac{\partial \theta}{\partial x} \right) + \frac{\partial}{\partial z} \left( \kappa \frac{\partial \theta}{\partial z} \right) + \frac{L \delta}{c_p \pi} \\
\frac{\partial q}{\partial t} = -J(\psi, \alpha_0 q) + \sigma q w + \frac{\partial}{\partial x} \left( \kappa \frac{\partial q}{\partial x} \right) + \frac{\partial}{\partial z} \left( \kappa \frac{\partial q}{\partial z} \right) - \delta \\
\frac{\partial c}{\partial t} = -J(\psi, \alpha_0 c) + \frac{\partial}{\partial x} \left( \kappa e \frac{\partial c}{\partial x} \right) + \frac{\partial}{\partial z} \left( \kappa e \frac{\partial c}{\partial z} \right) + \delta + \sigma c w c.
\]

Here \( u, v \) and \( w \) are the velocity components in \( x, y \) and \( z \) directions; \( \eta = \partial w / \partial x - \partial u / \partial z \) is the \( y \) component of vorticity; \( \psi \) the momentum streamfunction in the \((x, z)\) plane; \( \theta \) the potential temperature; \( g \) the gravity acceleration; \( q \) and \( c \) the water vapour and cloud water mixing ratios; \( \theta_0 = \theta(1 + 0.608 q) \); \( U_e \) the basic state geostrophic wind; \( L \) the latent heat of condensation; \( c_p \) the specific heat at constant pressure; \( \delta \) the condensation rate; \( v \) and \( K \) the eddy viscosity and eddy diffusivity coefficients; \( K \) is a constant used to enhance the horizontal diffusivity; \( \sigma_c = -\partial \ln \rho_0 / \partial z \) the stratification factor of the undisturbed density \( \rho_0 \); \( \pi \) the ratio of temperature to the potential temperature. Details of how to calculate the condensation rate \( \delta \) can be found in Ross and Orlanski (1978) and Kuo and Qian (1981).
In terms of the streamfunction $\psi$, the velocities $u$ and $w$ and the vorticity $\eta$ are given by

$$ u = -\alpha_0 \partial \psi / \partial z, \quad w = \alpha_0 \partial \psi / \partial x \quad (31a, b) $$

$$ \rho_0 \eta = \partial^2 \psi / \partial x^2 + \partial^2 \psi / \partial z^2 + \sigma_2 \partial \psi / \partial z \quad (32) $$

where $\alpha_0 = 1/\rho_0$ and $\rho_0 = \rho_{surf} \exp(-\sigma_2 z)$ is the vertically varying density.

In our calculations the grid spacing is such that $\Delta x \gg \Delta z$. Under these conditions the streamfunction has traditionally been calculated by neglecting the $\partial^2 \psi / \partial x^2$ term in Eq. (32) yielding the well-known hydrostatic approximation. Orlanski (1981) has introduced a quasi-hydrostatic approximation in which a correction containing part of the non-hydrostatic contribution is added to the hydrostatic solution. In this procedure the streamfunction is represented as

$$ \psi = \sum_{j=0}^{m} \psi_j \quad (33) $$

and the components of $\psi$ are obtained from the following equations:

$$ \partial^2 \psi_0 / \partial z^2 + \sigma_2 \partial \psi_0 / \partial z = \rho_0 \eta \quad (34a) $$

$$ \partial^2 \psi_j / \partial z^2 + \sigma_2 \partial \psi_j / \partial z = -\partial^2 \psi_{j-1} / \partial x^2, \quad j \geqslant 1. \quad (34b) $$

Here $\psi_0$ is the hydrostatic component and the $\psi_j, j \geqslant 1$, are an approximation to the non-hydrostatic contribution. The associated boundary conditions are

$$ \text{at } z = 0, \quad \psi_j = 0 \quad \text{for all } j \quad (35a) $$

$$ \text{at } z = H, \quad \psi_0 = \Psi \quad (35b) $$

$$ \psi_j = 0 \quad j \geqslant 1. \quad (35c) $$

The series for $\psi$ converges rapidly provided $\Delta x > \Delta z$. Taking $m = 1$ is sufficient for our purposes. Thus the solution of (32) is obtained successively from Eqs. (34a, b).

At the lateral boundaries the right hand side of Eq. (34b) must be approximated. Instead, as an alternative, we choose to let $\partial \psi_j / \partial x = 0, \quad j \geqslant 1$, i.e. the non-hydrostatic terms do not contribute to the vertical velocity at the lateral boundary. Otherwise there are no lateral boundary conditions required by this procedure.

For the problem in general at the lower boundary, $z = 0$, slip boundary conditions are utilized, e.g. $\eta = 0, \partial \theta / \partial z = 0$ and $\nu$ satisfies the thermal wind relation given by

$$ \partial \nu / \partial z = (g/f \theta_0)(\partial \theta / \partial x). \quad (36) $$

At the upper boundary, i.e. $H = 14$ km, we have a rigid lid so that $w = 0$, thus $\Psi$ is constant, and in addition, $h, \theta$ and $\nu$ keep their initial vertical gradients.

Initially, the thermal wind relation (36) is taken as satisfied by $\nu$ and $\theta$, and $U_0$ is given some vertical variation. Generally, our initial conditions and the values used in the eddy viscosity formulation as well as all pertinent details for $q$ and $c$ are similar to those used by Ross and Orlanski (1978) and will not be restated here.

The general numerical approach is composed of a lumped finite element scheme with a leap-frog time integration. The Arakawa representation of the Jacobian $J(\psi, \beta)$, where

$$ J(\psi, \beta) = (\partial \psi / \partial x)(\partial \beta / \partial z) - (\partial \psi / \partial z)(\partial \beta / \partial x), \quad (37) $$

is obtained when bilinear elements (chapeau) are used in the finite element formulation.
This was first identified by Jespersen (1974). The diffusion terms are lagged one time step and a Robert's filter is used to reduce the tendency of time splitting. A grid with $\Delta x = 20 \text{ km}$ and $\Delta z = 500 \text{ m}$ is used along with a $100 \text{ s}$ time step. The value of $\Psi$ used in Eq. (35b) depends only on the initial $u$ velocity field, the latter is a function of the $z$ coordinate only.

(f) Discussion of results of problem C

Problem C requires the numerical solution of several equations. Even though complicated by the release of the latent heat the procedure is nevertheless straightforward except for the conditions to be used at the open boundaries. At the lateral boundaries we tested a variety of different conditions for the various equations. These tests show that the procedure used by Clark (1979), and many others, is very satisfactory; i.e. the velocity component normal to the lateral boundary is computed at the boundary from the radiation condition while all other dependent variables have zero normal gradients. Thus in our problem at the lateral boundaries $\theta$, $q$, $c$ and $v$ satisfy $\partial(\ )/\partial n = 0$ while the radiation boundary condition is applied to the vorticity equation from which the normal velocity component is computed via the streamfunction.

Our results for the atmospheric cold front simulation are very similar to those obtained by Ross and Orlanski (1978). Details of the various fields, vorticity, streamfunction, etc., are very involved and it is not easy to gauge from these the influence that the lateral boundaries might have. A clearer picture showing the influence of the various radiation boundary condition formulations is gained by examining plots of an averaged quantity or spatial norm versus time. Figures 3 and 4 display our choices.

In Fig. 3 we plot $\tilde{u}'$ versus time where $\tilde{u}'$ is the norm of the perturbation velocity as defined by

$$
\tilde{u}' = \frac{1}{NM} \sum_{i=1}^{N} \sum_{j=1}^{M} |u_{i,j} - U_{ij}|.
$$

\[38\]

Figure 3. For problem C values of $\tilde{u}'$ versus time are shown when in the radiation condition $n = 2$ (solid and dotted curves), for $n = 1$ (dash-dot) and for an upstream time differencing scheme (dashed curve) using Eq. (14).
Here \( u \) is the computed \( x \) component of the velocity field while \( U_\parallel \) is the geostrophic wind which is a function of the vertical coordinate only. The norm is chosen because it allows for an easy interpretation of the magnitude of the perturbation velocity.

In Fig. 3 the solid curve displays the results obtained using the proposed radiation condition \((n = 2)\) for the extrapolation techniques while the dash-dot curve is the Orlanski procedure \((n = 1)\) and the dashed curve is the Miller–Thorpe technique using the improved estimates of \( r \) as given in Eq. (14). The dotted curve, which is essentially identical to the solid curve, uses coefficients \( C_x \) and \( C_y \) \((n = 2)\) centred correctly at time \( \tau \) for the leap-frog scheme but still computed by the extrapolation procedure. In these calculations only the radiation boundary condition has been changed. Also the area of computation coincides with the region over which the averaging is performed. The grid is as defined above for the \( x \) and \( z \) coordinate system utilizing \( N = 55 \) and \( M = 29 \). These curves are to be compared and contrasted with the ‘+’ curve, which gives an average over the same area but when the calculations are performed on a larger area having \( N = 73 \).

Two features are pronounced in comparing the various curves. First, there is some adjustment to the initial conditions and all the curves show an increase in magnitude which reaches a maximum at sixteen hours. During this process the curves are very similar. After the adjustment process the plus curve shows that a quasi-steady state exists. Here the small variations are due to the presence of internal gravity waves. The second feature is now apparent since the other curves reveal that the lateral boundary has a large impact if the disturbance is too close. The Miller–Thorpe procedure is particularly susceptible since boundary influences cause the value of \( \hat{u}' \) to more than double the maximum magnitude observed in the plus curve. The proposed \( n = 2 \) radiation boundary conditions (solid or dotted curves) display the smallest increases for times greater than thirty hours. The small differences between the curves computed using \( n = 1 \) and \( n = 2 \) radiation boundary conditions, before thirty-three hours, can be explained by the fact that away from the frontal zone the \( u \) component of the velocity is at least two orders of magnitude larger than the vertical velocity component. Thus the flow is essentially one dimensional except near the front. This is clearly seen by the magnitudes of \( \bar{w} \) in Fig. 4.

![Figure 4](image-url)  
Figure 4. As Fig. 3 except for \( \bar{w} \) and the upstream scheme uses Eq. (13).
In Fig. 4 values of $\tilde{w}$, where

$$\tilde{w} = \frac{1}{NM} \sum_{i=1}^{N} \sum_{j=1}^{M} |w_{ij}|,$$

are shown for calculations using the same radiation conditions as in Fig. 3 except now the dashed curve is the Miller-Thorpe upstream technique computed using Eq. (13). Again the lateral boundary effects are clearly evident after thirty-three hours in the leap-frog schemes which give very similar results and after twenty-six hours in the upstream procedure (dashed curve). The latter technique generates vertical velocity averages twice as large as the leap-frog scheme. The exact reason the upstream procedure is less effective is unclear in light of its performance in problem B (Table 3). The flow pattern is, however, much more complicated in problem C and it seems reasonable that a boundary procedure that uses the same time integration scheme (leap-frog) as used in the interior calculations would avoid computational modes and be the most compatible for long-time integrations. Williamson and Chen (1982), however, report no difficulties with the upstream procedure in their cloud modelling efforts. In contrast to our study where we allowed the disturbance to interact with the lateral boundary, they of course tried to avoid this and used additional fourth-order diffusion to dampen spurious noise and discourage growth of nonlinear instabilities.

In Eqs. (26) through (30) $w$ appears explicitly in one term in each equation. In our calculations we compute $w$ from calculated streamfunction values via the formula $w = \alpha_0 \psi / \partial x$ in finite element representation.

Along the lateral boundaries, however, the finite element technique predicts each $w$ using streamfunction values in a one-sided differencing scheme from six grid points. To test whether the reduced accuracy associated with the one-sided differencing scheme might be enhancing the error at the lateral boundaries we replaced the direct calculation of $w$ with the condition $\partial w / \partial x = 0$. This greatly reduces the noise generated at the lateral boundaries as seen by comparing values of $\tilde{w}$ in Table 5, computed using $\partial w / \partial x = 0$.

<table>
<thead>
<tr>
<th>Boundary condition schemes</th>
<th>Time (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>25</td>
</tr>
<tr>
<td>Upstream</td>
<td></td>
</tr>
<tr>
<td>Eq. (13)</td>
<td>0.319</td>
</tr>
<tr>
<td>Eq. (14)</td>
<td>0.413</td>
</tr>
<tr>
<td>Leap-frog</td>
<td></td>
</tr>
<tr>
<td>$n = 1$</td>
<td>0.298</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>0.299</td>
</tr>
<tr>
<td>Large area ($n = 2$)</td>
<td>0.247</td>
</tr>
</tbody>
</table>

against our earlier calculations of $\tilde{w}$ in Fig. 4 in which $w$ was calculated at the lateral boundaries. All categories of radiation boundary conditions show improvement with now almost no differences between the $n = 1$ and $n = 2$ cases. However, the leap-frog approach still remains superior to the upstream technique.

In our lumped finite element scheme every term in Eqs. (26) to (30) is expressed in a nine-point configuration except for the time term. Hence the contribution from
every term that includes $\omega$ can be interpreted in terms of the streamfunction as representing even more grid points since the calculation of each $\omega$ involves $\psi$ on nine grid points. Consequently using $\omega$ instead of the streamfunction representation explicitly obviously results in some smoothing. This fact is reflected in the values of $\tilde{\omega}$ shown in Table 6. In the last two lines in Table 6 it is clear that when the calculations are performed using $a_0 \partial \psi / \partial x$ explicitly in place of $\omega$ the values of $\tilde{\omega}$ are nearly a magnitude larger after forty-eight hours. The differences are small during the first six hours but as latent heat is released at grid point locations in the condensation process the differences grow. Unfortunately the intensified flow characteristics obtained when using the streamfunction representation explicitly also enhances all noise and the solutions become very noisy after twenty simulation hours. Thus it is difficult to test radiation conditions when the meteorological fields deteriorate beyond the point of interpretation. Nevertheless the result for $\tilde{\omega}$ displayed in Table 6 agrees essentially with our previous findings.

<table>
<thead>
<tr>
<th>Boundary condition schemes</th>
<th>Time (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Computed using $\psi$</td>
<td></td>
</tr>
<tr>
<td>Upstream</td>
<td></td>
</tr>
<tr>
<td>Eq. (13)</td>
<td>0.104</td>
</tr>
<tr>
<td>Eq. (14)</td>
<td>0.104</td>
</tr>
<tr>
<td>Leap-frog</td>
<td></td>
</tr>
<tr>
<td>$n = 1$</td>
<td>0.104</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>0.104</td>
</tr>
<tr>
<td>Large area ($n = 2$)</td>
<td>0.104</td>
</tr>
<tr>
<td>Computed using $\omega$</td>
<td></td>
</tr>
<tr>
<td>Large area ($n = 2$)</td>
<td>0.098</td>
</tr>
</tbody>
</table>

To determine if the error at the lateral boundaries can be reduced even further we tested several other ideas. For example, in an attempt to centre the coefficients $C_r$ and $C_v$ correctly in space we computed two columns of components adjacent to the boundary and then utilized a Taylor series expansion to predict the value at the boundary. This procedure is very similar to improvements suggested by Miller and Thorpe but applied to the $n = 2$ case. These calculations did not significantly improve the results presented in Table 5 for the $n = 2$ case. Using higher-order approximations of the derivatives also did not significantly change our results. Also averaging all quantities over six grid points, as computed via the finite element method with bi-linear basis elements, yields essentially the same results. Remembering that we are solving many equations explains why continued improvement is not obtained when higher-order-accurate calculations are made of the coefficients in the radiation boundary condition for the vorticity equation. Because the disturbance is so near the edge of our computational region the lateral boundary conditions for the other variables are less than satisfactory and are clearly responsible for a good part of the remaining noise found in our solutions. As previously mentioned the equating to the equation technique cannot be utilized in this problem because of the internal balance between two derivative terms in the vorticity equation, i.e. the thermal wind relation.
4. CONCLUSIONS

A radiation boundary condition has been tested on three two-dimensional problems. When the flow is outward across a lateral boundary and when the pattern of movement is multi-dimensional the proposed radiation condition has been found to be clearly superior to various formulations of the traditionally used Sommerfeld radiation condition. For outflow the new radiation boundary condition, as proposed, is equivalent to one-sided differencing of the governing equation provided the coefficients \( C_x \), \( C_y \) and \( C_z \) are correctly centred in time and space. When this centring procedure is not feasible or gives unrealistic results extrapolation procedures provide an alternative technique to determine the phase velocity components.

APPENDIX

Relations between the projections \( C_x, C_y, C_z \) of the vector phase velocity \( \mathbf{C} \) and the apparent phase velocities \( \dot{C}_x, \dot{C}_y, \dot{C}_z \)

According to the definition, the vector phase velocity \( \mathbf{C} \) of the disturbance is given by \( \mathbf{C} = \omega / \mathbf{k} = \left( \omega / k^2 \right) \mathbf{k} \), where \( \omega \) is the frequency of the disturbance and \( \mathbf{k} \) is the vector wavenumber whose components in \( x, y, z \) directions are \( k_x, k_y, k_z \), and

\[
k^2 = \mathbf{k} \cdot \mathbf{k} = k_x^2 + k_y^2 + k_z^2.
\]

Thus, the projection \( C_x \) of \( \mathbf{C} \) in the \( x_i \) direction is given by

\[
C_{x_i} = \left( \omega / k^2 \right) k_{x_i} = \left( \omega / k_{x_i} \right) \left( k_x^2 / k^2 \right) = \left( k_x^2 / k^2 \right) \dot{C}_{x_i}
\]

where \( \dot{C}_{x_i} = \omega / k_{x_i} \) is the apparent phase velocity in the \( x_i \) direction.

REFERENCES


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