A comparison of laboratory measurements and numerical simulations of baroclinic wave flows in a rotating cylindrical annulus

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SUMMARY

Quantitative and qualitative comparisons are made between laboratory measurements of rotating annulus flows and corresponding numerical model simulations. Two laboratory annuli, of similar dimensions but differing in instrumentation, are used. One contains a thermocouple array for temperature measurement; the other contains no sensor array but the working fluid is seeded with minute neutrally buoyant beads (600 µm diameter) which enable the horizontal velocity field to be measured. Each annulus has a rigid insulating lid in contact with the working fluid. The numerical model is a finite difference formulation based on the Navier–Stokes equations for baroclinic flow of a Boussinesq liquid. Although the atmosphere and the laboratory annulus are both rotating baroclinic fluid systems, the forcing processes acting in the annulus are much simpler than those acting in the atmosphere, and may be accurately represented by established formulae: under a wide range of conditions no parametrizations of subgrid-scale dynamical and diabatic processes are required. Comparison of numerical model results with laboratory measurements therefore enables the explicit dynamical formulation of numerical models of rotating, baroclinic flow to be verified to an extent which would be very difficult, if not impossible, to achieve using atmospheric data. Detailed quantitative comparisons for a steady wave flow reveal good agreement for major features of the temperature and horizontal flow fields, although a significant discrepancy in total heat flux is found. Qualitative comparisons are made by investigating the ability of the numerical model to reproduce the main flow types and phenomena of the laboratory system. Numerical simulations of intransitivity, hysteresis, wavenumber transitions, amplitude vacillation and a weak structural vacillation are described. Several suggestions for further comparative studies are made in conclusion.

1. INTRODUCTION

Various regimes of flow may be observed when an annulus of liquid rotating about its vertical axis of symmetry is subjected to a horizontal temperature gradient by maintaining the cylindrical sidewalls at different temperatures. At low rotation rates the motion is an axisymmetric overturning, but at higher rotation rates non-axisymmetric wave motions occur (unless the imposed temperature difference is very small); see Hide and Mason (1975) for a review. Because of the maintained horizontal temperature gradient all of the waves are baroclinic, but several different types are found. A brief account of regular and irregular waves, of steady waves and vacillation, and of the associated hysteresis and intransitivity phenomena, will be given in section 4. Here we note only that generically similar quasi-geostrophic wave motion occurs in the atmosphere and oceans. Studies of baroclinic waves in the laboratory annulus therefore contribute to a body of knowledge which is highly relevant to meteorological and oceanographical problems.

James, Jonas and Farnell (1981)—herein referred to as JJF—have described in this journal a combined laboratory and numerical study of baroclinic annulus waves. The present paper reports a continuation of their study. As JJF pointed out, numerical simulations can supply more extensive data than are practicably obtainable from experimental measurements (without gross distortion of the flow) and thus they can facilitate analysis of the waves’ dynamics. This aspect was one of JJF’s main concerns, and it remains perhaps the most important element of numerical simulation work. Here, however, a different and more practical aspect is the underlying theme: the verification of numerical models of baroclinic flow in rotating systems.

Atmospheric motion is subject to forcing processes which exhibit complicated spatial and temporal variability. (The term ‘forcing processes’ is used here in a general sense to include momentum and potential temperature sinks as well as sources.) Diabatic
heating contains contributions from radiative flux convergence, turbulent (subgrid-scale) flux convergence and latent heat release. These processes depend variously on the cloud cover, moisture budget and the nature of the underlying surface. Subgrid-scale transfer of momentum is the only important momentum forcing present in the atmosphere (discounting topographic effects), but it depends in a complicated way on the atmospheric structure and the state of the underlying surface. (It also includes the contributions of unresolved cumulonimbus circulations, which are not always dissipative—see Moncrieff (1981) and Le Mone (1983).) Representation of these forcing processes is one of the main preoccupations of weather forecasting and general circulation modellers. Apart from the sheer volume of the problem, an important consequence is the difficulty of testing numerical models: it is hard to tell whether, or in what combination, imperfect simulation reflects shortcomings in the basic dynamical formulation or inadequacies in the parametrization of physical processes.

The simplicity of the forcing processes in the laboratory annulus system is in marked contrast to the complexity of the forcing in the atmosphere. Under a wide range of conditions the only operative forcing processes are molecular viscosity and conductivity, both of which can be well represented in terms of the motion and temperature fields by using established and uncontroversial formulae. *Comparison of numerical model output with experimental measurements therefore enables the basic dynamical formulation of the numerical model to be tested unambiguously.* Our studies thus aim to investigate the ability of numerical models to reproduce the motions which occur in rotating baroclinic fluids. Quantitative comparisons are of clear importance, but qualitative tests are also valuable: in these we establish whether the different flow types and phenomena seen in the laboratory can be simulated by the numerical model.

The comparisons place demands on both experimentalists and numerical modellers. Experimentalists must ensure that their apparatus and methods of measurement and analysis are reliable to within acceptable tolerances; and numerical modellers must ensure that their models are correctly formulated and use numerical schemes of a similar standard to those used in meteorological models. Accordingly, sections 2 and 3 of this paper contain brief descriptions of experimental procedure and numerical model structure. Section 4 discusses the experimental regime diagram for the annulus system which is used for the comparisons presented in section 5. These comparisons involve the temperature and horizontal flow fields in steady wave flows. In section 6, a qualitative comparison is undertaken by examining the ability of the numerical model to simulate various different flow types. Conclusions and suggestions for future work are contained in section 7.

2. **Experimental details**

(a) **Apparatus**

Two annuli are used in this study; they have the same dimensions but differ in construction and instrumentation. All the velocity measurements are made in a modified version of the annulus used by JIF, which allows determination of the horizontal velocity field at five levels by a particle-tracking method. A second annulus is used for the measurement of temperatures (at mid-level and mid-radius) and the total heat transport by the fluid.

The annuli share the same basic design of sidewalls in which constant temperature water is made to circulate through spiral grooves, the flow rates (up to 50 cm$^3$s$^{-1}$) being chosen to give the smallest azimuthal and vertical temperature gradients at any particular
temperature difference. A schematic diagram of the annulus used in the velocity measuring system is shown in Fig. 1. Here the outer cylinder consists of a stack of six annular sections separated by transparent acrylic inserts to allow horizontal beams of light to illuminate the flow field and thus allow tracking of the beads. The annulus used for temperature measurements has an outer cylinder made from two sections placed one on top of the other and separated by an acrylic insert supporting an array of thirty-two equally spaced thermocouples. Each annulus has an insulating rigid lid which is in contact with the horizontal upper surface of the working fluid.

![Figure 1. A schematic cross-section through the velocity measurement annulus. The labelled features are: A, insulating lid; B, inner cylinder; C, outer cylinder; D, insulating base; E, water channels; F, transparent acrylic inserts.](image)

The wall temperatures can be maintained stable to typically ±0.01 K with vertical variations of temperature on the walls being less than 2% of the applied temperature difference—except for the outer cylinder of the velocity measurement annulus where variations are up to 3% of the temperature difference. The heat transport measurements are made in the same manner as described by Hignett (1982a)—i.e. the transport is inferred from the temperature rise and flow rate of the water circulating through the inner cylinder. This temperature difference is measured by a thermopile constructed from twenty-five copper–constantan junctions, the flow rate being measured and held constant to within 0.5%. The thermopile and flow meter have been calibrated against a known power dissipation in an electrical heater for a range of dissipations, flow rates and ambient water temperatures. The annuli are levelled (to within 10⁻⁴ radians) and centred on turntables driven directly by servo-controlled permanent magnet D.C. motors with a long-term rotation rate stability of better than 1 part in 10⁴.

The working fluid used is a water/glycerol solution whose density is adjusted to maintain the beads approximately neutrally buoyant at 20°C. In the preliminary com-
comparison described in section 5, which consists solely of temperature measurements, a solution was used that had the same physical properties as those adopted in the numerical simulation described by JIF. In practice it was found that this solution was not dense enough to maintain the beads in suspension for long periods and a denser solution is now employed; it is this fluid which is used in the main comparison of section 5.

The notation, relevant dimensions and fluid properties are summarized in Table 1.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>First comparison</th>
<th>Main comparison</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluid depth</td>
<td>(d)</td>
<td>14.0</td>
<td>14.0</td>
<td>cm</td>
</tr>
<tr>
<td>Outer cylinder radius</td>
<td>(b)</td>
<td>8.00</td>
<td>8.00</td>
<td>cm</td>
</tr>
<tr>
<td>Inner cylinder radius</td>
<td>(a)</td>
<td>2.50</td>
<td>2.50</td>
<td>cm</td>
</tr>
<tr>
<td>Rotation rate</td>
<td>(\Omega)</td>
<td>0.55</td>
<td>1.00</td>
<td>rad s(^{-1})</td>
</tr>
<tr>
<td>Temperature difference</td>
<td>(\Delta T)</td>
<td>1.0</td>
<td>1.0</td>
<td>K</td>
</tr>
<tr>
<td>Kinematic viscosity</td>
<td>(\nu)</td>
<td>(1.35 \times 10^{-2})</td>
<td>(1.66 \times 10^{-2})</td>
<td>cm(^2) s(^{-1})</td>
</tr>
<tr>
<td>Thermal diffusivity</td>
<td>(\kappa)</td>
<td>(1.30 \times 10^{-3})</td>
<td>(1.27 \times 10^{-3})</td>
<td>cm(^2) s(^{-1})</td>
</tr>
<tr>
<td>Expansion coefficient</td>
<td>(\alpha)</td>
<td>(2.52 \times 10^{-4})</td>
<td>(2.86 \times 10^{-4})</td>
<td>K(^{-1})</td>
</tr>
<tr>
<td>Fluid density</td>
<td>(\rho)</td>
<td>1.027</td>
<td>1.044</td>
<td>g cm(^{-3})</td>
</tr>
</tbody>
</table>

The values of the physical properties refer to a temperature of 20 °C; the fluid properties used in the numerical integrations are given by the following expressions:

**First comparison**

\[
\begin{align*}
\rho &= 1.0264\left(1 - 2.7704 \times 10^{-4} (T - T_0) - 5.215 \times 10^{-6} (T - T_0)^2\right) g \text{ cm}^{-3} \\
\nu &= 1.2981 \times 10^{-3} \left(1 - 2.6298 \times 10^{-3} (T - T_0)\right) \text{ cm}^2 \text{ s}^{-1} \\
\kappa &= 1.3354 \times 10^{-3} \left(1 + 2.1636 \times 10^{-3} (T - T_0)\right) \text{ cm}^2 \text{ s}^{-1}
\end{align*}
\]

**Main comparison**

\[
\begin{align*}
\rho &= 1.043 \left(1 - 3.07 \times 10^{-4} (T - T_0) - 7.83 \times 10^{-6} (T - T_0)^2\right) g \text{ cm}^{-3} \\
\nu &= 1.620 \times 10^{-2} \left(1 - 2.79 \times 10^{-4} (T - T_0) + 6.73 \times 10^{-6} (T - T_0)^2\right) \text{ cm}^2 \text{ s}^{-1} \\
\kappa &= 1.29 \times 10^{-3} \left(1 + 2.35 \times 10^{-3} (T - T_0)\right) \text{ cm}^2 \text{ s}^{-1}
\end{align*}
\]

with \(T_0 = 22^\circ\text{C}\).

(b) Data acquisition and analysis

(i) Temperature measurements. Temperatures at mid-level and mid-radius are measured using a ring of thirty-two equally spaced copper–constantan thermocouples each having a sensitivity of approximately 40 \(\mu\)V K\(^{-1}\). Although such an array has a perceptible effect on the flow fields (as discussed by JIF and in section 4) it does not change their basic character. Through its relative ease of use and construction a thermocouple ring offers a convenient means of gathering a large amount of data on the structure of the wave flows over a wide range of conditions.

Fourier analysis of the data gives the amplitude and phase of the azimuthal components of the temperature field at mid-height and mid-radius; but care is needed in interpreting this information. As well as the large amplitude drifting waves there exists a spectrum of stationary components which can arise from small irregularities in the shape of the thermocouple ring or small imperfections in the construction and alignment of the annulus walls. As described by Hide, Mason and Plumb (1977), to remove these stationary components the Fourier analysis should ideally be performed on the deviations of the thermocouple temperatures from their average values taken over a large number of drift periods (a drift period is the time taken for a wave to travel through one wavelength). Unfortunately, with the horizontal rigid upper and lower boundaries used
in these experiments the drift period can be too long to make this procedure practical, particularly if any short period motion has also to be resolved. Hence to obtain the time-averaged spectrum, examples of which are given in section 5, the averaging must be over an integral number of drift periods to minimize the errors. Failure to do this can lead to an erroneous modulation of the wave amplitudes at the drift period—see section 4 and Hignett (1982b).

(ii) Velocity measurements. Horizontal fluid velocities at five levels are measured by tracking the motion of 600 μm diameter polystyrene beads at heights above the base, in centimetres, of 1.6, 4.3, 7.0, 9.7 and 12.4; the thickness of the illuminating beam is approximately 0.3 cm. The reflected light from the beads is detected by a monochrome television camera mounted axially on the turntable. The video signal is converted to a binary signal by comparison with an adjustable threshold level. When a transition in the binary signal occurs, corresponding to a change in the video image from dark to light, the coordinates of the transition are recorded and stored in a PDP 11/34 minicomputer. The tracks of individual beads can be identified unambiguously from the stored coordinates of all the transitions in successive television frames. A complete scan consists of taking in data from all five levels in sequence. The time over which frames are recorded is adjusted to give maximum track lengths of approximately one to two centimetres. Radial and azimuthal velocity components of each particle are derived from their tracks in subsequent processing and then analysed to give a representation of the horizontal velocity field at a given level and time as an azimuthal Fourier series, the coefficients of which are expressed as a polynomial in the radial coordinate. This final part of the analysis is a modification of that described by Jonas and Kent (1979). Further details of the velocity measurement method are given in Jackson (1984).

The velocity measurement system has the highly desirable property of being effectively non-intrusive to the measured flow. For making detailed quantitative comparisons with numerical simulations it is therefore to be preferred to the temperature measurement system—in which probe effects (though generally minor) are inevitable.

3. NUMERICAL MODEL AND INTEGRATION PROCEDURE

In this study we have used the same numerical model as did JF. The following brief outline is given to emphasize the major features and to correct and clarify JF's description.

The model is based on the (non-hydrostatic) Navier–Stokes equation for baroclinic flow of a Boussinesq liquid, and corresponding continuity and thermodynamic equations. With the velocity \( \mathbf{u} \) measured relative to a frame rotating with uniform angular velocity \( \Omega \mathbf{k} \) (where \( \mathbf{k} \) is unit vector in the axial \((z)\) direction) these equations are:

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + 2\Omega \mathbf{k} \times \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{F} \tag{1}
\]

\[
\nabla \cdot \mathbf{u} = 0 \tag{2}
\]

\[
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \nabla \cdot (\kappa \nabla T). \tag{3}
\]

In Eq. (1), \( \mathbf{i} \) is unit vector in the radial \((r)\) direction. Note that \( \mathbf{k} \) is not in the direction of apparent vertical in the rotating frame of the apparatus, and so a reduced centrifugal force term appears in Eq. (1) as well as a reduced gravity term. The equation of state \( \rho = \rho_0(1 - \alpha(T - T_0)) \) has been assumed. \( T_0 \) is the value of the temperature \( T \) at which
the density $\rho$ of the liquid is $\rho_0$; $p$ is the deviation of the pressure from the reference function

$$p_0(z, r) = \rho_0 g (d - z) + \frac{1}{2} \rho_0 \Omega^2 r^2.$$  \(4\)

Here $d$ is the depth of the annulus (whose inner and outer cylindrical boundaries are at $r = a$ and $r = b$ respectively): see Fig. 1 and Table 1. The viscous term $F$ in Eq. (1) is a form which allows spatial variations of the kinematic viscosity $\nu$ and which is more complicated than the familiar $\nu \nabla^2 u$ term. The quantities $\nu$, $\kappa$ and $\alpha$ are all linear or quadratic functions of temperature $T$. Table 1 gives the numerical values of all the various constants in the comparisons to be described in sections 5 and 6.

In the integrations, the boundary conditions on $u$ are $u = 0$ on $r = a$, $b$ and $z = 0$, $d$; and on $T$ are $\partial T/\partial z = 0$ on $z = 0$, $d$ and $T = T_a$ on $r = a$, $T = T_b$ on $r = b$. These conditions represent the ideal experiment in which all boundaries are perfectly rigid and allow no slip, there are no temperature gradients on the side walls, and the end walls are perfect thermal insulators.

The finite difference structure of the model is similar to that of Williams’s (1969, 1971) model in that it consists of many grid cells within each of which the thermodynamic variables ($p$, $T$), and each of the radial, azimuthal and vertical velocity components ($u$, $v$, $w$), are held at different points. The physical boundaries of the system contain the points at which the normal velocities are held, but lie mid-way between the ($p$, $T$) points. The grid differs from that of Williams’s model in that the cell size is non-uniform: thus boundary layers are adequately resolved without committing a high density of points to the interior also. The size of the grid cells is determined by requiring that two ranks of cells lie within a notional boundary layer thickness $\delta$ of the boundary, and applying a hyperbolic tangent rule to ensure a smooth increase of cell size towards the interior region. ($\delta$ is the Ekman layer thickness for the end walls, and the smaller of the Stewartson and thermal boundary layer thicknesses for the side walls: see JF.) In all the integrations described here the total number of grid cells is $16$ (radial) $\times 16$ (vertical) $\times 64$ (azimuthal).

Finite differencing of Eq. (3) and the three components of Eq. (1) is achieved by using elementary expressions which are formally second-order accurate on the non-uniform grid. A flux form (Arakawa 1966) is used to represent the advection terms in Eq. (1), thus ensuring satisfactory treatment of the kinetic energy budget. For the advection terms in Eq. (3), however, an appropriate modification of the scheme of Piacsek and Williams (1970) is applied to ensure proper conservation of total temperature. Remaining terms are treated by schemes similar to those of Williams (1969).

Time-integration uses the leap-frog scheme with Du Fort–Frankel expression of the viscous terms and smoothing applied to all fields every 21 time steps (see JF). A time step $\Delta t$ of 0.05 s is used in all cases: computational instability has been found when $\Delta t$ exceeds 0.10 s, and the chosen value is a conservative compromise.

The pressure field is determined at each time step from the appropriate finite difference form of the Poisson equation

$$\nabla^2 p = \rho_0 \nabla \cdot \left[ F - [\partial u/\partial t + (u \cdot \nabla)u + 2\Omega k \times u - (gk - \Omega^2 r)\alpha] (T - T_0) \right].$$  \(5\)

Following Williams (1969, p. 738) the condition that the normal pressure gradient vanish at all boundaries is applied, in conjunction with a compensating change in the source term of Eq. (5). Solution is achieved by a direct method specially adapted for use with irregular grids: it is essentially that described by Farnell (1980).
JF's procedure is used to start an integration. An axisymmetric version of the model (with similar \( r, z \) resolution) is integrated out to a steady state under the chosen external conditions. The axisymmetric fields are then transferred to the full non-axisymmetric model and a small non-axisymmetric perturbation applied to the temperature field. This perturbation always contains an azimuthally localized component, and in some cases a component varying sinusoidally with azimuth is added. Both components have amplitude 0.1 K. Any changes in rotation rate that are made during an integration are applied discontinuously, rather than gradually over many time steps. The integrations described here were carried out on IBM 3081 and 360/195 computers.

4. EXPERIMENTAL REGIME DIAGRAM

As noted in the introduction, several different flow types occur in rotating annulus experiments. At low rotation rates the flow is axisymmetric, at higher rotation rates (unless the temperature difference is very small) large amplitude waves occur, and at the highest rotation rates these waves become irregular. Various subdivisions have also been reported, particularly of the regular wave regime. Figure 2 shows a regime diagram depicting the flow types which occur in the temperature measurement annulus (see section 2). The diagram is of the conventional type, the ordinate being a stability parameter determined by Hide (1958) and the abscissa a Taylor number (see Fowlis and Hide 1965; Fein and Pfeffer 1976; Matsuo et al. 1976; Pfeffer et al. 1980; and JF for examples of similar experimental regime diagrams). The 'lower symmetric regime' (\( \Delta T \approx 0.1\) K) has not been investigated in any detail and is not shown in Fig. 2. Six

![Figure 2. Experimental regime diagram for the temperature measurement annulus: \( \theta = ga\Delta Td/\Omega^2(b-a)^2 \), \( \tau = 4\Omega^2(b-a)^2/\nu^2d \). The lines of constant \( \Delta T \) are drawn for the fluid properties corresponding to the main comparison (section 5(b)). For an explanation of the flow type notation see section 4 and for points A, B and C refer to section 5.](image-url)
different flow types are labelled. Completely axisymmetric flow (a) occurs only for the lowest rotation rates and gives way to a region of weak waves (w) at higher rotation rates. These waves tend to have low dominant wavenumbers and a maximum amplitude in the lower part of the fluid; they have been described and discussed by Hide and Mason (1978) and Jonas (1980). The large amplitude regular waves are divided into three types. Steady waves (S) have no clear periodic variations in the wave amplitude or phase speed, although there may be small, irregular fluctuations particularly at the higher rotation rates. Amplitude vacillation (AV) is characterized by a periodic variation of the amplitude and phase speed of the dominant wave and its harmonics. Shape vacillation (SV), also known as structural or tilted-trough vacillation, appears as an approximately periodic tilting of the wave axis. Completely irregular flow (I) shows little coherent spatial or temporal structure; the transition to this region is vague and difficult to define and consequently it is marked on the diagram as a broad band.

Figure 2 differs from other published regime diagrams in that it shows all the possible wavenumbers and flow types rather than the distribution of most probable wavenumbers, as can be seen, for example, in Fowlis and Hide (1965). Each individual experiment was set up either by a rapid spin-up from rest after thermal equilibrium had been established, or by making small step changes in rotation rate at constant temperature difference. Most of Fig. 2 was in fact built up from the results of step change experiments. All the flows shown on the regime diagram can be achieved by a rapid spin-up from rest, but the probability of obtaining a specified flow in this manner varies throughout the diagram.

An important aspect of Fig. 2 is that, over a large part of the regular wave regime, more than one dominant wavenumber can exist for the same set of external conditions. Once equilibrated each of these flows is stable in that there is no observed tendency to change to another state; thus the system exhibits intransitive behaviour. (The phenomenon of intransitivity has interested climatologists because of the possibility in more complicated systems—such as the earth's atmosphere, ocean and cryosphere—of almost intransitivity (Lorenz 1975). In this mode of behaviour the system remains in one state for long periods of time but may occasionally transfer to another state and remain in it for a considerable time before making the reverse transition.) Furthermore, considerable hysteresis in the dominant wavenumber is possible at a transition: if, at constant temperature difference, the rotation rate is progressively reduced until the dominant wavenumber changes to the next lowest value, the rotation rate can then be increased (gradually) to much higher values without a corresponding increase in the wavenumber.

Figure 2 was derived from temperature measurements and hence from an annulus containing a thermocouple array. Comparable experiments using only flow visualization reveal some small probe effects. The main effect of the thermocouple ring seems to be to shift the wavenumber transitions to slightly higher rotation rates (up to 0.1 rad s\(^{-1}\) higher) than would otherwise be observed at that temperature difference. However, no clear qualitative differences have been recorded. One further effect of the probes, on the phase speed of the waves, will be discussed in section 5.

Comparison of Fig. 2 with the regime diagram of JF (Fig. 9 of that paper) reveals considerable qualitative differences between what should be two virtually identical diagrams. The geometries of the annuli used are the same and the small differences in fluid properties are insufficient to account for such large apparent discrepancies. The largest discrepancy lies in the relative occurrence of wave-two and wave-three amplitude vacillation. In JF these occur as a single mixed region whereas here they occur quite separately. In fact many of the 2AV flows in JF were incorrectly classified. It appears that, as described in section 2(b), after Fourier analysis the dominant wave temperature amplitudes were subject to an erroneous modulation of the drift period by failure to
average the thermocouple temperatures over an integral number of drift periods. This led to the classification of these flows as amplitude oscillations rather than steady waves. Other differences can possibly be attributed to an insufficient number of experiments or to the difficulty of identifying shape oscillation from temperature measurements alone. All the features of Fig. 2 have been verified by direct flow visualization. However, this does not mean that the diagram is necessarily complete; as more data are acquired further details and possibly other flow types could be added.

5. Steady wave comparisons

This section comprises: a comparison of temperature data from a repeat of a laboratory experiment of JIF and from the corresponding numerical simulation; and a more extensive comparison, for different external parameters, using both temperature and velocity data.

(a) A repeat of JIF's comparison

JIF compared the temperature spectrum of a steady 3-wave flow obtained from a laboratory experiment with that from a numerical integration conducted under similar, but not identical, external conditions. At the parameters chosen for their numerical experiment \((\Omega = 0.6 \text{ rad s}^{-1}, \Delta T = 2.0 \text{ K})\) JIF were unable to find a steady 3-wave flow in the laboratory. This experiment is shown as point A on the regime diagram of Fig. 2. It can be seen that this point lies within the region of 3-wave amplitude oscillation, hence the observations of JIF. However, the numerical integration using these parameters did produce a steady 3-wave flow. A possible explanation of this lies in the effect of the thermocouple ring on the regime diagram, as described in section 4. Its removal shifts the transitions to slightly lower rotation rates and transfers point A to the steady 3-wave region. To maintain consistency with this previous work the laboratory experiment of JIF was repeated \((\Omega = 0.55 \text{ rad s}^{-1}, \Delta T = 1.0 \text{ K})—\text{point B on Fig. 2}\) and a numerical integration performed with the same parameters. Steady 3-wave flows were produced in both cases.

In the laboratory experiment data were taken at 20-second intervals over two drift periods (=12000s) and analysed in the manner described in section 2(b) to give the time-averaged azimuthal wavenumber spectrum at mid-level and mid-radius \((r = 7.0 \text{ cm}, r = 5.25 \text{ cm})\). The corresponding spectrum from the numerical model data was formed by averaging over the period from 800 to 1000s after the start of the integration and linearly interpolating between the grid points adjacent to the mid-level and mid-radius. These two spectra are displayed on a log(amplitude) versus wavenumber plot in Fig. 3. The amplitudes of the dominant components agree to within 0.004 K, which is typical of the repeatability of the experimental measurements. Differences become progressively larger for wavenumbers 6 and 9 but compared to the dominant component these peaks are energetically very weak.

The dotted line on Fig. 3 is a representation of the effective background noise level in the laboratory experiment. The most practical way of determining this is to carry out an experiment at the same temperature difference but with a rotation rate low enough to produce an axisymmetric flow. The Fourier analysis is then carried out in the usual manner. In this way it can be seen that, with the possible exception of wavenumbers 1 and 2, the non-harmonic components and all wavenumbers greater than 9 are not significantly different from the effective noise level. Hence the relatively large differences between these components in the laboratory experiment and their counterparts in the
numerical simulation are also not significant. Overall the model gives a satisfactory quantitative representation of the main components of the temperature spectrum.

(b) Main comparison

The parameters chosen for this comparison were $\Delta T = 4.0 \text{ K}$, $\Omega = 1.0 \text{ rad s}^{-1}$ (point C on Fig. 2). This ensured that a steady 3-wave flow could be achieved by the numerical model and both laboratory systems. Temperature and total heat transport measurements were made at 10-second intervals over 2 drift periods ($\approx 3500$ s). The scan interval and the number of scans are limited in the velocity-measuring system by the data-handling rate and the available storage capacity. The horizontal velocity field was measured at 13 s intervals beginning at the top level and stepping down the levels in sequence before returning to the top; a total of 50 fields was recorded.

The heat transport measurements were converted to a Nusselt number defined as the ratio of the actual total transport to that which would be achieved by molecular conduction alone. In the numerical model this is calculated from the conductive heat transfer between the pairs of temperature grid points that straddle the side boundaries (see section 3). After 1000 seconds the Nusselt numbers at the hot and cold walls were within 0.1% of their mean value of 10.45. The corresponding experimental value was
9.2 (±2%). The difference between these values is greater than the estimated random errors of measurement but no systematic error has been found which could account for the difference; this remains an unsolved problem but may reflect the limited resolution of the numerical model. (It is worth noting that the measured Nusselt numbers vary by only a few per cent in the regular wave regime as the rotation rate is varied at constant temperature difference: see Kaiser 1973 and Hide and Mason 1975, Fig. 8. Thus the discrepancy between measured and simulated heat transports cannot be accounted for by the effect of the thermocouple array in displacing the regime diagram to slightly higher rotation rates.)

The drift frequency of the wave in the numerical model was taken from the interval 900 to 1000 seconds after the start of the integration and gave a value of 7.80 × 10^{-3} \text{ rad s}^{-1}. This is effectively equal to the corresponding value from the velocity-measuring system of 7.90 × 10^{-3} (±2%) \text{ rad s}^{-1}. However, the drift frequency measured in the temperature measurement annulus is smaller by more than a factor of 2 at 3.54 × 10^{-3} (±1%) \text{ rad s}^{-1}. This particular discrepancy results from a further effect of the thermocouple ring. Mid-level and mid-radius is typically just above the level of zero azimuthal velocity; consequently the thermocouple ring lies in a region where the fluid has 'westerly' relative angular momentum and the hypothesis is that therefore the ring exerts an 'easterly' drag on the flow resulting in a wave drift period which is longer than without the ring. Conversely if the ring is placed below the level of zero azimuthal velocity the drift periods are reduced (see Fig. 13 of JIF).

The very close agreement with the value from the velocity measuring system is probably fortuitous: repeated experiments show a scatter in the measured frequencies of about 10%, although in any one experiment the frequency is very steady (i.e. to within about 2%). This variation presumably arises from beads which become unavoidably attached to the boundaries and have an effect similar to that of the thermocouple ring described above. (It should be noted that wave frequencies in our annuli are small because of the rigid upper boundary to the flow. Phase speeds are typically less than 5% of the maximum azimuthal velocity differences occurring in the fluid.)

As in the previous comparison the time-averaged azimuthal wavenumber spectrum was calculated from the temperature data; it is compared with the equivalent numerical model output in Fig. 4. As before the agreement is best for the dominant component and its harmonic. However, the difference in the amplitudes of the dominant component is 0.03 K. This amounts to a discrepancy of about 10%, and is substantially larger than the observed scatter in the experimental measurements (which, as in the previous comparison, is about 0.004 K). There are several possible interpretations. Because of the presence of the thermocouple ring the two flows do not correspond to exactly the same point in parameter space (see section 4). Also, small errors in the positioning of the thermocouple ring will become more apparent as the temperature difference is increased (so that the temperature gradients in the fluid become larger) and would be more prominent on Fig. 4 than Fig. 3. However, it is thought equally likely that the discrepancy could be a consequence of the numerical model's limited resolution. As in Fig. 3, most of the non-harmonic components are not considered to be significantly different from the estimated background noise level. The largest of the non-harmonic components are the sidebands of the dominant component but energetically these are still very weak. As described by JIF these sidebands are thought to be generated via a wave-1 forcing arising from the presence of the thermocouple ring.

Data from the velocity-measuring system yield a great deal of information on the three-dimensional structure of the wave and mean flow. There are several ways of using this data in quantitative comparisons but it was decided to continue the theme of making
comparisons between Fourier components. The five levels used in the laboratory experiment were taken as the standard levels of comparison; the numerical model output 1000 s after the start of the integration was interpolated between the appropriate grid points (before Fourier analysis) to provide information at these five levels.

Figures 5(a) and 5(b) show the radial variation of the azimuthally averaged azimuthal velocity and the wave-3 Fourier component of the radial velocity at each of the five levels. Dashed lines refer to the laboratory experiment; because of the method of data analysis and storage the experimentally derived components will vary smoothly. As a measure of the uncertainty in these quantities the repeatability of the maximum value of each component is estimated at approximately 5%. From the numerical model data the grid point values are plotted and linked by straight lines. Agreement in both the shape and magnitude of these quantities is encouragingly good overall. In particular the characteristic ‘double-jet’ structure of the mean azimuthal flow is well represented in both laboratory experiment and numerical model. However, in Fig. 5(a) the numerical model does appear to over-emphasize the minimum which occurs near mid-radius; also, in Fig. 5(b), the model gives a profile which has less vertical variation than that shown by the experimental data.

Figure 6 compares the wavenumber-3 azimuthal velocity components (a) and the wavenumber-6 azimuthal (b) and radial (c) components, all at mid-level. As with the temperature spectra of Figs. 3 and 4 there is a tendency for differences to be more marked for the harmonic component: the comparisons in Fig. 6(c), and especially 6(b), are poor in local detail—although radial mean values are in fair agreement.

As well as the amplitudes the relative phases of the components may be compared. Radii corresponding to $\frac{1}{4}$, $\frac{3}{4}$, and $\frac{5}{4}$ of the gap-width were chosen to display the vertical phase variations of the dominant component; these are shown in Figure 7. Grid point values are plotted from the numerical model (full lines); the zero for both data sets was chosen to be mid-radius at the top level. All the curves show the familiar ‘westward’ tilt with height and agreement is encouragingly good.

The results of this section have demonstrated that for the major components of the flow and temperature fields good agreement can be achieved between a laboratory experiment and a numerical simulation of the same steady wave flow. Quantitatively the agreement is in most cases within or not substantially worse than the estimated errors of measurement. The 10% discrepancy in dominant wave temperature amplitudes in the main comparison may be partly attributed to probe effects. The 10% discrepancy in the Nusselt numbers is more significant, however. Since the measured variation of heat transport with rotation rate is very small in the regular wave regime, this discrepancy cannot be attributed to any known probe effects. Neither can it reflect any uncertainties in the positioning of probes, since the Nusselt number is a global, and not a local, property of the flow.

Further work will be necessary to establish whether these two discrepancies—and other smaller differences—can be reduced by improving the resolution of the model or the techniques of measurement and data analysis. It is encouraging that generally good agreement has been obtained between measured and modelled horizontal velocity fields, since in this comparison probe effects do not have to be allowed for.

6. Flow types in the numerical model

As well as qualitative comparisons such as those described in the previous section, also of interest is whether the main flow types and phenomena of the laboratory flows (section 4) can be reproduced by the numerical model. Such qualitative tests will be
Figure 5. (a) Radial variation of the azimuthally averaged azimuthal velocity at five levels in the main comparison experiment (section 5(b)), $\Omega = 1.00 \text{rad s}^{-1}$, $\Delta T = 4.0 \text{K}$; dashed lines are laboratory measurements, full lines numerical output. The heights, $z$, are: (i) 12.4 cm; (ii) 9.7 cm; (iii) 7.0 cm; (iv) 4.3 cm; (v) 1.6 cm. The ordinate is in cm s$^{-1}$ and the abscissa is a radial coordinate equal to zero at the inner cylinder and unity at the outer cylinder. (b) Details as for (a) except the data is the radial velocity wave-3 Fourier component.

Figure 6. Experimental and other details as for Fig. 5. (a) Azimuthal velocity wave-3 Fourier component at $z = 7.0 \text{cm}$. (b) Azimuthal velocity wave-6 Fourier component at $z = 7.0 \text{cm}$. (c) Radial velocity wave-6 Fourier component at $z = 7.0 \text{cm}$.
described in this section. The fluid properties assumed in the integrations are those used by JJF rather than those used for our main steady wave comparison (section 5(b)), and no detailed quantitative comparison between experiments and simulations will be undertaken here. (Such comparisons form an obvious subject for future study.) However, it will be noted when a flow type or phenomenon occurs in the numerical model under conditions which are noticeably different from those required in the laboratory—even allowing for the small differences in fluid properties noted above.

(a) Steady waves and intransitivity

The integrations described in section 5 and by JJF have amply demonstrated the ability of the numerical model to simulate steady waves. As noted in section 4, a common and outstanding feature of steady wave flows in the laboratory is intransitivity: the wavenumber of the flow need not be uniquely determined by the external conditions, but once a steady flow of a given wavenumber is set up it will not spontaneously evolve to another wavenumber. It is straightforward to reproduce this behaviour in the numerical model by biasing the initial conditions in favour of different wavenumbers in different integrations: in many cases the favoured wavenumber then dominates the final steady wave state. Figure 8 shows an example. Figure 8(a) gives the pressure field at an upper level in the steady wave-3 flow obtained at $\Omega = 0.6 \text{rad s}^{-1}$, $\Delta T = 2 \text{K}$ when the initial perturbation applied to the appropriate axisymmetric flow was the azimuthally-localized temperature perturbation described in section 3. Figure 8(c) gives the pressure field, at the same level, in the steady wave-2 flow obtained at the same rotation rate $\Omega$ and
Figure 8. Upper level pressure fields and zonal average zonal flow fields obtained in two steady-wave numerical integrations started from the same axisymmetric flow but with different initial perturbations. Rotation rate $0.6 \text{ rad s}^{-1}$ (direction of rotation anticlockwise); inner cylinder temperature 18°C, outer cylinder temperature 20°C. (a) Pressure field at height 11.998 cm (model level 11; depth of annulus 14 cm) 700 s after application of an azimuthally localized temperature perturbation to the axisymmetric flow. All fields are virtually steady by this time. Quantity plotted is deviation of $p/p_0$ (see Eq. (4)) from an arbitrary reference value. Units $10^{-4} \text{ cm}^2 \text{s}^{-2}$; contour interval $2 \times 10^{-4} \text{ cm}^2 \text{s}^{-2}$. (b) The corresponding zonal average zonal flow (drawn on the radius/height plane). Units mm s$^{-1}$; contour interval 0.5 mm s$^{-1}$. (c) and (d), as (a) and (b) but showing the steady fields obtained 700 s after application to the axisymmetric flow of a temperature perturbation consisting of an azimuthal wave-2 component (amplitude 0.1°C) as well as an azimuthally localized part.

temperature difference $\Delta T$ when the initial localized temperature perturbation was augmented by a wave-2 component applied all round the annulus near mid-channel and mid-depth. Both the 2- and 3-wave flows are steady and stable under the chosen external conditions. Although the two flows are noticeably different in their azimuthal structures, the zonal mean fields are very similar. Figures 8(b) and 8(d) show the zonal average zonal flows which correspond to Figs. 8(a) and (c). Their 'double-jet' structure is typical of the steady wave regime when the upper surface is rigid (see section 5, Fig. 5(a) and White 1984).

Several other examples of intransitivity have been investigated. In the laboratory, steady 2-, 3- or 4-wave flows can occur when $\Delta T = 4 \text{ K}$ and $\Omega = 1.3 \text{ rad s}^{-1}$ (see Fig. 2), and each of these flows has been reproduced in the numerical model. This case is the subject of a detailed quantitative comparison which will be reported in due course.
(b) Wavenumber transitions and hysteresis

The usual behaviour in a steady-wave integration is that described by JF: a wave with a well-defined structure grows to a large amplitude initially and equilibration proceeds via a damped oscillation of the main azimuthal Fourier component and its harmonics, and a slow decay of non-harmonic components (see JF, Fig. 2). Cases have been found, however, in which the dominant wavenumber changes during the equilibration. Of greater interest are wavenumber transitions which occur when the external conditions are changed. Two such examples, both involving only rotation rate changes, will be described here.

Figure 9. (a) Pressure field at height 11-933 cm (model level 11) 750 s after application of an azimuthally localized temperature perturbation to the axisymmetric flow obtained for $\Omega = 0.5 \text{ rad s}^{-1}$, $\Delta T = 2 \text{ K}$ (inner/outer cylinder temperature = 18/20°C). Quantity plotted is deviation of $p/p_0$ from an arbitrary reference value. Units $10^{-3}$ cm$^3$s$^{-2}$, contour interval $2 \times 10^{-3}$ cm$^3$s$^{-2}$. (b) The corresponding zonal average zonal flow. Units mm s$^{-1}$; contour interval 0.5 mm s$^{-1}$. (c) Pressure field at height 11-933 cm in the flow simulated for $\Omega = 0.4 \text{ rad s}^{-1}$, $\Delta T = 2 \text{ K}$. This flow was obtained from that shown in (a) by reducing the rotation rate in steps of 0.05 rad s$^{-1}$—see text for further details. (d) Zonal average zonal flow field corresponding to (c)—note the ‘single-jet’ structure. Units and contour intervals for (c) and (d) as for (a) and (b).

Figure 9(a) shows the pressure field at an upper level in the steady wave-2 flow obtained when $\Omega = 0.5 \text{ rad s}^{-1}$, $\Delta T = 2 \text{ K}$, and the standard initial perturbation (see section 3) had been applied. The corresponding zonal mean flow field is shown in Fig.
9(b). It has the familiar double jet structure, though less markedly so than at higher rotation rates (Figs. 8(c) and (d)). Even allowing for the differences in fluid properties, the conditions applied in this integration locate it close to the upper transition line on the experimental regime diagram. Starting from the flow as shown in Fig. 9(a) the rotation rate in the numerical model was reduced discontinuously in steps of 0.05 rad s\(^{-1}\) to 0.35 rad s\(^{-1}\). 300 seconds after the first rotation rate reduction the wave had weakened only slightly but the double jet structure of the zonal mean flow had almost disappeared. The rotation rate was then immediately reduced to 0.40 rad s\(^{-1}\). After a further 300 s the flow was nearly axisymmetric (Fig. 9(c)) and the zonal mean flow (Fig. 9(d)) had a clear single jet form. The trend towards axisymmetric, single jet flow was continued upon reduction of the rotation rate to 0.35 rad s\(^{-1}\).

Without further integrations (at intermediate rotation rates) the upper transition in the numerical model cannot be located precisely; but it evidently lies between 0.40 and 0.45 rad s\(^{-1}\). In this integration the double jet structure of the mean zonal flow disappears before the wave amplitude is noticeably reduced. However, from many other integrations there appears to be a close association between steady waves and double jet zonal average flows.

Figure 10 shows a sequence of upper level pressure fields during a transition from a wave-3 to a wave-2 flow. The integration was begun from a steady wave-3 flow obtained at \(\Omega = 1.0\) rad s\(^{-1}\), \(\Delta T = 4\) K, and a succession of discontinuous rotation rate reductions were applied. At \(\Omega = 0.58\) rad s\(^{-1}\) a weak wave-3 amplitude vacillation (see sections 4 and 6(c)) was obtained. Upon reduction of \(\Omega\) to 0.56 rad s\(^{-1}\) (at time \(t = 0\)) a spectacular sequence of flow changes occurred, as shown in the upper level pressure fields of Fig. 10. As soon as the rotation rate is reduced, the wave amplitude begins to decline—but it is still fairly large after 50 s (Fig. 10(a)). After 300 s the wave has all but disappeared (Fig. 10(b)) and the evolution so far looks like a transition to axisymmetric flow. Nevertheless, the wave amplitude then begins to increase again, until at 650 s (Fig. 10(c)) it is larger than before the rotation rate reduction. Marked horizontal tilts are present in the trough lines. By 850 s (Fig. 10(d)) the wave has weakened again; and the evolution up to this time is suggestive of a wave-3 amplitude vacillation. However, a slight asymmetry between the wave lobes is evident in Fig. 10(d), and by 950 s the flow has developed into an uneven 2-lobe structure (Fig. 10(e)). At 1050 s (Fig. 10(f)) the two trough axes are approximately 180° apart, but one is noticeably broader than the other. Over the next 400 s the flow relaxes towards a wave-2 structure having lobes of equal size and form (Fig. 10(g)). The integration has been carried on for a further 1000 s. No change of flow pattern occurs, and the initial fluctuations in the amplitude of wave 2 are progressively damped out.

Hysteresis at the transition was investigated by progressively increasing the rotation rate after the 1000-second period noted above. Steps of 0.04, 0.05 or 0.10 rad s\(^{-1}\) were applied, and the intervals between steps varied from 300 to 800 s. The wave-2 flow persisted up to \(\Omega = 1.8\) rad s\(^{-1}\) at which value unsteadiness suggestive of shape vacillation began to appear. Clearly there is extensive hysteresis at the 3/2 wave transition in the numerical model.

The qualitative behaviour described in this section is in agreement with that found in the laboratory system. In particular, hysteresis at the 3/2 (and many other) wave transitions is a marked feature of the experimental flows. However the 3/2 transition occurs at a significantly lower rotation rate in the model (0.56 rad s\(^{-1}\)) than in experiments with a thermocouple ring present (0.80 rad s\(^{-1}\)). Up to 0.1 rad s\(^{-1}\) of the discrepancy may be accounted for by probe effects, but the remainder cannot be explained by the difference in fluid properties. The discrepancy between the measured and simulated
Figure 10. (a)–(g) Sequence of upper level (height 13.10 cm—model level 12) pressure fields during a simulated wavenumber transition at $\Omega = 0.56 \text{rad s}^{-1}$, $\Delta T = 4 \text{K}$ (inner/outer cylinder temperature = 17/21 $\text{°C}$). Quantity plotted is deviation of $p/p_0$ from an arbitrary reference value. Units $10^{-8}\text{cm}^2\text{s}^{-2}$; contour interval $5 \times 10^{-8}\text{cm}^2\text{s}^{-2}$. Times, in seconds, are (a) 50, (b) 350, (c) 650, (d) 850, (e) 950, (f) 1050, (g) 1450 after reduction of rotation rate from 0.58 rad s$^{-1}$ (for which a weak wave-3 amplitude vacillation had been obtained).
upper transitions—0.60 rad s\(^{-1}\) as against 0.40 to 0.45 rad s\(^{-1}\)—can be accounted for by such considerations, however.

(c) Amplitude vacillation

In the laboratory, amplitude vacillation is most easily obtained by setting up a steady wave and then reducing the rotation rate progressively. The vacillation then occurs—if

Figure 11. Upper level (height 11-998 cm—model level 11) pressure fields at a minimum (a) and maximum (b) of the wave-2 amplitude vacillation obtained at \( \Omega = 0.51 \) rad s\(^{-1}\), \( \Delta T = 4 \) K (inner/outer cylinder temperature 17/21°C). Quantity plotted is deviation of \( p/\rho_0 \) from an arbitrary reference value. Units 10^{-2} cm\(^2\) s\(^{-1}\); contour interval 5\times10^{-2} cm\(^2\) s\(^{-1}\). Times are 2450s (a) and 2750s (b), after reduction of rotation rate from 0.53 rad s\(^{-1}\) (for which a steady wave-2 flow had been obtained). (c) Time variations of the main azimuthal Fourier component (2) and its first harmonic (4) of the temperature field at radius 5.770 cm and height 4.852 cm during the simulated amplitude vacillation. (Rotation rate reduction was made at \( t = 0 \).)
at all—before the transition to a lower wavenumber. (See Fig. 2 for cases in which wavenumber transitions occur without the appearance of amplitude vacillation.) This procedure has been used to produce a wave-2 amplitude vacillation in the numerical model. A steady wave 2 was first set up at $\Delta T = 4 \, \text{K}$, $\Omega = 0.6 \, \text{rad} \, \text{s}^{-1}$. Upon reduction of $\Omega$ to $0.55 \, \text{rad} \, \text{s}^{-1}$ no tendency towards vacillation was seen. Further reduction, to $0.53 \, \text{rad} \, \text{s}^{-1}$ (after 400s) excited a long period oscillation in wave-2 amplitude, but it was heavily damped. After a further 600s, $\Omega$ was decreased to $0.51 \, \text{rad} \, \text{s}^{-1}$; a strong amplitude vacillation, with a period of about 700s, quickly set in. The vacillation was followed through four complete cycles, during which time no diminution of vacillation amplitude was seen. (Subsequently it was found that the same amplitude vacillation could be produced simply by perturbing the axisymmetric flow obtained for $\Delta T = 4 \, \text{K}$, $\Omega = 0.51 \, \text{rad} \, \text{s}^{-1}$.)

Figures 11(a) and (b) show the pressure field at an upper level at a minimum, (a), and a maximum, (b), of the vacillation cycle at that level (see below). The change from wavy to nearly axisymmetric flow is clear. Figure 11(c) shows the time variation of the second and fourth azimuthal Fourier components ($T_2$ and $T_4$) of the temperature at a point near mid-radius and mid-height during several cycles of the simulated vacillation. The fluctuation in amplitude of $T_2$ (the main component) is by a factor of about five. Although the fluctuation is periodic, the growth phase is more rapid than the decay phase—in keeping with experimental evidence (Hignett 1984). Away from mid-height, however, the asymmetry of the time variation is less marked. Minima of wave amplitude are closely in phase at all heights, but maxima near mid-level lead maxima in the top and bottom of the annulus. Component $T_4$—the first harmonic of the main wave—fluctuates by an order of magnitude during the vacillation cycle (Fig. 11(c)). Its variation is strictly periodic, in spite of the presence of much short time-scale detail; and the longest time-scale component is clearly in phase with that of the main wave. Even more intricate—and still periodic—features are present in the higher harmonics.

The amplitude vacillation is associated with marked variations in main wave phase speeds (not shown). At mid-height and mid-radius, the phase speed associated with $T_2$ varies by a factor of 2 during the cycle; it is largest during the growth phase and smallest during the decay phase of the vacillation.

Various numerical experiments have been carried out to test the behaviour of this vacillating flow. Upon increase of $\Omega$ back to $0.53 \, \text{rad} \, \text{s}^{-1}$ the vacillation died out within two cycles. Thus there appears to be no hysteresis in the transition from steady wave 2 to vacillating wave 2. The same is also true of the weak wave-3 vacillation noted in section 6(b), and of amplitude vacillations in the laboratory. Decrease of the rotation rate to $0.49 \, \text{rad} \, \text{s}^{-1}$ led to a lengthening and strengthening of the simulated vacillation cycle—again in qualitative agreement with observed behaviour.

The conditions under which this amplitude vacillation occurs in the numerical model are not significantly different from those found experimentally. In the laboratory the transition from steady to vacillating wave 2 occurs near $0.70 \, \text{rad} \, \text{s}^{-1}$ in the presence of a thermocouple ring. The difference from the numerical model value ($0.51 \, \text{rad} \, \text{s}^{-1}$) can be accounted for by probe effects and the different fluid properties.

(d) Behaviour at higher rotation rates

The integrations described above have attempted to reproduce those regular wave phenomena which are found in the laboratory at rotation rates of $1 \, \text{rad} \, \text{s}^{-1}$ and less. Several integrations have however been carried out assuming higher rotation rates. It has been found that equilibration to a steady wave becomes progressively slower at the higher rotation rates if the integration is started by perturbing the corresponding axi-
Figure 12. (a)-(f) Sequence of upper level (height 13-147 cm—model level 12) pressure fields at 20 s intervals in the wave-3 flow obtained at $\Omega = 1.8 \text{ rad s}^{-1}$, $\Delta T = 4 \text{ K}$ (inner/outer cylinder temperature = 17/21°C). See text for further details. Quantity plotted is deviation of $\rho/\rho_0$ from an arbitrary reference value. Units $10^{-3} \text{ cm}^2 \text{s}^{-2}$; contour interval $3 \times 10^{-3} \text{ cm}^2 \text{s}^{-2}$.

symmetric flow. Indeed, in some cases it is not clear that equilibration is occurring at all, the flows sometimes being spatially and temporally chaotic although the rotation rates are much less than those associated with irregular flows in the laboratory. It is generally more convenient to set up a steady wave flow at a lower value of $\Omega$ and then to increase it, in steps, to the desired value. This is especially so if a particular wavenumber is required, for the wavenumber of a flow is generally resilient to increase in rotation rate (see section 6(b)).
One integration of this type, which shows features similar to those of a shape vacillation, will be described here. A steady wave-3 flow was set up at $\Delta T = 4 K$, $\Omega = 1.2 \text{rad s}^{-1}$ and the rotation rate then increased in steps of 0.1 rad s$^{-1}$ (at intervals of several hundred seconds) to 1.8 rad s$^{-1}$. Figure 12 shows a sequence of upper level pressure fields at intervals of 20 s. Compared with those shown in earlier sections the pressure fields show much less regular spatial structure—although a wave-3 structure is apparent throughout. Temporal fluctuations are also evident. Although these are to a considerable extent irregular, a periodic—or quasi-periodic—element can be seen. The period is about 100 s, and may be recognized in Fig. 12 by the similarity of (a) and (f). The sequence is reminiscent of a shape vacillation, albeit a weak one, as seen in the laboratory system. Perhaps the most interesting aspect, however, is that the flow retains its basic 3-wave form in spite of the fluctuations which occur. Thus, although the flow is evidently unstable, incipient instabilities do not become dominant: the gross structure of the flow remains unchanged.

7. Conclusions and Suggestions for Further Work

In this paper an account has been given of quantitative and qualitative comparisons between experimental measurements and numerical simulations of rotating annulus flows. The quantitative comparisons have been limited to steady waves, but have revealed an encouraging level of agreement in zonal average and main wave velocity fields. The qualitative comparisons have consisted of attempts to reproduce numerically the main experimental flow types and phenomena: simulations of intransitivity, wavenumber transitions, hysteresis, amplitude vacillation and a weak shape vacillation have been obtained. Our numerical model thus appears able to reproduce the general behaviour of the rotating annulus system, although the parametric location of a simulated wavenumber transition is in significant disagreement with experiment.

Many avenues for further work may be recognized. In addition to further comparisons of steady waves (see section 6(a)) it will be of interest to make quantitative comparisons between measurements and simulations of the various other flow types. Construction of a regime diagram for the numerical model, for comparison with its experimental counterpart, is also desirable. In this connection it will be important first to construct an experimental regime diagram from the velocity measurements: the regime diagram used in the present study (Fig. 2) has been constructed from temperature measurements, and probe effects—although small—have complicated the comparison of experimental and simulated transitions.

A clear requirement is to run the numerical model at different spatial resolutions, so as to investigate the convergence of the numerical solutions and to establish the resolution required for a desired accuracy of simulation. These results should be of considerable value to atmospheric numerical modellers. Of equal, if not greater, interest would be to establish the effect (at fixed resolution) of using various different finite difference schemes.

A matter of some general interest is the extent to which the behaviour of the Navier–Stokes equation numerical model used in this study can be reproduced by simpler formulations. This could be investigated by constructing a hierarchy of hydrostatic, quasi-geostrophic and other dynamical models and operating them under the appropriate conditions.

So far, the behaviour of the numerical model at rotation rates greater than about 2 rad s$^{-1}$ has not been examined. Thus flows showing shape vacillation have been explored only tentatively, and irregular flows not at all. At higher rotation rates the degree of
inhomogeneity of the grid becomes severe (because of the thinness of the Ekman layers) and it will be necessary to investigate the ability of the model to handle this situation. Any comparison with experiment at these higher rotation rates will also have to take account of the reduced accuracy of the velocity measurement technique in the more time-dependent, irregular flows (see Jackson 1984).

The results presented in this paper have encouraged us in our original belief that the rotating annulus experiments provide a useful way of verifying the dynamical formulation of large-scale numerical models. The experimental measurements and numerical integrations also, of course, provide data which are invaluable in testing theories of the various flow phenomena, and which should assist in the construction of improved theories. This aspect has been described elsewhere (Hignett 1984; Read 1984; White 1984) and is the subject of continuing study.

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