Super-rotation and diffusion of axial angular momentum: I. 'Speed limits' for axisymmetric flow in a rotating cylindrical fluid annulus

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SUMMARY

'Super-rotation' can be defined with respect to the limitations on the magnitude of specific angular momentum, $m$, in an inviscid fluid. In a viscous fluid, a steady, super-rotating axisymmetric flow is shown to require diffusion of $m$ against its gradient $\nabla m$ in the meridional plane. The conditions under which $m$ can be diffused in an incompressible fluid by molecular viscosity against $\nabla m$ in a cylindrical system are shown to be consistent with the normal properties of isotropic Newtonian viscosity (i.e. appropriate for the laminar flow of a viscous liquid in the laboratory) when cylindrical curvature is fully represented. A series of numerical simulations of thermally-driven axisymmetric circulations in a cylindrical fluid annulus, subject to various mechanical boundary conditions, is then presented. The role of diffusion in the angular momentum budget of the simulations is examined by diagnosing $m$, its diffusive flux $\mathbf{F}$ (due to molecular viscosity) and divergence $\nabla \cdot \mathbf{F}$ from the final equilibrium (steady) flow. Up-gradient diffusion (with respect to $\nabla m$) is found to be particularly important for flows in a system with stress-free top and side boundaries and a non-slip base. Similar up-gradient diffusion can also result in angular momentum expulsion effects in a system confined entirely by stress-free boundaries. The characteristic dynamics of super-rotation in a viscous fluid, and the associated limitations upon its magnitude and dependence upon the external conditions and fluid properties, are then explored in a scale analysis for the cylindrical annulus with a non-slip base (based on a scheme due to Higlett, Ilbeisbron and Kittsworth). The most rapid super-rotation (with zonal Rossby number $> 1$) is found to be favoured at moderate rotation rates, with strong cylindrical curvature, and a large meridional aspect ratio and Prandtl number. The results of the scale analysis are verified by means of further numerical experiments.

1. INTRODUCTION

It has long been appreciated that the tendency for a fluid approximately to conserve angular momentum can place important constraints on the form and intensity of the flow pattern in a rotating system. This is especially true of an axisymmetric flow since, in the absence of viscosity, the absolute angular momentum per unit mass, $m$, defined in cylindrical geometry by

$$m = r(\Omega r + \nu)$$

where $r$ is the cylindrical radius from the axis of rotation, $\Omega$ is the system rotation rate and $\nu$ the azimuthal component of velocity, is then precisely conserved following the motion of the fluid (e.g. Hide 1969, 1971). Under these conditions, the flow pattern obtained at all subsequent times is severely constrained by the angular momentum distribution of the initial state. Thus, for example, if a fluid is initially at rest in a frame rotating at angular velocity $\Omega$, the total angular momentum, $\int \int \int m \, dV$ (where $dV$ is an element of volume), must remain constant at all times, while the local value of $m$ can never exceed $\Omega r_{\text{max}}^2$ (where $r_{\text{max}}$ is the maximum radial distance from the rotation axis). The latter constraint, of course, entirely precludes the possibility of prograde ($\nu > 0$) motion near $r = r_{\text{max}}$ itself (or, conversely, retrograde motion ($\nu < 0$) near $r = r_{\text{min}}$). It follows, therefore, that limitations on the magnitude of $m$ and its budget, set by the initial conditions, can be determined which, if found to be exceeded in a given flow (corresponding to 'super-rotation', as defined more quantitatively below), must imply that one or more of the initial assumptions (such as axisymmetry, or that the fluid is inviscid) is invalid.

The present paper is concerned primarily with the possible role of molecular viscosity in an axisymmetric, incompressible flow, in generating and maintaining a steady super-rotation with respect to the corresponding inviscid flow. Molecular viscosity is an impor-
tant reason for the non-conservation of $m$ in the laminar flow of liquids in the laboratory, but the results may also be expected to have wider implications. This is because the mathematical formalism which describes the flow of a viscous liquid is frequently used in the formulation, e.g., of simple theoretical models of atmospheric circulation systems (see part II, Read 1986a), where the diffusion term parametrizes the effects of non-axisymmetric eddies. Of particular interest is whether viscous effects alone can be sufficient to maintain a very rapid super-rotation in a rotating fluid. Very strongly super-rotating flow is observed in the Venus atmosphere, for example (e.g. see Schubert 1983; part II), yet the generation of such a flow in simple atmospheric models by axisymmetric (or indeed, non-axisymmetric, see Rossow (1983)) processes has hitherto proved elusive, despite an indication of its feasibility in principle from theoretical considerations (Gierasch 1975). The present work is intended to provide some insight into this problem, and to determine what further limits can be placed on super-rotation in a viscous flow, by examining the dynamics of a well-studied analogous system, i.e. the rotating, cylindrical annulus.

We examine in section 2 the properties of axisymmetric diffusion of $m$ by an isotropic viscosity in steady rotating flows with cylindrical geometry. A detailed examination is then made in section 3 of some specific examples of flow in a differentially-heated rotating fluid annulus, obtained from an axisymmetric numerical model using the full Navier-Stokes equations for a Boussinesq, incompressible liquid, showing how a significant local and global super-rotation (as suitably defined below) can be set up and maintained in a real fluid under appropriate external conditions. In section 4 we explore the dependence of this super-rotation on some of the external conditions and determine quantitative limitations on its magnitude, with reference to a scale analysis of the basic equations and some further numerical simulations, while some concluding remarks are given in section 5. Part II reviews the problem of super-rotation in planetary and stellar atmospheres, including some of the implications of the present work for quasi-axisymmetric models of atmospheric circulations.

2. ANGULAR MOMENTUM DIFFUSION IN CYLINDRICAL GEOMETRY

If viscosity is active in an axisymmetric flow, axial angular momentum is no longer formally conserved, although the form taken by the departures from conservation may still place useful constraints on the flow. In the presence of viscosity, the equation governing the specific axial angular momentum, $m$, may be written for an incompressible fluid (e.g. see Williams 1968) as

$$ m_{\tau} + \nabla \cdot (mu) = \mathcal{F}/\rho $$  \hspace{1cm} (2)

where $u$ is the fluid velocity in the meridional plane with components $(u, w)$ in $(r, z)$, $\mathcal{F}/\rho$ is the local source/sink of angular momentum due to viscosity, and subscripts denote partial differentiation. $\mathcal{F}/\rho$ can often be written as $-\nabla \cdot \mathbf{F}$ (so that a convergence of $\mathbf{F}$ corresponds to an acceleration of the flow), where $\mathbf{F}$ is the diffusive flux of $m$ due to viscosity (see below). We now consider the budget of $m$ (obtained by integrating Eq. (2)) for a toroidal volume enclosed by a closed contour of $m$ in the meridional plane surrounding a local maximum or minimum at $m = m_0$ (or possibly including segments consisting of impermeable boundaries, see Plumb (1977) and part II). It is straightforward to show (e.g. see part II) that, for a steady state, the constraints of mass conservation require

$$ \int \int \mathbf{F} \cdot \mathbf{n} = 0 $$  \hspace{1cm} (3)
(where $dn$ is the outward normal vector element of surface area). A crucial factor necessary for the maintenance of a local maximum in $m$ implied by Eq. (3), therefore, is the diffusion of axial angular momentum up-gradient (i.e. $\mathbf{F} \cdot \nabla m > 0$) somewhere in the flow.

In this context, it is important to note that Newtonian viscosity diffuses linear and angular momentum in a circularly-symmetric system so as to tend to remove gradients of angular velocity (i.e. $\mathbf{F}$ is anti-parallel to the gradient of relative angular velocity $\gamma$, where $\gamma = \nu/r$ in cylindrical geometry). The reason for this lies in the anisotropy imposed by the geometry and the constraints of rotation (e.g. see Batchelor 1967). Thus, in cylindrical geometry, $\mathbf{F}$ takes the form in an axisymmetric, incompressible flow with constant kinematic viscosity $\nu$,

$$\mathbf{F} = -\nu r^2 \nabla \gamma$$  \hspace{1cm} (4)

$$= -\nu r^2 \nabla (m/r^2).$$ \hspace{1cm} (5)

Note that $\mathbf{F}$ is anti-parallel to $\nabla \gamma$ and not to $\nabla m$. Thus, the projection, $P$, of $\mathbf{F}$ down the gradient of $m$ in the meridional plane is given by

$$P = -\mathbf{F} \cdot \nabla m = \nu [\nabla m]^2 - (1/r)(m^2),.$$ \hspace{1cm} (6)

We see, therefore, that $\mathbf{F}$ can be effectively up-gradient for $m$, yet still be anti-parallel (and hence down-gradient) with respect to $\nabla \gamma$, provided

$$(1/r)(m^2), > |\nabla m|^2$$ \hspace{1cm} (7)

implying that metric effects (due to the cylindrical curvature) cannot be ignored when considering the effects of diffusion on axial angular momentum. Furthermore, the form of $\mathbf{F}$ given in Eq. (5) implies that only its radial component can be up-gradient with respect to $\nabla m$ since, from the definition of $\mathbf{F}$ in Eqs. (4) and (5), the vertical components of $\nabla \gamma$ and $\nabla m$ must be in the same sense.

3. SOME EXAMPLES OF VISCOS FLows IN THE CYLINDRICAL ANNULUS

Having established the conditions for up-gradient viscous diffusion of $m$ in a cylindrical system, we now examine the role of internal diffusion in some actual examples of axisymmetric flows. Steady axisymmetric flows occur under certain conditions in the laboratory, in a cylindrical fluid annulus subject to differential heating at the sidewall boundaries (e.g. Hide and Mason 1975). Accordingly, we first consider configurations of thermal and mechanical boundary conditions appropriate to this laboratory system.

(a) Numerical model

To obtain sufficiently detailed data for the required flow diagnostics, we make use of a numerical model to simulate the flow. The model used throughout this paper is a two-dimensional, axisymmetric version of the model described by James et al. (1981) and Hignett et al. (1985), which integrates the full (non-hydrostatic) equations for the time evolution of a Boussinesq incompressible fluid in cylindrical annular geometry. The model uses a grid-point finite-difference formulation, with a typical resolution of 16$\times$16 points in $(r, z)$. A stretched mesh is used to enhance the resolution in boundary layers adjacent to the side- and endwalls, and the diffusion of heat and momentum by molecular conduction and viscosity are fully represented (e.g. as in Batchelor 1967). The finite-difference schemes are designed to conserve most quantities of interest to second-order accuracy, and the model has been extensively verified against laboratory data (for further details see Hignett et al. 1985). All the examples given below were obtained by specifying
the appropriate external and boundary conditions at \( t = 0 \), and then integrating the model forwards in time from an initially isothermal state at rest in the rotating frame, until a steady final state was obtained.

\[(b) \quad \text{An example with laboratory boundary conditions}\]

The first example (case A) consists of a typical axisymmetric flow with isothermal sidewalls at \( T = T_a \) and \( T = T_b \) at \( r = a \) and \( r = b \) respectively, thermally insulating endwalls, and non-slip rigid boundary conditions applied at all boundaries except at the top. The upper boundary is equivalent to that obtained without a lid in contact with the fluid, conventionally represented in numerical models by a rigid, stress-free condition, i.e.

\[
w = u_z = v_z = 0
\]

(8)
on \( z = H \) (cf. Williams 1967a). The relevant physical parameters for this, and subsequent examples, are listed in Table 1. Figure 1 shows a selection of fields obtained from the numerical simulation for the final steady state. As found in previous work (e.g. Williams 1967a,b), the temperature field in Fig. 1(a) is dominated by thin boundary layers adjacent to the side-walls, with a weakly baroclinic interior. The meridional circulation (represented by the Stokes streamfunction \( \chi \) in Fig. 1(d)) is largely confined to the boundary layers adjacent to the three non-slip boundaries, but with significant radial motion in the upper part of the interior along a detached shear layer. The azimuthal velocity field is shown as the relative angular velocity \( \gamma = v/r \) in Fig. 1(b), and consists of a strong prograde jet close to the inner sidewall at the top surface, with a much weaker retrograde jet in the lower regions of the annulus.

The angular momentum properties of the flow may be examined in various ways. In the present paper, we define the ‘super-rotation’ of the flow in terms of the relative and absolute angular momentum per unit mass. ‘Global super-rotation’ represents the volume-integrated relative angular momentum, normalized by the absolute angular momentum of the initial rest state of the fluid in the rotating frame, and is therefore defined by

\[
S = \int_0^H \int_a^b 4\pi r^2 dr \, dz / [\Omega H (b^4 - a^4)]
\]

(9)

where \( H \) is the depth of the annulus. \( S > 0 \) then represents the extra angular momentum introduced into the fluid during the spin-up process by interactions with the boundaries.

<table>
<thead>
<tr>
<th>TABLE 1. EXPERIMENTAL PARAMETERS FOR THE NUMERICAL SIMULATIONS</th>
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<tbody>
<tr>
<td>Inner radius, ( a ) (cm)</td>
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<tr>
<td>Outer radius, ( b ) (cm)</td>
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<tr>
<td>Depth, ( H ) (cm)</td>
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<tr>
<td>Rotation rate, ( \Omega ) (s(^{-1}))</td>
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<td>Temperature difference, ( \Delta T (=</td>
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<td>Coefficient of cubical expansion, ( \alpha ) (K(^{-1}))</td>
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<td>Kinematic viscosity, ( \nu ) (cm(^2)s(^{-1}))</td>
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<tr>
<td>Thermal diffusivity, ( \kappa ) (cm(^2)s(^{-1}))</td>
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<tr>
<td>Mean density, ( \rho_0 ) (g cm(^{-3}))</td>
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<tr>
<td>Rayleigh number, ( Ra ) (10(^9))</td>
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<td>Ekman number, ( E ) (10(^{-3}))</td>
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<td>Boundary layer parameter, ( Q )</td>
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Figure 1. Contour maps of the steady-state fields for a numerical simulation of the thermally-driven axisymmetric circulation in a rotating fluid annulus with rigid, non-slip base and side boundaries, and a stress-free top surface (case A, see text): (a) temperature (contour interval, 0.5 K); (b) relative angular velocity $\gamma = \nu/r$ (see text, contour interval, 0.02 s$^{-1}$); (c) absolute angular momentum $m$, normalized by $\Omega b^3$ (see text, contour interval, 0.1); (d) meridional streamfunction $\chi$ (contour interval, 0.01 cm$^2$s$^{-1}$); (e) local torque, $-\nabla \cdot F$ (see text, contour interval, 0.0125 cm$^2$s$^{-1}$). The diffusive flux $F$ of $m$ (see text) is shown in (c), superimposed upon the distribution of $m$ as vectors whose length is proportional to $|F|$. Negative contours are dashed.
We may also define a ‘local super-rotation’, $s$, which represents the absolute specific angular momentum of the fluid, normalized by the absolute angular momentum of stationary fluid at the furthest point in the system from the axis of rotation (i.e. $m = \Omega b^2$). For compatibility with the definition of $S$ in Eq. (9), we define $s$ by

$$s = m/\Omega b^2 - 1$$

so that $s > 0$ represents excess angular momentum beyond that possessed by any fluid element in the initial rest state.

For the example in Fig. 1, the flow is characterized by a modest global super-rotation ($S = 0.084$), but with scarcely any local super-rotation ($s_{\max} < 0.01$), indicating little net change in the angular momentum distribution from the initial state (in solid-body rotation at $\Omega$). This is confirmed by Fig. 1(c), which shows contours of $m$ which are significantly distorted from the initial vertical form only in the Ekman layer on the lower boundary. Also shown in Fig. 1(c) are vectors representing the diffusive flux, $\mathbf{F}$, of $m$ due to viscosity (defined by Eqs. (4) and (5)). $|\mathbf{F}|$ is seen to be largest in the lower Ekman layer and close to both the sidewalls, but is also quite strong throughout the interior. At the boundaries, there is a significant (down-gradient) flux of angular momentum into the fluid through the lower boundary and out of the fluid through the inner sidewall boundary (cf. the discussion of Plumb 1977) as well as at the outer sidewall (where $\mathbf{F}$ is up-gradient). This is further confirmed by the distribution of local torque $\mathbf{T}/\rho$ on the fluid ($=-\nabla \cdot \mathbf{F}$, see Eqs. (2) and (5)), shown in Fig. 1(e), which exhibits a pronounced source of $m$ along the lower boundary and a sink of $m$ adjacent to the upper regions of the inner sidewall. Up-gradient diffusion of $m$ is seen to occur in the interior of the flow, as required to satisfy the integral condition implied by Eq. (3), but plays a relatively minor role in the angular momentum budget of the flow (in the sense that it is not associated with regions of significant torque on the fluid). It is also of interest to note that only the horizontal component of $\mathbf{F}$ is seen to be up-gradient in certain regions of Fig. 1(c), as suggested by the discussion in section 2, while the vertical component always acts down-gradient with respect to $\nabla m$.

(c) Stress-free sidewalls

The previous example clearly indicates the importance of the boundary friction in the laboratory annulus, especially associated with the sidewalls. This has a strong influence on the angular momentum budget, preventing much significant storage of angular momentum in the interior and any more than a modest global super-rotation, while the non-slip condition at the outer sidewall limits the maximum local super-rotation to very small values. The dominant influence of sidewall boundary layers on the laboratory annulus was noted by Williams (1968), who suggested removing some of the frictional constraints associated with the sidewalls in a numerical model by making them stress-free. Such a configuration was supposed by Williams to be more closely analogous to a planetary atmosphere in allowing significant frictional interaction with the fluid only at the lower boundary. The appropriate boundary conditions become those given by Eq. (8) at the upper boundary, the usual rigid, non-slip conditions at the lower boundary, and

$$u = (v/r)_r = w_r = 0$$

on $r = a, b$ (cf. Williams 1968).

(i) ‘Direct’ circulation and super-rotation. Figure 2 shows contour maps of some of the fields equivalent to Fig. 1 for a steady flow obtained under the same external conditions as the flow discussed in sub-section 3(b) (case B). The temperature field and stream-
function for the meridional flow (Figs. 2(a) and (c)) are largely identical to the fields produced by Williams (1968, case A4), including the pronounced frontal structure of the temperature field in the interior, and the three main cells (one direct and two indirect in the thermal sense) in the meridional circulation. The field of $\gamma$ is shown in Fig. 2(b), and clearly shows significant prograde relative motion throughout most of the annulus except for a narrow strip close to the outer regions of the lower boundary. The latter is clearly necessary to produce zero net resultant torque on the fluid from the lower boundary. The most rapid prograde motion occurs at the inside upper corner of the domain, although significant motion is found right up to the outer boundary itself, implying that some of the fluid elements have acquired angular momentum well in excess of $\Omega r_{\text{max}}^2 = \Omega b^2$. This is confirmed by the field of $m$ shown in Fig. 2(d), in which the region of fluid where $m$ exceeds $\Omega b^2$ (i.e. $s > 0$) is shaded. The steady flow is seen to be characterized by a relatively large maximum local super-rotation ($s_{\text{max}} = 0.35$) and global super-rotation ($S = 0.36$), so that viscous friction must play a crucial role in establishing the flow.
To investigate this, we again show vectors of $\mathbf{F}$ superimposed upon contours of $m$ in Fig. 3(a). Because of the stress-free condition imposed on $v$ (and hence on $\gamma$) at the top and side boundaries, $\mathbf{F}$ is constrained to lie parallel to those boundaries, and the net flux of $m$ across them is therefore zero. The constraint expressed in Eq. (3), together with the form of $m$ in Fig. 2(d), demands that $\mathbf{F}$ be up-gradient for $m$ somewhere in the flow as before, and this is clearly illustrated in Fig. 3(a). $\mathbf{F}$ is again everywhere anti-parallel to $\nabla \gamma$ (cf. Fig. 2(b)), and is up-gradient for $m$ where the contours of $m$ are most nearly vertical. The local torque, represented again by $-\nabla \cdot \mathbf{F}$, is shown in Fig. 3(b) and, unlike the previous case, is clearly not negligible in the interior of the fluid. Pronounced sources and sinks are found adjacent to the lower boundary (associated with down-gradient $\mathbf{F}$), while somewhat weaker features are found near the inner and outer side boundaries, which extend well into the interior and are associated with regions of up-gradient diffusion. Despite the apparent complexity of Fig. 3(b), the volume-integrated torque calculated from the field in Fig. 3(b) is approximately zero (i.e. to within the limits of numerical precision). Individual fluid elements follow closed streamlines in the meridional plane, as indicated by Fig. 2(c), and are alternately accelerated and decelerated as they move through the pattern of local torque in Fig. 3(b). It is of interest to note that the most strongly super-rotating fluid never comes into contact with the main sources and sinks of $m$ near the lower boundary, but remains in the upper regions of the annulus (see Fig. 2(d)) and is then affected only by the interior torque features shown in Fig. 3(b). They must clearly obtain their excess of angular momentum during the spin-up process.

The setting up of the flow is also followed by the model, of course, and the behaviour of several of the globally-integrated properties of the flow in case B during its evolution are shown in Fig. 4. The initial thermally-driven meridional circulation is set up relatively quickly after setting the temperature difference at the sidewalls, as indicated by the Nusselt number, which reaches its equilibrium value by $t \sim 40$ s. The volume-integrated torque, $\tau$, is found to rise rapidly to a peak value $\sim 3.4 \text{ cm}^2\text{ s}^{-2}$ after $\sim 30$ s of model time after switching on the differential heating, and then decays slowly towards zero. The excess global super-rotation $S$, meanwhile, rises at a rate, $dS/dt$, corresponding to $\tau$, levelling out beyond $t \sim 300$ s towards its final value $S = 0.36$. The peak in $\tau$ corresponds to the interaction of fluid in the initial meridional circulation with the non-slip lower boundary. The initial circulation, driven by the differential heating, results in outward

![Figure 3](image-url). Contour maps of $m$ and local torque ($-\nabla \cdot \mathbf{F}$) for the numerical simulation case B (see Fig. 2). (a) $m/\Omega b^2$ (contour interval, 0-1), with vectors of $\mathbf{F}$ superimposed; (b) $-\nabla \cdot \mathbf{F}$ (contour interval, 0-0.025 cm$^2$ s$^{-2}$). Negative contours are dashed.
radial motion along the lower boundary. The tendency of the fluid to conserve $m$ results in sub-rotating flow throughout the lower Ekman layer, causing positive angular momentum to be pumped into the fluid. As the flow develops, this angular momentum is carried in the same direction as the heated fluid, upwards and inwards toward the upper corner of the annulus (in the way envisaged, e.g., by Eady (1953) for the earth's atmosphere), generating a positive peak in angular velocity as $m$ is approximately conserved. This region continues to accumulate angular momentum, spinning up until internal viscous diffusion of $m$ acting outwards from the maximum in $\gamma$ balances the transport of $m$ into the region by advection in the meridional circulation.

To achieve zero net torque at the lower boundary in the final steady state, the fluid must develop a positive (prograde) component of relative azimuthal motion during the evolution of the flow somewhere along the boundary, associated with thermally-indentirect (inward) radial flow. Thus, the steady meridional flow pattern in Fig. 2(c) includes at least one thermally-indentirect circulating cell which arises primarily because of the frictional constraints, and which evidently develops from an axisymmetric diffusive instability when the Prandtl number $\sigma \gg 1$ (Williams 1970). A similar pattern of meridional cells was predicted by the analytical model of a steady, zonally-symmetric circulation by Eliassen (1952), discussed in this context by Williams (1968), and which provides a further rationale for the interior sources and sinks of $m$, found in the field of $-\mathbf{V} \cdot \mathbf{F}$ in Fig. 3(b).

The present example, illustrated in Figs. 2-4, thus demonstrates a relatively strongly super-rotating flow in which up-gradient diffusion of $m$ plays a crucial role in redistributing angular momentum, obtained from viscous interactions with the lower boundary, towards the upper and outer regions of the flow. Super- (rather than sub-) rotation is obtained.
because the main thermally-driven meridional circulation is 'direct' for angular momentum in the same way as it is for heat, in that positive anomalies of both quantities are transported upwards and inwards towards the rotation axis.

(ii) 'Indirect' circulation and sub-rotation. If the meridional circulation were to be reversed relative to Fig. 2(c) by reversing the imposed thermal gradient, the flow would again transport heat vertically upwards (being driven by buoyancy forces), but also radially outwards towards the top outside corner of the annulus. Because the initial radial gradient of $m$ is always positive, such a circulation would effectively transport low angular momentum fluid upwards and outwards, resulting in a net transfer of $m$ inwards and downwards, i.e. in an 'indirect' sense by analogy with the transfer of heat. Thus, fluid with initially high $m$ would be brought rapidly into contact with the lower boundary through inward radial motion along the lower Ekman layer. The tendency for fluid to conserve $m$ would result in retrograde relative motion, and $m$ in this high angular momentum fluid would then be pumped out of the system to produce a sub-rotating flow (i.e. with $s$ and $S < 0$ everywhere). An example of such a sub-rotating flow is given by Read (1985). Super- and sub-rotation are therefore seen to be associated with 'direct' and 'indirect' meridional circulations relative to the initial distribution of $m$, by analogy with the transport of heat by buoyancy-driven flows. Up-gradient diffusion may be much

![Figure 5. Contour maps of the steady-state fields for a numerical simulation of the thermally-driven axisymmetric circulation in a rotating fluid annulus with rigid, stress-free boundaries (case C, see text): (a) $T$ (contour interval, 0.5 K); (b) $\gamma$ (contour interval, 0.05 s$^{-1}$); (c) $\chi$ (contour interval, 0.01 cm$^3$ s$^{-1}$); (d) $m/\Omega b^2$ (contour interval, 0.1; the region where $m > \Omega b^2$ is shown shaded). Negative contours are dashed.](image)
more important for the angular momentum budget of a super-rotating flow, however, than for a sub-rotating flow, since the local sources and sinks of \( m \) in the sub-rotating flow discussed by Read (1985) were found to be associated with down-gradient diffusion of \( m \).

\[(d) \text{ A completely stress-free experiment} \]

The final example in this section illustrates the role of up-gradient viscous diffusion of angular momentum in maintaining an axisymmetric local super-rotation without a global super-rotation (i.e. \( S = 0 \)). If the numerical model is run with all its boundaries maintained stress-free, no net torque between the fluid and its surroundings can occur at any time during the simulation, and the globally-integrated angular momentum should therefore remain constant throughout the experiment. The steady state fields for an experiment under the same external conditions as in sections 3(b) and (c), but with all boundaries kept stress-free (case C), are shown in Fig. 5. The temperature field (Fig. 5(a)) is only slightly changed from Fig. 2(a), being somewhat less frontal in character. The angular velocity field (Fig. 5(b)) is divided almost exactly diagonally between the outside lower half of the annulus, where \( \gamma < 0 \), and the inside upper half, where \( \gamma > 0 \), reflecting the zero torque condition which allows no net global super-rotation. The meridional streamfunction in Fig. 5(c) has a number of features in common with Fig. 2(c), being dominated by a thermally-direct component lying along the concentration of isotherms in the interior (cf. Fig. 5(a)). The removal of the frictional constraint along the lower boundary, however, renders a lower, thermally-indirect cell unnecessary, permitting radially outwards motion nearly everywhere along the lower (free surface) Ekman layer. The angular momentum distribution in Fig. 5(d) is quantitatively changed from Fig. 2(d) although the contours are distorted from the vertical in the same way qualitatively as in Fig. 2(d) due to advection of \( m \) by the similar meridional circulation. In particular, \( s \) is found to be almost entirely confined between \( s = 0 \) and \( s = a^2/b^2 - 1 \) (i.e. corresponding to the upper and lower limits on \( m \) in the initial rest state) except in the outside top corner (where \( s \) exceeds zero by a small factor) and in the inside bottom corner (where \( s < a^2/b^2 - 1 \), i.e. a small local sub-rotation). Thus, although \( S \) is constrained to zero by the boundary and initial conditions, local sub- and super-rotation still occur which, if physically correct, must represent the effects of up-gradient viscous diffusion of \( m \) in the interior of the fluid. This hypothesis is supported by the distribution of \( F \) in Fig. 6(a), in which the diffusive flux of \( m \) is seen to be predominantly outwards (and hence up-gradient with respect to \( \nabla m \)) as is consistent with Eq. (3) and the distribution of \( \gamma \) in Fig. 5(b). The resultant effect of the flow on \( m \) is to redistribute it so that the centroid of angular momentum is shifted outwards during the spin-up, i.e. a weak angular momentum expulsion occurs towards the outer boundary. It is of interest to note that, despite the removal of any net local viscous stress at any of the boundaries, the local torque on fluid elements is not negligible in the flow (see Fig. 6(b)), but is only required to balance to zero at equilibrium when integrated over the entire volume of the annulus.

In view of the possible implications of the inferred expulsion of angular momentum towards the outer boundary (see part II), it is important to assess whether such an effect is real, or simply an artifact of the numerical model. A completely stress-free experiment is an extremely severe test of a numerical model and its conservation properties, since small systematic departures from perfect conservation of \( m \) could result in significant errors in \( s \) and \( S \). Slowly growing errors of this form were found by Weir (1976), for example, in his numerical simulations of flow in a rotating fluid sphere, which manifested themselves as a slow but steady growth of \( S \) with time. In the present example, a similar
effect was detectable, but resulted in a departure of $S$ from zero of less than one part in $10^4$ throughout the duration of the experiment (i.e. up to 6000 s of model time), which is not sufficient to account for the magnitude of the effects observed. Similarly, the volume-integrated torque was calculated during the spin-up of the flow, and remained less than $3 \times 10^{-3} \text{cm}^2\text{s}^{-2}$ throughout the experiment. Other experiments were conducted to test the effects of numerical precision, using a 64-bit arithmetic (i.e. double precision) version of the model, but produced essentially the same result. We may conclude, therefore, that the angular momentum expulsion effect is a real phenomenon, and is a natural consequence of the up-gradient viscous diffusion of $m$ acting to balance the advection of $m$ in a meridional circulation which is 'direct' both for heat and angular momentum.

4. DEPENDENCE OF SUPER-ROTATION ON EXTERNAL CONDITIONS

The previous section has explored a number of examples in the cylindrical annulus which demonstrate varying degrees of local and global super-rotation (as suitably defined in Eqs. (9) and (10)), and which elucidate the role of viscous diffusion in transferring $m$ against its gradient in the meridional plane under certain circumstances. In view of the possibility of some more general applications of these results, it is also of importance to determine the quantitative limitations which may be placed upon the magnitude of super-rotation obtainable in the annulus, its dependence upon the external conditions, and the characteristic dynamics associated with the various possible regimes of flow. We now go on, therefore, to examine the dependence of super-rotation on some of the external conditions relevant to the annulus flows (geometry, fluid properties and magnitude of the differential heating and cooling) for the configuration of boundary conditions investigated in section 3(c). For this purpose, we make use of a scale analysis of the governing equations and some further numerical experiments, placing particular emphasis on the determination of limits on the magnitude of $S$ obtainable in an axisymmetric flow.

(a) Scale analysis in cylindrical geometry

For a cylindrical fluid annulus heated and/or cooled at its boundaries, it is convenient to use a form of the boundary layer analysis similar to that used by Gill (1966), McIntyre (1968) and Hignett et al. (1981), and implicit in the work of Hignett (1982). The full
governing equations for steady, axisymmetric flow of an incompressible Boussinesq fluid in cylindrical geometry can be written as an equation for the specific azimuthal linear momentum,

\[ \nu(\nabla^2 v - v/r^2) = (1/r)(\chi_z(v_r + v/r) - \chi_r v_z) + (f_0/r)\chi_z \]  

(12)

where \( f_0 = 2\Omega \) and subscripts again denote differentiation; an azimuthal vorticity equation,

\[ \nu(\nabla^2 \zeta - \zeta/r^2) = J(\chi_r, \zeta/r) + g\alpha T_r - f_0 v_z - (1/r)(v^2)_z \]  

(13)

where \( \alpha \) is the coefficient of cubical expansion, \( g \) is the acceleration due to gravity and \( \zeta \) is the vorticity:

\[ \zeta = u_r - w_z = (1/r)(\chi_{rr} - \chi_r/r + \chi_{zz}); \]  

(14)

and a thermodynamic equation

\[ \kappa \nabla^2 T = (1/r)J(\chi, T). \]  

(15)

\( \kappa \) is the thermal diffusivity, and \( \chi \) is the streamfunction for the meridional flow, so that

\[ u = (1/r)\chi_z, \quad w = -(1/r)\chi_r \]  

(16)

(which implies the usual form of the continuity equation, e.g. see McIntyre (1968)), and

\[ J(a, b) = a_z b_z - a_z b_r. \]  

(17)

The boundary conditions appropriate to the problem are

\[ T = T_0 - \Delta T/2 \quad \text{at } r = a \]  

(18a)

\[ T = T_0 + \Delta T/2 \quad \text{at } r = b \]  

(18b)

and

\[ u = w_r = (v/r)_r = 0 \quad \text{at } r = a, b \]  

(18c)

\[ u = v = T_z = 0 \quad \text{at } z = 0 \]  

(18d)

\[ u_z = v_z = w = T_z = 0 \quad \text{at } z = H. \]  

(18c)

The problem may then be described fully by five dimensionless parameters, namely a Rayleigh number

\[ Ra = g\alpha \Delta T L^2/\kappa \nu \]  

(19)

where \( L = (b - a) \), the horizontal length scale imposed by the cylinder gap width; an Ekman number

\[ E = \nu/f_0 H^2; \]  

(20)

the Prandtl number

\[ \sigma = \nu/\kappa; \]  

(21)

and the two aspect ratios

\[ \varepsilon = H/L \]  

(22)

\[ \eta = L/\bar{r} \]  

(23)

where \( \bar{r} \) is a ‘typical’ radius, and \( \eta \) thus measures the effect of cylindrical curvature. It is also convenient to define a subsidiary dimensionless parameter—a ‘curvature’ Rossby
\[ Ro_c = \frac{V}{f_0 \rho}, \]

where \( V \) is a measure of the horizontal velocity scale. The significance of the latter may be seen from a comparison with the definition of the global super-rotation \( S \) in Eq. (9), which was obtained from

\[
S = \iint vr^2 \, dr \, dz / \iint \Omega r^3 \, dr \, dz
= \iint (v/r) r^3 \, dr \, dz / \iint \Omega r^3 \, dr \, dz
\sim V/\Omega \rho \sim Ro_c. \tag{25}
\]

The aim of the analysis may then be seen as obtaining constraints and predictions as to the magnitude of \( Ro_c \), and its dependence on the imposed conditions as measured by the five main dimensionless parameters.

We follow Hignett et al. (1981) and Hignett (1982) in making use of a combination of the main parameters which measures the ratio of the thickness of the (non-rotating) thermal boundary layer adjacent to the sidewalls to that of the Ekman layer above the bottom surface (and also adjacent to the stress-free upper surface, cf. Hide (1964), although its influence on the meridional transport properties of the flow is much weaker). The thermal boundary layer is characterized by an advective/diffusive balance in the thermodynamic equation (applicable even to a stress-free sidewall boundary), whose thickness is given by

\[ l_T = L(Ra/e)^{-1/4} \tag{26} \]

(cf. Gill 1966; McIntyre 1968; Hignett 1982), while the thickness of each Ekman layer is given by

\[ h_E = HE^{1/2}. \tag{27} \]

Thus, the boundary layer ratio \( Q \) is defined by

\[ Q = (l_T/h_E)^2 = Ra^{-1/2} E^{-1} e^{-3/2} \tag{28} \]

which is conveniently proportional to \( \Omega \), and ranges from \( 0 \to \infty \). For the following analysis, we assume that \( \varepsilon \) and \( \eta \) are not too different from unity, while for most purposes, \( Ra \gg 1 \) and \( E < 1 \) (to obtain distinct boundary layers and an interior) and \( \sigma \gg 1 \). Under these circumstances, the dynamical balances are found to depend largely on \( Q \), and six main regimes are discernible (cf. Hignett et al. 1981):

(i) zero rotation \( (Q = 0) \);
(ii) 'very weak' rotation \( (0 \ll Q \ll e^2 \sigma^{-2}) \);
(iii) 'weak' rotation \( (e^2 \sigma^{-2} \ll Q \ll 1) \);
(iv) 'moderate' rotation \( (Q \sim 1) \);
(v) 'strong' rotation \( (1 \ll Q \ll e^{1/2} Ra^{1/6}) \);
(vi) 'very strong' rotation \( (Q \gg e^{1/2} Ra^{1/6}) \).

Each regime is characterized by a different dynamical balance, reflecting the varying effects of rotation on the boundary layers and interior flow. For regimes (i)–(iii), the meridional flow is dominated by the sidewall boundary layers, so that the streamfunction \( \frac{\chi}{\hat{r}} \sim \kappa e^{3/4} Ra^{1/4} \). The total (advective plus conductive) heat transfer of the flow is measured relative to that due to thermal conduction alone by the Nusselt number, \( Nu \), which is defined for the present system by
\[ \text{Nu} = 1 + \tilde{\chi}(\tilde{f}e)^{-1}. \] (29)

For regimes (i)–(iii), therefore, \((\text{Nu} - 1) = (Ra/e)^{1/4}\). The main difference between regimes (ii) and (iii) lies in the role of the Ekman layer and its dynamical balances. For \(Q \ll e^2\sigma^{-2}\), the most significant horizontal boundary layer is a passive layer, characterized by an advective/diffusive balance in the azimuthal momentum equation, of thickness \(h_H = H\sigma^{-7/4}Ra^{-1/4} < h_E\) (cf. Read 1986b). Over length scales comparable with the Ekman layer depth, however, the dynamics are characterized by a nonlinear/Coriolis balance in the azimuthal momentum equation. The importance of this result for the angular momentum balance may be realized by non-dimensionalizing the angular momentum equation relative to the meridional advection timescale \(HLF/\tilde{\chi}\) over the Ekman layer depth, so that

\[ (1/r_*)J(\chi_*, m_*) = (\sigma^2 Q/e^2)[\nabla \cdot (r^2 \nabla (m_*/r^2))]. \] (30)

(where the * subscript denotes dimensionless functions of order unity). Thus, for regime (ii), the entire flow (apart from the thin, passive lower boundary layer) is characterized by local angular momentum conservation to leading order, implying that \(V \sim f_0 e^2\) and hence \(Roe \sim 1\).

If \(Q \gg e^2\sigma^{-2}\), \(h_H > h_E\) and the Ekman layer becomes the dominant horizontal boundary layer. Equation (30) implies that \(m\) may no longer be regarded as a conserved quantity in the Ekman layer, although it may be approximately conserved in the interior. Two balances in the azimuthal momentum equation are possible in the Ekman layer, depending upon the curvature parameter, \(\eta\). The two extreme possibilities are either a Coriolis/viscous balance (i.e. a simple, linear Ekman layer), obtained for small curvature \((\eta \ll R^2 Q e^{-3})\), or an inertial/viscous balance, obtained for larger curvature \((\eta \gg e^2 \sigma^{-3})\). For the former case, \(V \sim \tilde{\chi} Re^{1/2} / L^{-1}\) and \(Roe \sim e^2 (\rho^2 / \eta)^{1/3}\) and the isotherm slope is \(-1\), \(Roe\) can be \(\gg 1\), implying a centrifugal/buoyancy balance in the interior vorticity equation (i.e. a cyclostrophic balance). \(V\) is then \(\sim \nu (Ra \tilde{H}/\sigma)^{1/2} L^{-2}\) (i.e. independent of rotation) and \(Roe \sim e^2 \rho^{1/2} Q^{-1} \sigma^{-1/2}\). Since the meridional circulation is hardly affected by rotation in this regime, however, the assumption of unit isotherm slope in the interior is unlikely to be fulfilled except near the boundary of regimes (iii) and (iv).

For the intermediate regime (iv) where \(Q \sim 1\), the Ekman layer begins to exert an influence on the flow. The meridional circulation is then rescaled to \(\tilde{\chi} / L \sim \tilde{\chi} Re^{1/4} Q^{-1/2}\) and \((\text{Nu} - 1) \sim Ra^{1/4} e^3 Q^{-1/2}\). Both the azimuthal momentum and vorticity balances discussed for regime (iii) can apply in principle to this regime, except that the additional influence of the Ekman layer on the heat transfer of the flow makes the constraint that the isothermal slope \(-1\) in the interior, as required for cyclostrophic balance in the vorticity equation, more likely. The analysis therefore suggests that \(V\) will rise to its maximum value \(V \sim \nu (Ra \tilde{H}/\sigma)^{1/2} L^{-2}\) as implied by cyclostrophic balance, provided the cylindrical curvature is sufficiently large, while the largest value of \(Roe\) \((- e^2 \rho^{1/2} Q^{-1} \sigma^{-1/2}\)) is obtained for the lowest value of \(Q\) compatible with the required balance. The regime(s) most likely to exhibit a large super-rotation are therefore those centred around where \(Q \approx 1\) (i.e. between regimes (iii) and (iv)). Under these conditions, a strong viscous interaction between the fluid and the lower boundary can occur in the Ekman layer, while the interior flow is characterized by a cyclostrophic balance, implying that

\[ (\nu^2) / r = g \alpha T. \] (31)

Such a flow should be obtained for large curvature and aspect ratio, and is significantly
affected by the value of the Prandtl number (which must still exceed $\varepsilon^{2/3} \eta^{-1/3}$).

When $Q \gg 1$, the meridional flow is dominated by the lower Ekman layer transports, so that $\tilde{\chi}/\tilde{r} = \kappa \varepsilon^{3/4} Ra^{1/4} Q^{-3/2}$ and $(Nu - 1) = Ra^{1/4} \varepsilon^{3/4} Q^{-3/2}$. The sidewall thermal boundary layer is forced to expand to $l_T = L Ra^{-1/4} \varepsilon^{-3/4} Q^{3/2}$ until, in regime (vi), it becomes comparable with $L$, and the resultant heat flow is dominated by conduction everywhere. In the interior, the azimuthal momentum and vorticity balances are primarily geostrophic, so that $V \sim \kappa Ra^{1/2} \varepsilon^{3/2} L^{-1} Q^{-1}$ (i.e. the thermal wind scale), so that $Ro_c \sim \varepsilon^2 \eta \sigma^{-1} Q^{-2}$ which is always $\ll 1$.

For the global super-rotation properties of the flow, the analysis therefore predicts three main regimes which may be summarized as follows:

(I) 'slow' rotation ($Q \ll \varepsilon^2 \sigma^{-2}$), where $V = f_\sigma \varepsilon$ (i.e. proportional to $Q$) and $Ro_c \sim 1$ (i.e. the 'angular momentum conserving' regime);

(II) 'moderate' rotation ($\varepsilon^2 \sigma^{-2} \ll Q < 1$), in which two cases are possible, depending upon $\varepsilon$, $\eta$ and $\sigma$: (a) a 'weak' super-rotation regime for small curvature ($\eta \ll \sigma^2 Q \varepsilon^{-2}$), where $V = \kappa \varepsilon Ra^{1/2} Q^{1/2} L^{-1}$ and $Ro_c = \eta \sigma^{-1} Q^{-1/2} \ll 1$; or (b) a 'strong' super-rotation regime, obtained for large curvature ($\eta \gg \varepsilon^2 \sigma^{-2}$) where $V = \nu (Ra \delta H/\sigma)^{1/2} L^{-2}$ and $Ro_c = \varepsilon \sigma^{-1} Q^{-1}$ (which can be $\gg 1$);

(III) 'rapid' rotation ($Q \gg 1$), where $V$ is geostrophic and given by the thermal wind scale $V = \kappa Ra^{1/2} \varepsilon^{3/2} L^{-1} Q^{-1}$, so that $Ro_c = \varepsilon^2 \eta \sigma^{-1} Q^{-2} \ll 1$.

![Figure 7](image_url)

Figure 7. The variation of the steady-state globally-integrated flow parameters with the boundary layer parameter $Q$ (see text), for thermally-driven axisymmetric circulation in a rotating fluid annulus subject to the configuration of boundary conditions used in case B (with $\varepsilon = 1$). Results are calculated from numerical simulations, and the derived parameters are scaled according to the analysis of section 4(a): (i) super-rotation $S$; (ii) Nusselt number $(Nu - 1)$; (iii) zonal velocity scale $V = (\sigma^2 \varepsilon^{3/2})$, see text). Characteristic gradients inferred from the scale analysis in 4(a) are indicated next to the curves for $S$ and $V$. 
(b) Numerical verification

The above analysis has indicated ways in which the super-rotation is expected to depend upon external conditions, as measured by the boundary layer ratio, $Q$, the aspect ratios, $\varepsilon$ and $\eta$, and the Prandtl number. Because the numerical model used for the experiments described above in section 3 uses the full Navier–Stokes equations (in effect Eqs. (12)–(18)), we may also use it to verify the analysis given above, by conducting a series of numerical experiments over a sufficiently wide range of rotation rate and aspect ratio. Figure 7 shows the variation of the three main global properties of the flow: $(\bar{\phi})^{1/2}$ (a measure of $V$, where $\bar{}$ implies an average over the whole annulus); $S$; and $(Nu - 1)$, for a series of experiments carried out at $\varepsilon = 1$ and $\Delta T = 5$ K (as for case B in 3(c)), spanning a range in $Q$ from $10^{-4}$ to nearly 4-0. Both $V$ and $(Nu - 1)$ are scaled in the way suggested by the analysis in 4(a) above.

As predicted by the scale analysis, $(Nu - 1)$ is almost unaffected by rotation until $Q$ exceeds about 0-05. Beyond $Q = 0-2$, $(Nu - 1)$ drops away sharply, indicating the strong effect of the lower Ekman layer on the meridional circulation and advective heat transfer as its thickness approaches that of the thermal boundary layers, until by $Q = 4-0$, most of the heat is being carried by molecular conduction. The velocity scale $V$ is proportional to $Q$ (and $\Omega$) until $Q \sim 0-02$ (as suggested by the scale analysis, since $e^2\sigma^{-1} \sim 0-01$, see above). $S$ is accordingly a constant in this region ($S = 0-3$), and exhibits no dependence on $Q$. Between $Q = 0-02$ and 0-2, $V$ rises above the straight-line dependence on $Q$ extrapolated from the lower rotation regime, reaching its maximum value around $Q = 0-3$. Beyond $Q = 0-3$, $V$ drops away sharply, tending towards the predicted geostrophic dependence upon $Q$ as $Q^{-1}$ at high rotation ($Q > 2$). $S$ is found to rise somewhat between $Q = 0-02$ and 0-2, reaching its maximum value $S_{\text{max}} = 0-68$ around $Q = 0-2$, before dropping away very sharply toward the predicted geostrophic dependence as $Q^{-2}$ at high rotation. The overall behaviour of these global properties thus appears to bear out the qualitative predictions of the scale analysis, and highlights the region just below $Q = 1$ as being the range of parameters associated with the strongest super-rotation (i.e. $S \sim 1$).

According to the numerical results, the maximum values of $S$ and $Ro_c$ occur around $Q = 0-2$, where the Ekman layer is beginning to exert its influence upon the flow (so that the interior isotherm slope approaches unity), while the constraints of rotation in the interior are insufficient to prevent the metric term dominating the Coriolis term. The maximum possible value of $Ro_c$ can therefore be obtained by substituting $Q \sim 0-2$ into the estimate for $Ro_c$ from the scale analysis, given by

$$\max(Ro_c) \sim 5\varepsilon^{1/2}\sigma^{-1/2}$$

subject to the limitation that

$$\sigma \gg \varepsilon^{2/3}\eta^{-1/3}.$$

The largest super-rotation is therefore obtained for a flow with large aspect ratio, $\varepsilon$, and curvature, $\eta$, and the smallest Prandtl number consistent with the limit in Eq. (33). A numerically large super-rotation is therefore seen to require a relatively large Prandtl number, $\sigma \gg 1$, in order to maximize $Ro_c$ in Eq. (32) at the appropriate value of $Q\sigma^2\varepsilon^{-2}$.

The limits on $Ro_c$ and $S$ set by $\varepsilon = 1$, as used in the series of experiments illustrated in Fig. 7, are comparatively modest (the value obtained being $S_{\text{max}} \sim 0-7$, cf. $Ro_c < 1-6$ suggested by Eq. (32)), and so it is of interest to see whether the scalings for $Ro_c$ can be invoked to obtain a flow in which $S > 1$. Figure 8 shows the main fields from a numerical experiment (case D) conducted under similar external conditions as for the cases illus-
Figure 8. Contour maps of the steady-state fields for a numerical simulation of the thermally-driven axisymmetric circulation in a rotating fluid annulus with rigid, non-slip base, and stress-free side and top boundaries (case D, see text): (a) temperature (contour interval, 0.5 K); (b) $\gamma$ (contour interval, 0.2 s$^{-1}$; the region where $\gamma < 0$ is shown shaded); (c) $\Gamma/\beta a^2$ (contour interval, 0.5; the region where $\Gamma > \beta a^2$ is shown shaded); (d) $\chi$ (contour interval, 0.01 cm$s^{-1}$); (e) $\eta/\Omega b^2$ (see (c)) with vectors of $\mathbf{F}$ superimposed; (f) $-\nabla \cdot \mathbf{F}$ (contour interval, 0.0125 cm$s^{-2}$). Negative contours are dashed.

trated in Fig. 7, at the rotation rate appropriate to set $Q = 0.2$, but in which the mean radius has been decreased from 3.5 to 2 cm (while keeping the gap width $(b - a)$ constant, thereby roughly doubling the curvature, $\eta$), and the aspect ratio $e$ has been increased from 1 to 2. According to Eq. (32), this is expected to raise the maximum value of $S$ by a factor of $\sim 2 \times 2^{1/2}$, i.e. from 0.7 to 2.1. In keeping with the scaling, the value of $S_{\text{max}}$ found for the steady state has risen from 0.68 to 2.21, in excellent agreement with Eq. (32).

Most of the fields in Fig. 8 are little changed qualitatively from those in Figs. 2 and 3, except that the meridional streamfunction in Fig. 8(d) does not exhibit a thermally-indirect cell adjacent to the lower boundary (though a small patch where $\gamma < 0$ does
occur near mid-radius next to the lower boundary, see Fig. 8(b), consistent with the zero net torque requirement discussed above). The temperature field (Fig. 8(a)) is more spread out and less frontal in character than in Fig. 2(a), and the isotherms are oriented at nearly 45° to the horizontal in the upper half of the interior flow, as consistent with the requirements discussed in section 4(a) for cyclostrophic balance. Near the top of the annulus, γ is nearly independent of r except near the inner side boundary, confirming the role of viscous diffusion in smoothing out gradients in γ (see Fig. 8(b)). The maximum in γ again lies at the innermost top corner of the flow, and reaches a value of 1.9 s⁻¹, as compared with the system rotation rate Ω = 0.15 s⁻¹. The field of m in Fig. 8(c) shows the region where s > 0 extending over much more of the annulus than before, with the maximum value of the local super-rotation, s_{max}, rising to ~3.0 and located in the outermost top corner as in earlier examples. From a comparison of Figs. 8(c) and (e), it is clear that up-gradient horizontal viscous diffusion of m is again taking place to maintain a very high degree of super-rotation.

The primary dynamical balances for this flow are illustrated in Fig. 9, which shows vertical profiles calculated from the numerical simulation (cf. Williams 1967a, b, 1968), near mid-radius, of the most significant terms in the meridional and azimuthal momentum equations, and the thermodynamic and azimuthal vorticity equations, for comparison with the scale analysis in section 4(a) above. The predominance of the centrifugal term over the Coriolis term in the meridional momentum and vorticity equations is clearly

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**Figure 9.** Profiles of the terms in (a) the meridional momentum equation, (b) the azimuthal momentum equation, (c) the thermodynamic equation and (d) the azimuthal vorticity equation, calculated from the numerical simulation shown in Fig. 8 (case D). Terms are plotted as a function of height at mid-radius, and correspond to (a) $c = + f_0 \omega, m = v^2/r, p = -p_0/r_0$; (b) $a = v u, c = - f_0 \omega, m = -u v/r, v = \nu (\nabla^2 v - v/r^2)$; (c) $a(w) = w T, \ a(\psi) = u T, \ d = x \nabla^2 T$; (d) $m = (v^2)_x/r, c = f_0 \omega, b = g a T, \ v = \nu (\nabla^2 c - \xi/r^2)$. The fiducial mark in (b) labelled E indicates the Ekman layer thickness ($\nu/\Omega)^{1/2}$. 
apparent in Figs. 9(a) and (d), except in the lower Ekman layer where the viscous term becomes significant in the vorticity balance. The thermodynamic equation (Fig. 9(c)) is largely characterized by a balance between vertical and horizontal advection, implying local conservation of $T$ by the meridional flow except in a small region in the interior just below mid-level where the diffusion term becomes significant. This is consistent with the relatively large Nusselt number in this regime (see Fig. 7), in which the lower Ekman layer exerts only a weak influence on the meridional transport properties of the flow. In the azimuthal momentum equation (Fig. 9(b)), the interior balance involves a mixture of the metric, advective and Coriolis terms, consistent with angular momentum conservation, while the Ekman layer (thickness indicated by $E$ in Fig. 9(b)) is characterized by a nonlinear advective/viscous balance as suggested above. The present example clearly shows that a very high super-rotation can readily be achieved in a moderately rotating fluid, provided the distribution of sources and sinks of $m$ is appropriately chosen and a suitable meridional circulation is driven by differential heating and cooling. The parameters chosen for the case in Figs. 8 and 9 are by no means particularly extreme, suggesting that much higher values of $S$, $\text{Ro}_c$, $s_{\text{max}}$, and $\gamma_{\text{max}}/\Omega$ could also be achieved with a suitable combination of parameters, involving even larger values of $\varepsilon$, $\eta$, and $\sigma$, as indicated in Eqs. (32) and (33).

5. Concluding remarks

In exploring the properties of super-rotation and the angular momentum budget of axisymmetric, thermally-driven circulations in the cylindrical annulus, we have obtained a number of important results which shed useful insight into a variety of dynamical problems. Because of the form of the stress tensor in a circularly-symmetric system, molecular viscosity acts to diffuse $m$ down the local gradient of strain, which acts along $-\nabla \gamma$. Any theoretical model which uses a form of diffusion similar to viscosity should take this into account, especially since $\nabla \gamma$ can possess a component anti-parallel to $\nabla m$ (resulting in up-gradient transfer). It is important to note also that this property is not restricted to an incompressible fluid (see part II).

Super-rotation, as defined with respect to the limiting values of $m$ and its volume-integrated form in an inviscid fluid, is associated with a meridional circulation which is 'direct' for $m$ in the same sense as for the transport of heat (i.e. transferring positive anomalies with respect to the global mean vertically upwards, and horizontally down the initial gradient of $m$). A circulation in the opposite, 'indirect', sense will, in general, result in local and/or global sub-rotation, depending upon the imposed boundary conditions. For boundary conditions appropriate to the laboratory annulus, the non-slip sidewalls provide significant sinks for $m$, thereby limiting the degree of super- or sub-rotation obtained. By allowing the sidewall boundary condition to be stress-free, these limitations can be overcome, resulting in a relatively large super-rotation under certain conditions. It is important to note that, as demonstrated in case C above, local super-rotation can occur in an entirely isolated system (wholly confined by stress-free boundaries), as a result of the angular momentum expulsion associated with the combined effects of advection and up-gradient diffusion by viscosity. The latter result may have some important implications for the super-rotation behaviour of simple models of the atmospheric circulation of entirely fluid planets, such as Jupiter and Saturn, and stars (see part II).

The scale analysis in 4(a) provides a useful and comprehensive means of analysing the dependence of various global parameters for an axisymmetric flow on external conditions and fluid properties. Three main global super-rotation regimes were deduced
for the system with only one non-slip boundary, depending upon the rotation rate (as measured by the boundary layer parameter \( Q \), see 4(a)). Moderate super-rotation (\( S < 1 \)) occurs at the lowest rotation rates, but only a very small super-rotation in the rapidly rotating, geostrophic regime (\( Q \gg 1 \)). The largest values of \( S (>1) \) were obtained at moderate rotation rates (\( Q \approx 1 \)), provided \( \sigma, \varepsilon \) and \( \eta \) were appropriately scaled. Under these conditions, the flow was characterized by a cyclostrophic balance (between buoyancy torques and centrifugal effects) in the vorticity equation at upper levels, and a nonlinear/viscous balance in the angular momentum equation within the Ekman layer. The latter balances are very similar to that deduced from the scale analysis of Gierasch (1975) in his simple model for rapid super-rotation in the Venus atmosphere. Indeed, the numerical example presented above as case D may be regarded as a demonstration of the viability of this model, in generating a steady, rapid super-rotation in a dynamically-consistent way from a plausible initial state. A more detailed discussion of the wider implications of these results is given in part II.

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