On the relative dispersion of two particles in homogeneous stationary turbulence and the implications for the size of concentration fluctuations at large times

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SUMMARY

A model for the separation of two particles in homogeneous stationary turbulence is derived by considering the analogous process of one-particle dispersion in inhomogeneous turbulence. Numerical simulations suggest that the intensity of concentration fluctuations tends to zero at large times.

1. INTRODUCTION

Several papers have appeared recently on the use of two-particle dispersion models to estimate the size of fluctuations in the concentration of a tracer dispersing in a turbulent flow (e.g. Durbin 1980 and Sawford 1982, 1983). These models consider turbulence in one dimension only and assume that the evolution of the particle positions, \( z_1 \) and \( z_2 \), and velocities, \( w_1 \) and \( w_2 \), is Markovian in \((z_1, z_2, w_1, w_2)\) space. The models of Durbin (1980) and Sawford (1982) give rise to different predictions for the size of concentration fluctuations at large times. In particular the intensity of the concentration fluctuations, \( \overline{(c^2 - \overline{c}^2)}^{1/2} / \overline{c} \), where \( c \) is the concentration and an overbar denotes an ensemble average, tends to zero in Sawford's model but remains finite in Durbin's.

The purpose of this note is to draw attention to a connection between two-particle dispersion in homogeneous turbulence and one-particle dispersion in inhomogeneous turbulence, and to discuss the implications for concentration fluctuations.

2. ONE- AND TWO-PARTICLE DISPERSION MODELS

Most random walk models of one-particle dispersion in inhomogeneous stationary turbulence take the form of a Markovian model for the evolution of \((z,w)\) where \( z \) is the position and \( w \) the velocity of a tracer particle. Let \( g_s(z,w) \) denote the density function of the distribution in \((z,w)\) space of all the particles of air in an ensemble of flows. For the model to give acceptable results it is necessary that the model admits a steady state particle distribution with density function \( g_s \) (Thomson 1984).

Most models of two-particle dispersion in homogeneous stationary turbulence give rise to a Markovian model for the evolution of \((\Delta z, \Delta w)\) where \( \Delta z = z_1 - z_2 \) and \( \Delta w = w_1 - w_2 \). Consider an ensemble of flows. For every pair of air particles in every flow we can define the particle separation \( \Delta z \) and the relative velocity of the particles \( \Delta w \). Let \( g_s^\Delta(\Delta z, \Delta w) \) denote the density function of the distribution in \((\Delta z, \Delta w)\) space of all pairs of particles of air in the ensemble of flows. Suppose that the tracer is well mixed at \( t = 0 \) and hence that the density function of the \((\Delta z, \Delta w)\) distribution of tracer is proportional to \( g_s^\Delta \). The evolution of the \((\Delta z, \Delta w)\) distribution of tracer can be calculated from the Markovian model for \((\Delta z, \Delta w)\) by allowing each particle pair to move independently through \((\Delta z, \Delta w)\) space as prescribed by the model (this is analogous to calculating the evolution of the \((z,w)\) distribution of tracer by assuming each particle travels independently in \((z,w)\) space). In reality the \((\Delta z, \Delta w)\) distribution of tracer will remain unchanged.

Hence, for a two-particle model to be acceptable, it should admit a steady state particle distribution in \((\Delta z, \Delta w)\) space with density function \( g_s^\Delta \).

Thomson (1984) showed how one-particle models for the evolution of \((z,w)\) could be designed to yield a steady state distribution of particles with given density function \( g_s \). These same methods can be used to design models for the evolution of \((\Delta z, \Delta w)\) which admit a steady state distribution with a given density function \( g_s^\Delta \).

3. A MODEL FOR THE SEPARATION OF TWO PARTICLES IN HOMOGENEOUS STATIONARY TURBULENCE

The density function \( g_s^\Delta \) has a number of properties which we note here. Firstly, on obvious physical grounds, \( \int_{-\infty}^{\infty} g_s^\Delta d\Delta w \) is independent of \( \Delta z \) (i.e. the marginal distribution of \( \Delta z \) for all
pairs of particles of air is uniform), and secondly
\[
\int_{-\infty}^{\infty} (\Delta w)^2 g_{\Delta z} d\Delta w \int_{-\infty}^{\infty} g_{\Delta w} d\Delta w = R(\Delta z)
\]
where \( R \) is the structure function which is zero at small \( \Delta z \) and becomes equal to twice the fixed point velocity variance, \( \sigma_{w}^2 \), at large \( \Delta z \). If we assume that the difference between the velocities at two fixed points is normally distributed, then
\[
g_{\Delta w}^2(\Delta z, \Delta w) = \frac{1}{(2\pi R)^{1/2}} \exp\{- (\Delta w)^2 / 2R(\Delta z)\}. \tag{1}
\]
In reality \( g_{\Delta w}^2 \) is not Gaussian, especially for small \( \Delta z \) (Batchelor 1953, pp. 170–173). However, Eq. (1) may be regarded as not too unrealistic for many purposes.
Durbin’s (1980) model for \( (\Delta z, \Delta w) \) is
\[
d\Delta z / dt = (R(\Delta z))^{1/2} U(t), \quad dU / dt = -U / \tau + (2 / \tau)^{1/2} \xi(t) \tag{2}
\]
where \( \tau \) is the Lagrangian time-scale and \( \xi \) is white noise with \( \langle \xi \rangle = 0 \) and \( \langle \xi(t_1) \xi(t_2) \rangle = \delta(t_1 - t_2) \). \( \delta \) is the Dirac delta function and an overbar denotes an ensemble average. This model does not admit a steady state solution with \( \int_{-\infty}^{\infty} g_{\Delta w}^2 d\Delta w \) independent of \( \Delta z \) (see the analysis of the inhomogeneous one-particle equivalent of Eq. (2) by Thomson (1984)). Sawford’s (1982) model does not give a realistic steady state solution either, and the evolution of \( (\Delta z, \Delta w) \) depends unphysically on the ensemble mean of \( \Delta z \).

We propose the following model for \( (\Delta z, \Delta w) \):
\[
d\Delta z / dt = (R(\Delta z))^{1/2} U(t), \quad dU / dt = -U / \tau + (2 / \tau)^{1/2} \xi(t) + \partial R^{1/2} / \partial \Delta z. \tag{3}
\]
This is a modified version of (2) which admits the steady state solution (1) (Thomson 1984). (If the fact that \( g_{\Delta w}^2 \) is actually non-Gaussian has an important influence on the separation of particles then this model will require modification—this requires further investigation.) \( R \) is taken to be \( 2\sigma_{w}^2(\Delta z^2 / (\Delta z^2 + L^2))^{1/3} \), where \( L \) is related to the integral length-scale. This form is the same as that used by Durbin (1980); it is suitable for high Reynolds number flows and has the correct inertial subrange form at small \( \Delta z \) (i.e. for small \( \Delta z \), \( R \) depends only on \( \Delta z \) and the dissipation which is assumed proportional to \( \sigma_{w}^2 / L \)). In contrast to Durbin’s work, the time-scale \( \tau \) is also chosen to obey inertial subrange scaling at small \( \Delta z \) and is taken equal to \( R(\Delta z)L / 2\sigma_{w}^2 \). This is reasonable because, for small \( \Delta z \), the eddies responsible for dispersing the particles have a size of order \( \Delta z \) and are characterized by the dissipation and \( \Delta z \) alone.

In high Reynolds number turbulence the accelerations are very large and have a very short correlation scale in both space and time (Monin and Yaglom 1975, p. 370; Sawford 1984). The model (3) idealizes this, and with the result that the auto-covariance of the relative acceleration of the two particles (i.e. \( \Delta a(t) \Delta a(t + \tau) \)) where \( \Delta a \) is the relative acceleration and an overbar denotes an average over particle pairs with a given separation at time \( t \) contains a delta function singularity at \( \tau = 0 \). Because of the short length-scale of the acceleration, the size of this delta function peak should be independent of \( \Delta z(t) \). This is true only if \( \tau \) depends on \( \Delta z \) as given above.

4. SMALL-TIME BEHAVIOUR OF PARTICLE PAIRS

The small-time behaviour of a pair of particles can be calculated exactly without reference to a random walk model, but, to get the right answer, it is necessary to consider more than one dimension. Consider a pair of particles which, at \( t = 0 \), are positioned at \( x_1 \) and \( x_2 \) where \( x_1 \) and \( x_2 \) are two points separated in the \( z \) direction (this separation is not assumed to be small). If we consider all such particle pairs in the ensemble of flows, then \( \Delta w = 0 \) at \( t = 0 \). Let \( u_1 \) and \( u_2 \) be the velocities at \( x_1 \) and \( x_2 \) respectively and let \( \Delta u = u_1 - u_2 \). Superscripts will be used to denote Cartesian components. The rate of change of \( \Delta w \) at \( t = 0 \) is given by
\[
d\Delta w / dt = D\Delta u / Dt = \partial \Delta u / \partial t + u_1 \cdot \nabla_1 (\Delta u^2) + u_2 \cdot \nabla_2 (\Delta u^2)
\]
where \( D / Dt \) denotes the material derivative. Using incompressibility and taking the ensemble average yields
\[
\overline{d\Delta w / dt} = \nabla_1 \cdot (\overline{u_1 \Delta u^2}) + \nabla_2 \cdot (\overline{u_2 \Delta u^2}).
\]
Changing coordinates to \( \Delta x = x_1 - x_2 \) and \( X = x_1 + x_2 \) gives
\[
\overline{d\Delta w / dt} = (\nabla_X + \nabla_{\Delta x}) \cdot (\overline{u_1 \Delta u^2}) + (\nabla_X - \nabla_{\Delta x}) \cdot (\overline{u_2 \Delta u^2}) = \nabla_X \cdot (\overline{(u_1 - u_2) \Delta u^2}). \tag{4}
\]
Now, in a homogeneous incompressible flow, \( \nabla \Delta \bar{w}_1 \bar{w}_2 = 0 \) (see e.g. Batchelor (1953, p. 27)). When combined with (4) this implies \( d\Delta w/dt = 0 \). In a one-dimensional model however it is impossible to allow correctly for the correlations between the three components of \( \Delta u \). The model (3) yields \( d\Delta w/dt = \partial R/\partial \Delta z \) (which is consistent with neglecting the correlations between the components of \( \Delta u \) in (4)) while (2) gives \( d\Delta w/dt = \partial R/\partial \Delta z \). It is not clear whether the incorrect value of \( d\Delta w/dt \) given by the model (3) is a serious error—if so it can only be corrected by considering two- or three-dimensional models.

5. NUMERICAL SIMULATIONS USING EQ. (3)

Some numerical simulations were carried out using Eq. (3). For numerical reasons \( R(\Delta z) \) was replaced by \( R(\Delta z_{crit}) \) whenever \( |\Delta z| \) was less than \( \Delta z_{crit} \). The results presented below were obtained with \( \Delta z_{crit} = 10^{-4}L \). Sensitivity tests showed the results to be insensitive to changes in \( \Delta z_{crit} \) of an order of magnitude. The time-step \( \Delta t \) used to solve (3) was variable and equal to \( \tau/10 \). This also ensures that \( \Delta t \partial R/\partial \Delta z \ll 1 \) and hence that the fractional change in \( R^{1/2} \) over a time-step is small.

The initial particle separation was zero. The results presented for \( t \leq 10L/\sigma_w \) were obtained by simulating 40000 particle pairs; beyond \( t = 10L/\sigma_w \) only 15000 particle pairs were used.

In Fig. 1 \( \Delta z^2 \) is plotted. This shows \( \tau^2 \) behaviour at small times as expected in an inertial subrange. In Fig. 2 the probability density function of \( \Delta z \), \( P(\Delta z) \), is presented for two values of \( \tau \). At large times \( P(\Delta z) \) is very close to Gaussian. Figure 3 shows \( P(\Delta z) \) at \( t = 50L/\sigma_w \), plotted on a scale which displays the behaviour near \( \Delta z = 0 \) in detail. The scatter is due to the limited number of particle pairs which were used in the simulation. There is no hint of the \( \Delta z^{-1/3} \) behaviour at small \( \Delta z \) which occurs in Durbin's (1980) model. Unfortunately the model presented here is less amenable than Durbin's model to analytic treatment and so we cannot be certain that \( P(\Delta z) \) becomes exactly Gaussian at large times; however, the numerical simulations suggest that this is so.

Figure 1. The mean square particle separation as given by the model.
6. IMPLICATIONS FOR THE INTENSITY OF CONCENTRATION FLUCTUATIONS

The size of concentration fluctuations can be calculated from the statistics of the motion of particle pairs as described by Durbin (1980). Unfortunately the model (3) does not give enough information to allow this to be done—in particular it says nothing about the motion of the particle centroid, \((z_1 + z_2)/2\). However, Sawford (1983) showed that, given some very plausible assumptions, the intensity of concentration fluctuations tends to zero if \(P(\Delta z)\) becomes Gaussian at large times. We have already seen that in our model \(P\) becomes at least very close to, and possibly exactly, Gaussian—this suggests that the intensity of fluctuations tends to zero at large times.

Of course the model presented here is only a model, but one which we believe contains a
Figure 3. The probability density function, $P(\Delta z)$, of the particle separation at $t = 50L/\sigma_v$, plotted on a scale which displays the behaviour near $\Delta z = 0$ in detail. The scatter is a result of the limited number of particle pairs used in the simulation.

A truer representation of the physics than other models. Two improvements which could be made to the model are the incorporation of skewness in $g_3 v^3$ and the extension to three dimensions. It is not clear whether these improvements will make qualitative or merely quantitative differences to the results presented above; hence these results must be regarded as only preliminary. In order to test the model against data it will be necessary to extend it to inhomogeneous and non-stationary situations where the behaviour of the concentration fluctuations may well be qualitatively different (for example, in a surface layer the length-scale of the turbulence increases with height, which will prevent the cloud of tracer ever becoming much larger than the length-scale).

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